# Approximation Theorems for spherical monogenics of complex degree

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**Abstract**

Spherical monogenics of complex degree correspond to local eigenfunctions of the (Atiyah-Singer) Dirac operator on the unit sphere $S^{m-1}$ of $\mathbb{R}^m$. In this paper we will consider Runge approximation Theorems and some of their consequences for this class of functions.

## 1 Introduction

Let $(e_1, \ldots, e_m)$ be an orthonormal basis of Euclidean space $\mathbb{R}^m$ endowed with the inner product $\langle x, y\rangle = \sum_{i=1}^m x_i y_i,\ x, y \in \mathbb{R}^m$. By $\mathbb{C}_m$ we denote the complex $2^m$-dimensional Clifford algebra over $\mathbb{R}^m$ generated by the relations $e_i^2 = -1,\ i = 1, \ldots, m$ and $e_i e_j + e_j e_i = 0,\ i \neq j$. An element of $\mathbb{C}_m$ is of the form $a = \sum_{A \subset M} a_A e_A,\ a_A \in \mathbb{C},\ M = \{1, \ldots, m\}$ and $e_\phi = e_0 = 1$. The elements $a \in \mathbb{C}_m$ such that $a_A \in \mathbb{R}$ for all $A \subset M$ determine a real subalgebra of $\mathbb{C}_m$ denoted by $\mathbb{R}_m$; this is the *real* Clifford algebra over $\mathbb{R}^m$ generated by the above relations. Conjugation on $\mathbb{C}_m$ is the anti-involution on $\mathbb{C}_m$ given by $\bar{a} = \sum_{A \subset M} \bar{a}_A \bar{e}_A$ where $\bar{e}_A = \bar{e}_{\alpha_h} \ldots \bar{e}_{\alpha_1}$ and $\bar{e}_j = -e_j,\ j = 1, \ldots, m$. Vectors $x \in \mathbb{R}^m$ are identified with Clifford numbers $x = \sum_{j=1}^m x_j e_j$. For vectors $x, y \in \mathbb{R}^m$,

$$xy = x \cdot y + x \wedge y$$

where the inner product and outer product are given by

$$x \cdot y = -\langle x, y\rangle = -\sum_{j=1}^m x_j y_j, \quad x \wedge y = \sum_{i<j} (x_i y_j - x_j y_i) e_{ij} \ .$$

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A norm $|\ |_0$ on $\mathbb{C}_m$ is given by $|a|_0^2 = [a\bar{a}]_0$ and satisfies $|a+b|_0 \leq |a|_0 + |b|_0$, $|ab|_0 \leq 2^{\frac{m}{2}}|a|_0|b|_0$.
Let $\partial_x = \sum_{i=1}^m e_i\partial_{x_i}$ be the Dirac operator on $\mathbb{R}^m$. In spherical coordinates $x = \rho\omega,\ \rho = |x| = (x_1^2 + \ldots + x_m^2)^{1/2}$ and $\omega \in S^{m-1}$, the Dirac operator admits the polar decomposition $\partial_x = \omega(\partial_\rho + \frac{1}{\rho}\Gamma_\omega)$ where $\Gamma_\omega = -x \wedge \partial_x$ is the spherical Dirac operator on $S^{m-1}$. In terms of the momentum operators $L_{ij} = x_i\partial_{x_j} - x_j\partial_{x_j},\ i,j = 1,\ldots,m$ on $\mathbb{R}^m$ the $\Gamma$-operator is given by $\Gamma = -\sum_{i<j} e_{ij}L_{ij}$. In [18] we studied spherical monogenics of degree $\alpha$, $(\alpha \in \mathbb{C})$ on the unit sphere $S^{m-1}$ in $\mathbb{R}^m$. Let us recall the following

**Definition**

Let $\Omega \subset S^{m-1}$ be open. A $C^1$-function $f : \Omega \to \mathbb{C}_m$ satisfying $(\Gamma+\alpha)f = 0$ in $\Omega$ is called a spherical monogenic of order (degree) $\alpha$ in $\Omega$. The right module of this class of functions is denoted by $M^\alpha_{(r)}(\Omega)$.
The value $\alpha = \frac{-m+1}{2}$ plays a special role in the scheme presented. The corresponding spherical monogenics are null solutions of the (Atiyah-Singer) Dirac operator $\omega(\Gamma_\omega + \frac{-m+1}{2})$ on $S^{m-1}$ used in differential geometry (see [10]). Amongst the operators $\omega(\Gamma+\alpha)$ this particular operator has the special property that it is conformally invariant. The family of operators $\omega(\Gamma+\alpha)$, $\alpha \in \mathbb{C}$, can be regarded as a holomorphic perturbation of the Dirac operator on the sphere. Spherical monogenics of degree $\alpha \neq \frac{-m+1}{2}$ can be regarded as *(local) eigenfunctions* of the (A-S) Dirac operator on $S^{m-1}$. If e.g. $(\Gamma+\alpha)f = 0$ in $\Omega$, then $\omega(\Gamma + \frac{-m+1}{2})(1 \pm \omega)f = \mp(\alpha + \frac{m-1}{2})(1\pm\omega)f$ in $\Omega$. If on the other hand $\omega(\Gamma + \frac{-m+1}{2})g = \lambda g$ in $\Omega$, then $(\Gamma \pm (\lambda + \frac{-m+1}{2}))(1\pm\omega)g = 0$ in $\Omega$. Hence eigenfunctions of the $\Gamma$-operator (which is the submanifold Dirac operator on $S^{m-1}$ induced by the Dirac operator on the embedding space $\mathbb{R}^m$) correspond to eigenfunctions (with shifted eigenvalue) of the Dirac operator on the sphere (see also [3]).

In this paper we prove the following type of Runge approximation Theorems. Let $\Omega, \Omega' \subset S^{m-1}$ be open and let $K \subset S^{m-1}$ be compact. Then $M^\alpha_{(r)}(\Omega)$ is dense in $M^\alpha_{(r)}(K),\ K \subset \Omega$ and $M^\alpha_{(r)}(\Omega)$ is dense in $M^\alpha_{(r)}(\Omega'),\ \Omega' \subset \Omega$ iff $\Omega\backslash K$ and $\Omega\backslash\Omega'$ satisfy some topological condition. As our proof of these Theorems relies on the existence of a Cauchy kernel for the operator $\Gamma+\alpha$, we impose the condition $\alpha \in \mathbb{C}\setminus(\mathbb{N}\cup(-m+1-\mathbb{N}))$. As a consequence we solve the equation $(\Gamma+\alpha)f = g,\ g \in C^\infty(\Omega),\ \Omega$ open and prove Mittag-Leffler's Theorem for the operator $\Gamma+\alpha$ on $S^{m-1}$. As a result we solve the inhomogeneous equation $(\Gamma+\alpha)f = g$ in $\Omega$, $g \in C^\infty(\Omega)$, $\Omega \subset S^{m-1}$ open. This leads to Mittag-Leffler's Theorem for the operators $\Gamma+\alpha$.

## 2 Some introductory Lemmas

The following lemmas of a topological nature are of importance. We list them without proof.

Let $u \in S^{m-1}$. Then we define the ball $B_S(u,\delta) = B(u,\delta) \cap S^{m-1} = \{\omega \in S^{m-1} : \sqrt{2(1-\langle\omega,u\rangle)} < \delta\}$. Obviously the sets $B_S(u,\delta),\ 0 < \delta \leq 2$, form a fundamental system of connected neighbourhoods of $u$ on $S^{m-1}$.

**Lemma 1.** *Let $K \subset \Omega \subset S^{m-1}$, $K$ compact and $\Omega$ open. Then the following conditions are equivalent:*

*(i)* $\Omega \backslash K$ *has no components of which the closure (in* $S^{m-1}$*) is contained in* $\Omega$

*(ii) For each component* $W$ *of* $S^{m-1} \backslash K$ *:* $\overline{W} \cap (S^{m-1} \backslash \Omega) \neq \emptyset$ .

**Lemma 2.** $V$ *is a component of* $\Omega \backslash K$ *satisfying* $\overline{V} \subset \Omega$ *iff* $\partial V \subset K$.

**Lemma 3.** *Let* $\Omega, \Omega'$ *be open subsets of* $S^{m-1}$ *such that* $\Omega' \subset \Omega$. *Then the following conditions are equivalent:*

*(i)* $\Omega \backslash \Omega'$ *has no components which are closed in* $S^{m-1}$

*(ii) For each component* $W$ *of* $S^{m-1} \backslash \Omega'$ *:* $\overline{W} \cap (S^{m-1} \backslash \Omega) \neq \emptyset$

*(iii) For each component* $G$ *of* $\Omega \backslash \Omega'$ *:* $\overline{G} \cap \partial \Omega \neq \emptyset$ .

**Lemma 4.** *Let* $\Omega$, $\Omega'$ *be open subsets of* $S^{m-1}$, $\Omega' \subset \Omega$ *and let* $V$ *be a component of* $\Omega \backslash \Omega'$ *which is closed in* $S^{m-1}$. *Then* $\partial V \subset \partial \Omega'$.

**Lemma 5.** *(Exhaustion of open sets on* $S^{m-1}$ *by means of compacta)*
*Define for* $j \in \mathbb{N}_0$*:*

$$\begin{aligned} K_j &= \{\omega \in S^{m-1} : d(\omega, S^{m-1} \backslash \Omega) \geq \frac{1}{j}\} \\ G_j &= \{\omega \in S^{m-1} : d(\omega, S^{m-1} \backslash \Omega) > \frac{1}{j}\} \end{aligned}$$

*where* $d(\omega, \xi) = |\omega - \xi| = \sqrt{2(1 - \langle \omega, \xi \rangle)}$, $\omega, \xi \in S^{m-1}$. *Then:*

*(i)* $K_j \subset \overset{\circ}{K}_{j+1}$, $\Omega = \cup K_j = \cup \overset{\circ}{K}_j$

*(ii) Each compact set* $K \subset \Omega$ *is contained in some* $K_{j_0}$

*(iii) Each component of* $S^{m-1} \backslash K_j$ *contains a component of* $S^{m-1} \backslash \Omega$

*(iv)* $\Omega \backslash K_j$ *has no components of which the closure is contained in* $\Omega$

*(v) Put* $H_1 = G_2$, $H_j = G_{j+1} \backslash \overline{G}_{j-1}$, $j \geq 2$*; then* $\{H_j\}_{j \geq 1}$ *is a locally finite cover of* $\Omega$ .

**Theorem 6.** *Let* $\Omega$ *be a proper open subset of* $S^{m-1}$ *and let* $K \subset \Omega$ *be compact such that* $\Omega \backslash K$ *has no components of which the closure is contained in* $\Omega$. *Then there exists a fundamental system* $\{F_i\}$ *of compact neighbourhoods of* $K$ *in* $\Omega$ *such that for each* $i$*:*

*(i)* $F_i$ *has piecewise smooth boundary*

*(ii)* $\Omega \backslash F_i$ *has no components whose closure is contained in* $\Omega$ .

*Proof.*
Consider the sets

$$K_j = \{\omega \in S^{m-1} : d(\omega, K) \le \frac{1}{j}\} = \cup_{\omega\in K}\overline{B}_S(\omega, \frac{1}{j})$$
$$G_j = \{\omega \in S^{m-1} : d(\omega, K) < \frac{1}{j}\} = \cup_{\omega\in K}B_S(\omega, \frac{1}{j}) \ , \quad j \in \mathbb{N}_0 \ .$$

Each $K_j$ is compact while each $G_j$ is open and $K_{j+1} \subset G_j \subset \overset{\circ}{K}_j$. By compactness of $K$; $\{G_j\}_{j\in\mathbb{N}_0}$ and $\{K_j\}_{j\in\mathbb{N}_0}$ are fundamental systems of neighbourhoods of $K$ on $S^{m-1}$. For each $j, G_j$ is an open cover of $K$ and therefore has a finite subcover $\cup_{i_j=1}^{n_j} B_S(\xi_{i_j}, \frac{1}{j}) \subset G_j$ covering $K$. Put $\tilde{K}_j = \cup_{i_j=1}^{n_j}\overline{B}_S(\xi_{i_j}, \frac{1}{j})$, then $\tilde{K}_j$ is compact and has piecewise smooth boundary. As $K \subset \tilde{K}_j \subset K_j$, $\{\tilde{K}_j\}_{j\in\mathbb{N}_0}$ is a fundamental system of neighbourhoods of $K$ on $S^{m-1}$; since $K_{j_0} \subset \Omega$ for some $j_0$, $\{\tilde{K}_j\}_{j\ge j_0}$ is then a fundamental system of neighbourhoods of $K$ in $\Omega$. Consider now an arbitrary $\tilde{K}_j$, $j \ge j_0$. Since $\Omega\backslash\tilde{K}_j \subset \Omega\backslash K$, each component of $\Omega\backslash\tilde{K}_j$ is contained in a component of $\Omega\backslash K$. As $\tilde{K}_j$ is the union of a finite number of closed balls, $\Omega\backslash\tilde{K}_j$ has only a finite number of components; say $W_i^l$, $i = 1, \ldots, n_j, 1 \le l \le k_j$ satisfying $\overline{W}_i^l \subset \Omega$. Call $\Omega_i$ the components of $\Omega\backslash K$ such that $W_i^l \subset \Omega_i$, $i = 1, \ldots, n_j$. Suppose now that $G$ is a component of $\Omega\backslash K$ which does not contain any component of $\Omega\backslash\tilde{K}_j$, then $G\cap(\Omega\backslash\tilde{K}_j) = \emptyset$ or $G \subset (\Omega\backslash K)\backslash(\Omega\backslash\tilde{K}_j) = \tilde{K}_j\backslash K \subset \tilde{K}_j$, hence $\overline{G} \subset \Omega$ which contradicts the assumption in the Lemma; therefore each component of $\Omega\backslash K$ contains a component of $\Omega\backslash\tilde{K}_j$. In the same way each component of $\Omega\backslash K$ contains a component of $\Omega\backslash L$ where $L$ is the compact set given by $L = \tilde{K}_j \cup (\cup_{i,l}\overline{W}_i^l)$; as $\Omega\backslash L$ and $\Omega\backslash\tilde{K}_j$ have the same components of which the closure is not contained in $\Omega$, it follows that each $\Omega_i$ contains a component $G_i$ of $\Omega\backslash\tilde{K}_j$ such that $\overline{G}_i \cap \partial\Omega \ne \emptyset$. Choose in each $\Omega_i$ containing a component $W_i^l$, a set $G_i$ satisfying $\overline{G}_i \cap \partial\Omega \ne \emptyset$ and choose points $a_i \in \overline{G}_i \cap \partial\Omega$, $b_i^l \in W_i^l$. As $\Omega_i$ is open and connected, $\Omega_i$ is path connected and for each $l$ there is an arc $L_i^l$ in $\Omega_i$ connecting $a_i$ and $b_i^l$. Choose for each $\xi \in L_i^l$ an open ball $B_S(\xi, r_\xi) \subset \Omega_i$, the union of these balls forms an open cover of $L_i^l$; as $L_i^l$ is compact, there exists a finite subcover $T_i^l = \cup_{j=1}^{N(l,i)} B_S(\xi_j, r_{\xi_j}) \subset \Omega_j$. Put $F_j = \tilde{K}_j\backslash\cup_{i,l}T_i^l$, then $F_j$ is compact and has piecewise smooth boundary. Since $T_i^l \cap K = \emptyset$ it follows that $K \subset F_j \subset \tilde{K}_j$. Applying this construction to each $j \ge j_0$ we thus obtain a fundamental system $\{F_j\}_{j\ge j_0}$ of compact neighbourhoods of $K$ in $\Omega$. Call $H_{jk}$ the remaining components of $\Omega\backslash\tilde{K}_j$. Then:

$$\Omega\backslash F_j = \Omega\backslash(\tilde{K}_j\backslash(\cup_{i,l}T_i^l)) = (\Omega\backslash\tilde{K}_j) \cup (\cup_{i,l}T_i^l) = \cup_{i,l}(W_i^l \cup T_i^l \cup G_i) \cup (\cup_k H_{jk}) \ .$$

Put $\tilde{W}_i^l = W_i^l \cup T_i^l \cup G_i$; then $\tilde{W}_i^l$ is connected and $\overline{\tilde{W}}_i^l \cap \partial\Omega \ne \emptyset$. Since each component of $\Omega\backslash F_j$ contains some connected set $\tilde{W}_i^l$ or $H_{jk}$, $\Omega\backslash F_j$ has no components whose closure is contained in $\Omega$. This proves the Theorem. ∎

The following is proved in [11].

**Theorem 7.** *Let $Y$ be a locally compact Hausdorff space, let $X$ be a closed subset of $Y$ and let $K$ be a connected component of $X$ which is compact. Then there exists a fundamental system of neighbourhoods $U$ of $K$ in $Y$ such that*

$$(\partial U) \cap X = \emptyset \ ,$$

*$\partial U$ denoting the boundary of $U$ in $Y$.*

In particular, this Theorem is valid when we put $Y = \Omega,\ X = \Omega \backslash \Omega',\ \Omega' \subset \Omega \subset S^{m-1};\ \Omega', \Omega$ open.

## 3 Runge Theorems

The Cauchy kernel for spherical monogenics of complex degree $\alpha$ is denoted by $E_\alpha(\xi,\omega)$ and satisfies $(\Gamma + \alpha)E_\alpha(\xi,\omega) = \delta(\omega - \xi)\xi$, $\alpha \in \mathbb{C} \setminus (\mathbb{N} \cup (-m+1-\mathbb{N}))$ (see also [18]).

**Definitions.**

(i) Let $K \subset S^{m-1}$ be compact and let $\mu$ be a $\mathbb{C}_m$-valued regular Borel measure on $S^{m-1}$ with support $[\mu]$ contained in $K$. Then the Cauchy transform $CT_\alpha(\mu)$ of the measure $\mu$ is defined by:

$$CT_\alpha(\mu)(\xi) = \int_{S^{m-1}} d\mu(\omega) E_\alpha(\xi,\omega), \quad \omega \in S^{m-1}.$$

By a standard argument $(CT_\alpha(\mu)(\xi))(\Gamma_\xi - \beta) = 0$ in $S^{m-1} \backslash [\mu]$, $\alpha + \beta = -m+1$. By means of the Riesz Representation Theorem the dual of the right module $C^0_{(r)}(K)$ of continuous functions on $K$ can be identified with the left module of $\mathbb{C}_m$-valued regular Borel measures on $S^{m-1}$ having support contained in $K$ and

$$\langle \mu, h \rangle = \int_{S^{m-1}} d\mu(\omega) h(\omega), \quad h \in C^0_{(r)}(K)\ .$$

(ii) Let $K \subset S^{m-1}$ be compact. Then $M^\alpha_{(r)}(K)$ consists of the elements $f$ which are null solutions of $\Gamma + \alpha$ in some open neighbourhood of $K$. On $M^\alpha_{(r)}(K)$ we consider two different topologies. First of all, $M^\alpha_{(r)}(K)$ is a subspace of $C^0(K)$. The space $C^0(K)$ endowed with the supremum norm $\| f \| = \sup_K |f|_0$ is a Banach space and $M^\alpha_{(r)}(K)$ can be given the topology inherited from the Banach space $C^0(K)$. In general $M^\alpha_{(r)}(K)$ is not a closed subspace of $C^0(K)$. Secondly one can consider $M^\alpha_{(r)}(K) = \lim \mathrm{ind}_{K \subset \Omega} M^\alpha_{(r)}(\Omega)$, i.e. $M^\alpha_{(r)}(K)$ is given the inductive limit topology determined by the Fréchet modules $M^\alpha_{(r)}(\Omega),\ K \subset \Omega$ .

The following Lemma plays an important role in the sequel.

**Lemma 8.** *Let $K \subset S^{m-1}$ be compact and suppose that $\mu$ is a $\mathbb{C}_m$-valued regular Borel measure on $S^{m-1}$ having support contained in $K$. Then:*

$$\int_{S^{m-1}} d\mu(\omega) f(\omega) = 0 \textit{ for all } f \in M^\alpha_{(r)}(K) \textit{ iff } CT_\alpha(\mu)(\xi) = 0 \textit{ in } S^{m-1} \backslash K.$$

*Proof.*
(Necessary condition.) Take $\xi \in S^{m-1}\backslash K$ and put $f(\omega) = E_\alpha(\xi, \omega)$. Then $f \in M^\alpha_{(r)}(K)$, hence $CT_\alpha(\mu)(\xi) = 0$ in $S^{m-1}\backslash K$.
(Sufficient condition.) Let $f \in M^\alpha_{(r)}(K)$; then $f \in M^\alpha_{(r)}(\Omega_f)$ for some open neighbourhood $\Omega_f$ of $K$. Consider a compact neighbourhood $K'$ having piecewise smooth boundary and such that $K \subset \overset{\circ}{K}{}' \subset K' \subset \Omega_f$. By Cauchy's Theorem:

$$f(\omega) = \int_{\partial K'} E_\alpha(\xi, \omega) n ds f(\xi), \quad \omega \in K,$$

and by Fubini's Theorem:

$$\int_{S^{m-1}} d\mu(\omega) f(\omega) = \int_{\partial K'} (CT_\alpha(\mu))(\xi) n ds f(\xi) = 0 \ .$$

■

**Lemma 9.** *Let $\alpha \in \mathbb{C} \setminus (\mathbb{N} \cup (-m+1-\mathbb{N}))$ and put $M^\omega_{ij} = L^\omega_{ij} - \frac{1}{2}e_{ij}$, $L^\omega_{ij} = \omega_i \partial_{\omega_j} - \omega_j \partial_{\omega_i}$ being the momentum operators. Then:*

*(i) The $\Gamma$-operator and $M_{ij}$-operators commute, i.e. $[\Gamma_\omega, M^\omega_{ij}] = 0$*

*(ii) $M^\omega_{ij} E_\alpha(\xi, \omega) = -E_\alpha(\xi, \omega) \overline{M}^\xi_{ij}$ .*

*Proof.*
(i) See [16].
(ii) Up to a constant $E_\alpha(\xi, \omega)$ is given by $\xi C^{\frac{m}{2}}_\alpha(-\langle \omega, \xi \rangle) + \omega C^{\frac{m}{2}}_{\alpha-1}(-\langle \omega, \xi \rangle)$ and

$$\begin{aligned} & L^\omega_{ij}[\xi C^{\frac{m}{2}}_\alpha(-\langle \omega, \xi \rangle) + \omega C^{\frac{m}{2}}_{\alpha-1}(-\langle \omega, \xi \rangle)] \\ = & \ -(\omega_i \xi_j - \omega_j \xi_i)[\xi C^{\frac{m}{2}}_\alpha{}'(-\langle \omega, \xi \rangle) + \omega C^{\frac{m}{2}}_{\alpha-1}{}'(-\langle \omega, \xi \rangle)] \\ & +(\omega_i e_j - \omega_j e_i) C^{\frac{m}{2}}_{\alpha-1}(-\langle \omega, \xi \rangle) \end{aligned}$$

while

$$\begin{aligned} & [\xi C^{\frac{m}{2}}_\alpha(-\langle \omega, \xi \rangle) + \omega C^{\frac{m}{2}}_{\alpha-1}(-\langle \omega, \xi \rangle)] L^\xi_{ij} \\ = & \ (\omega_i \xi_j - \omega_j \xi_i)[\xi C^{\frac{m}{2}}_\alpha{}'(-\langle \omega, \xi \rangle) + \omega C^{\frac{m}{2}}_{\alpha-1}{}'(-\langle \omega, \xi \rangle)] + (\xi_i e_j - \xi_j e_i) C^{\frac{m}{2}}_\alpha(-\langle \omega, \xi \rangle) \ , \end{aligned}$$

where ' denotes derivation with respect to the variable $-\langle \omega, \xi \rangle$. Hence

$$\begin{aligned} & M^\omega_{ij}[\xi C^{\frac{m}{2}}_\alpha(-\langle \omega, \xi \rangle) + \omega C^{\frac{m}{2}}_{\alpha-1}(-\langle \omega, \xi \rangle)] + [\xi C^{\frac{m}{2}}_\alpha(-\langle \omega, \xi \rangle) + \omega C^{\frac{m}{2}}_{\alpha-1}(-\langle \omega, \xi \rangle)] \overline{M}^\xi_{ij} \\ = & \ [\frac{1}{2}[\xi, e_{ij}] + (\xi_i e_j - \xi_j e_i)] C^{\frac{m}{2}}_\alpha(-\langle \omega, \xi \rangle) + [\frac{1}{2}[\omega, e_{ij}] + (\omega_i e_j - \omega_j e_i)] C^{\frac{m}{2}}_{\alpha-1}(-\langle \omega, \xi \rangle) \\ = & \ 0 \ , \end{aligned}$$

since $\frac{1}{2}[\omega, e_{ij}] = -\omega_i e_j + \omega_j e_i$. ■

**Lemma 10.** *Let $\Omega$ be an open connected subset of $S^{m-1}$, let $\xi \in \Omega$ and let $f \in M^\alpha_{(r)}(\Omega)$. If $M_{i_1 j_1} \ldots M_{i_k j_k} f(\omega)|_{\omega = \xi} = 0$ for all couples $(i_l, j_l)$, $i_l < j_l$, $1 \leq i_l, j_l \leq m$, $0 \leq l \leq k$, $k \in \mathbb{N}$, then $f \equiv 0$ in $\Omega$.*

*Proof.*
Extend $f$ to an $\alpha$-homogeneous null solution $\tilde{f}$ of $\partial_x$ in the connected cone $\mathbb{R}_+\Omega$. By assumption all derivatives of $\tilde{f}$ in $\xi$ vanish. Since $\tilde{f}$ is real analytic in $\mathbb{R}_+\Omega$, $\tilde{f} \equiv 0$ in $\mathbb{R}_+\Omega$. ∎

**Definition.**

Let $\xi \in S^{m-1}$ and let $\alpha \in \mathbb{C} \setminus (\mathbb{N} \cup (-m+1-\mathbb{N}))$. Then we define the set

$$R^\alpha(\xi) = \{M^\omega_{i_1 j_1} \ldots M^\omega_{i_k j_k} E_\alpha(\xi, \omega),\ (i_l, j_l) \in \{1, \ldots, m\} \times \{1, \ldots, m\},$$

$$i_l < j_l,\ 0 \le l \le k,\ k \in \mathbb{N}\}\ .$$

In view of Lemma 9(i) the operators $M_{ij}$ and $\Gamma$ commute; therefore each element of this set belongs to $M^\alpha_{(r)}(S^{m-1} \backslash \{\xi\})$.
Consider a set $V = \{\xi^i,\ \xi^i \in S^{m-1},\ i \in I\}$ of points $\xi^i$ on $S^{m-1}$, then we put:

$$R^\alpha(V) = \cup_{i \in I} R^\alpha(\xi^i)$$

and we call $R^\alpha_{(r)}(V)$ the right $\mathbb{C}_m$-span of the set $R^\alpha(V)$, i.e. $R^\alpha_{(r)}(V)$ is the space of finite right $\mathbb{C}_m$-linear combinations of elements of $R^\alpha(V)$. Clearly $R^\alpha_{(r)}(V)$ is a right $\mathbb{C}_m$-module of null solutions of $\Gamma + \alpha$ having singularities in the set $V$.

**Theorem 11.** *Let $K \subset S^{m-1}$ be compact and let $S^{m-1} \backslash K = \cup_{i=1}^\infty \Omega_i$ be the decomposition of $S^{m-1} \backslash K$ in connected components. Choose in each $\Omega_i$ a point $\xi^i$ and put $V = \{\xi^i,\ i \in \mathbb{N}_0\}$. Then:*
*$R^\alpha_{(r)}(V)$ is dense in $M^\alpha_{(r)}(K)$ with respect to the topology given by the supremum norm on $K$.*

*Proof.*
Let $C^0_{(r)}(K)$ be the right module of $\mathbb{C}_m$-valued continuous functions on $K$ endowed with the supremum norm on $K$. Then we have the following inclusions where the supremum norm on $K$ is restricted to the corresponding subspaces of $C^0_{(r)}(K)$:

$$R^\alpha_{(r)}(V) \subset M^\alpha_{(r)}(K) \subset C^0_{(r)}(K)\ .$$

By the Hahn-Banach Theorem each continuous linear functional on $R^\alpha_{(r)}(V)$ can be extended to a continuous linear functional on $M^\alpha_{(r)}(K)$ and hence also to $C^0_{(r)}(K)$. In view of the Riesz Representation Theorem the dual of $C^0_{(r)}(K)$ can be identified with the left module of $\mathbb{C}_m$-valued regular Borel measures on $S^{m-1}$ having support in $K$. The space $R^\alpha_{(r)}(V)$ will be dense in $M^\alpha_{(r)}(K)$ iff the zero functional on $R^\alpha_{(r)}(V)$ has only the zero functional on $M^\alpha_{(r)}(K)$ as continuous extension. To prove this it is sufficient to prove that each regular Borel measure on $S^{m-1}$ having support in $K$ which annihilates the space $R^\alpha_{(r)}(V)$ also annihilates $M^\alpha_{(r)}(K)$. Consider such a measure $\mu$. By assumption $\mu$ annihilates in particular $R^\alpha(\xi^i)$, hence for all couples $(i_l, j_l) \in \{1, \ldots, m\} \times \{1, \ldots, m\},\ i_l < j_l,\ 0 \le l \le k,\ k \in \mathbb{N}$:

$$\langle \mu(\omega), M^\omega_{i_1 j_1} \ldots M^\omega_{i_k j_k} E_\alpha(\xi^i, \omega) \rangle = 0\ .$$

By Lemma 9(ii):

$$\begin{aligned} M^{\omega}_{i_1j_1}\dots M^{\omega}_{i_kj_k}E_\alpha(\xi^i,\omega) &= -M^{\omega}_{i_1j_1}\dots M^{\omega}_{i_{k-1}j_{k-1}}[E_\alpha(\xi,\omega)\overline{M}^{\xi}_{i_kj_k}]|_{\xi=\xi^i} \\ &= (-1)^k[E_\alpha(\xi,\omega)\overline{M}^{\xi}_{i_1j_1}\dots\overline{M}^{\xi}_{i_kj_k}]|_{\xi=\xi^i}\ , \end{aligned}$$

hence

$$\langle\mu(\omega),E_\alpha(\xi,\omega)\rangle\overline{M}^{\xi}_{i_1j_1}\dots\overline{M}^{\xi}_{i_kj_k}|_{\xi=\xi^i}=0\ .$$

Since $CT_\alpha(\mu)(\xi)=\langle\mu(\omega),E_\alpha(\xi,\omega)\rangle$ is a right null solution of $\Gamma_\xi-\beta$ in $S^{m-1}\backslash K$ ($\alpha+\beta=-m+1$) one has by Lemma 10 that $CT_\alpha(\mu)(\xi)\equiv 0$ in $\Omega_i$, $i$ being chosen arbitrarily, hence $CT_\alpha(\mu)\equiv 0$ in $S^{m-1}\backslash K$. By Lemma 8 $\mu$ annihilates $M^\alpha_{(r)}(K)$, q.e.d. ■

We will now determine under which conditions $M^\alpha_{(r)}(\Omega)$ is dense in $M^\alpha_{(r)}(K)$, $K\subset\Omega$ compact, $\Omega$ open. In view of the previous Theorem such a result will hold when we can choose $V$ such that $V\cap\Omega=\emptyset$. This will only be possible if $\Omega$ satisfies some further topological condition with respect to $K$. This is formulated in the following

**Theorem 12.** *(First Approximation Theorem of Runge)*
*Let $K\subset\Omega\subset S^{m-1}$, $\Omega$ open and $K$ compact. Then the following conditions are equivalent:*

*(i) $M^\alpha_{(r)}(\Omega)$ is dense in $M^\alpha_{(r)}(K)=\text{lim ind}_{K\subset\Omega}M^\alpha_{(r)}(\Omega)$*

*(ii) $\Omega\backslash K$ has no components of which the closure (in $S^{m-1}$) is contained in $\Omega$ .*

*Proof.*
(ii) $\Rightarrow$ (i) First of all we prove that $M^\alpha_{(r)}(\Omega)$ is dense in $M^\alpha_{(r)}(K)$ for $\sup_K$. Let $G_i$, $i\in\mathbb{N}_0$ be the components of $S^{m-1}\backslash K$. By the topological condition on $\Omega\backslash K$ and Lemma 1: $\overline{G}_i\cap(S^{m-1}\backslash\Omega)\neq\emptyset$. Choose for all $i\in\mathbb{N}_0$ points $\xi^i\in\overline{G}_i\cap(S^{m-1}\backslash\Omega)\subset\overline{G}_i\cap(S^{m-1}\backslash K)=G_i$ and put $V=\{\xi^i,\ i\in\mathbb{N}_0\}$. By the previous Theorem $R^\alpha_{(r)}(V)$ is dense in $M^\alpha_{(r)}(K)$ for $\sup_K$. Since $V\subset S^{m-1}\backslash\Omega$, $R^\alpha_{(r)}(V)$ is a subspace of $M^\alpha_{(r)}(\Omega)$; thus $M^\alpha_{(r)}(\Omega)$ is dense in $M^\alpha_{(r)}(K)$ for $\sup_K$.
Let $f\in M^\alpha_{(r)}(K)$; then there is an open set $\Omega_f$, $K\subset\Omega_f\subset\Omega$ such that $f\in M^\alpha_{(r)}(\Omega_f)$. In view of Theorem 6 one can always find a compact set $F_{i_0}$, $K\subset F_{i_0}\subset\Omega_f$ such that $\Omega\backslash F_{i_0}$ has no components of which the closure is contained in $\Omega$. Hence there is a sequence $(f_i)_{i\in\mathbb{N}_0}$, $f_i\in M^\alpha_{(r)}(\Omega)$ such that $f_i\to f$ in $\sup_{F_{i_0}}$ and thus $f_i\to f$ in the Fréchet module $M^\alpha_{(r)}(\overset{\circ}{F}_{i_0})$. As the inductive limit topology on $M^\alpha_{(r)}(K)$ is the strongest locally convex topology on $M^\alpha_{(r)}(K)$ which is weaker than the topology on any $M^\alpha_{(r)}(\Omega)$, $K\subset\Omega$, the sequence $(f_i)_{i\in\mathbb{N}_0}$ converges to $f$ in $\lim\text{ind}_{K\subset\Omega}M^\alpha_{(r)}(\Omega)$.
(i) $\Rightarrow$ (ii) Suppose that $W$ is a component of $\Omega\backslash K$ such that $\overline{W}\subset\Omega$; by Lemma 2, $\partial W\subset K$. Take a fixed point $\nu\in W$ and consider the function $f(\omega)=E_\alpha(\nu,\omega)$; then $f\in M^\alpha_{(r)}(K)$. By assumption there is a compact set $F$, $K\subset\overset{\circ}{F}\subset F\subset\Omega\backslash\{\nu\}$ and a sequence of functions $(f_j)_{j\in\mathbb{N}_0}$, $f_j\in M^\alpha_{(r)}(\Omega)$ such that $f_j\to f$ for $\sup_F$. Since $\partial W\subset K\subset\overset{\circ}{F}$ and $\nu\in W\backslash F$ it follows that $\overline{W}\backslash\overset{\circ}{F}=W\backslash\overset{\circ}{F}$ is a non empty compact subset of $W$. Therefore one can always find a compact set $C$ which covers $\overline{W}\backslash\overset{\circ}{F}$

and has piecewise smooth boundary $\partial C$ contained in $W\backslash(\overline{W}\backslash \overset{\circ}{F}) = W\backslash(W\backslash \overset{\circ}{F}) = W\cap \overset{\circ}{F}\subset\overset{\circ}{F}$. For all $\omega \in \overline{W}\backslash \overset{\circ}{F}$:

$$(f_i - f_j)(\omega) = \int_{\partial C} E_\alpha(\xi,\omega) n ds (f_i - f_j)(\xi) \ .$$

Hence

$$\sup_{\omega\in\overline{W}\backslash\overset{\circ}{F}} |(f_i - f_j)(\omega)|_0 \leq A(\partial C) \sup_{\omega\in\overline{W}\backslash\overset{\circ}{F},\ \xi\in\partial C} |E_\alpha(\xi,\omega)|_0 \sup_{\xi\in\partial C} |(f_i - f_j)(\xi)|_0 \ ,$$

$A(\partial C)$ denoting the area of $\partial C$. Since $\overline{W}\backslash \overset{\circ}{F}$ and $\partial C$ are compact and $\partial C \subset F$ there is a constant $K(\overline{W}, F, \partial C)$ such that

$$\sup_{\omega\in\overline{W}\backslash\overset{\circ}{F}} |(f_i - f_j)(\omega)|_0 \leq K \sup_{\omega\in F} |(f_i - f_j)(\omega)|_0 \ .$$

From $F \cup \overline{W} \subset F \cup (\overline{W}\backslash \overset{\circ}{F})$ it follows that

$$\sup_{\omega\in F\cup\overline{W}} |(f_i - f_j)(\omega)|_0 \leq (1 + K) \sup_{\omega\in F} |(f_i - f_j)(\omega)|_0 \ .$$

Since $(f_i)_{i\in\mathbb{N}_0}$ is a Cauchy sequence in $M^\alpha_{(r)}(F)$ for $\sup_F$, $(f_i)_{i\in\mathbb{N}_0}$ is also Cauchy in $M^\alpha_{(r)}(F \cup \overline{W})$ for $\sup_{F\cup\overline{W}}$. Consequently there is an $\tilde{f}$ such that $f_i \to \tilde{f}$ for $\sup_{F\cup\overline{W}}$ and $\tilde{f} \in M^\alpha_{(r)}(\overset{\circ}{F} \cup W)$ where $\tilde{f}|_F = f$. Since $\overset{\circ}{F} \cap W \neq \emptyset$ there is a component $G$ of $\overset{\circ}{F}$ such that $G\cap W \neq \emptyset$, therefore $G\cup W$ is connected and $\tilde{f}$ is the unique extension of $f$ to the region $G \cup W$; but $\nu \in W$ which leads to a contradiction ($f$ is singular in $\nu$) . ■

**Theorem 13.** *(Second Approximation Theorem of Runge)*
*Let $\Omega, \Omega'$ be open subsets of $S^{m-1}$, $\Omega' \subset \Omega$. Then the following conditions are equivalent:*

*(i) $M^\alpha_{(r)}(\Omega)$ is dense in $M^\alpha_{(r)}(\Omega')$ in the sense of Fréchet modules*

*(ii) $\Omega\backslash\Omega'$ has no components which are closed in the topology on $S^{m-1}$ .*

*Proof.*
(ii) ⇒ (i) Consider the exhaustion of $\Omega'$ by means of the compact sets

$$K'_j = \{\omega \in S^{m-1} : d(\omega, S^{m-1}\backslash\Omega') \geq \frac{1}{j}\}, \quad j \in \mathbb{N}_0 \ .$$

The space $M^\alpha_{(r)}(\Omega)$ will be dense in $M^\alpha_{(r)}(\Omega')$ in the sense of Fréchet modules if for each $j \in \mathbb{N}_0$ the space $M^\alpha_{(r)}(\Omega)$ is dense in $M^\alpha_{(r)}(K'_j)$ for $\sup_{K'_j}$. Choose an arbitrary fixed $j \in \mathbb{N}_0$; then $S^{m-1}\backslash\Omega' \subset S^{m-1}\backslash K'_j$ and by Lemma 5 each component of $S^{m-1}\backslash K'_j$ contains some component $G_i$ of $S^{m-1}\backslash\Omega'$. By assumption and Lemma 3: $\overline{G}_i \cap (S^{m-1}\backslash\Omega) \neq \emptyset$ for each component $G_i$ of $S^{m-1}\backslash\Omega'$. Choose for each component $G_i$, $i \in I$ points $\xi^i \in \overline{G}_i \cap (S^{m-1}\backslash\Omega) \subset \overline{G}_i \cap (S^{m-1}\backslash\Omega') = \overline{G}_i \cap (\cup_{j\in I} G_j) = G_i$ and

put $V = \{\xi^i,\ i \in I\}$, then $V$ intersects each component of $S^{m-1}\backslash K'_j$. By Theorem 11 the space $R^{\alpha}_{(r)}(V)$ is dense in $M^{\alpha}_{(r)}(K'_j)$ for $\sup_{K'_j}$. The particular choice of points $\xi^i$ shows that $R^{\alpha}_{(r)}(V) \subset M^{\alpha}_{(r)}(\Omega)$; since $j$ was chosen arbitrary, this proves the first part.
(i) $\Rightarrow$ (ii) Suppose that $\Omega\backslash\Omega'$ has some compact component $V$. By Theorem 7 there exists an open $U$ such that $V \subset U \subset \overline{U} \subset \Omega$ and $\partial U \subset \Omega'$. Choose for each $\xi \in \partial U$ a ball $B_S(\xi, r_\xi) \subset \Omega'$. By compactness of $\partial U$ there exists a finite number of $\xi^i \in \partial U,\ i = 1, \ldots, N$ such that $\partial U \subset \cup_{i=1}^N B_S(\xi^i, r_{\xi^i})$. Put $U_1 = U \cup (\cup_{i=1}^N B_S(\xi^i, r_{\xi^i}))$ and $U_2 = U \cup (\cup_{i=1}^N B_S(\xi^i, \frac{r_{\xi^i}}{2}))$; then $U_1$ has piecewise smooth boundary $\partial U_1 \subset \Omega'$ and $V \subset U_2 \subset \overline{U}_2 \subset U_1$. Choose a fixed point $\xi \in V$ and put $f(\omega) = E_\alpha(\xi, \omega)$; then $f \in M^{\alpha}_{(r)}(\Omega')$. By assumption there is a sequence $(f_j)_{j\in\mathbb{N}_0}$ in $M^{\alpha}_{(r)}(\Omega)$ such that $f_j \to f$ uniformly on each compact subset of $\Omega'$, in particular $f_j \to f$ for $\sup_{\partial U_1}$. Hence

$$\sup_{\omega\in\overline{U}_2} |f_i(\omega) - f_j(\omega)|_0 = \sup_{\omega\in\overline{U}_2} |\int_{\partial U_1} E_\alpha(\xi, \omega) n ds (f_i - f_j)(\xi)|_0 \to 0,\ \inf(i, j) \to \infty\ ,$$

thus $(f_i)_{i\in\mathbb{N}_0}$ is a Cauchy sequence in the Fréchet space $M^{\alpha}_{(r)}(U_2)$. Call $W$ the connected component of $U_2$ which contains $V$ and let $\zeta \in \partial V$; since $W$ is open, $B_S(\zeta, \delta) \subset W$ for sufficiently small $\delta$. By Lemma 4, $\partial V \subset \partial\Omega'$, hence $B_S(\zeta, \delta) \cap \Omega' \neq \emptyset$ or $W \cap \Omega' \neq \emptyset$. Therefore there is a component $G$ of $\Omega'$ such that $W \cap G \neq \emptyset$; this implies that $W \cup G$ is connected. As each closed subset of $W \cup G$ can be written as the union of a closed subset of $W$ and $G$, $(f_j)_{j\in\mathbb{N}_0}$ is a Cauchy sequence in $M^{\alpha}_{(r)}(W \cup G)$. By the principle of analytic continuation $f_j \to f = E_\alpha(\xi, \omega)$ in $M^{\alpha}_{(r)}(W \cup G)$, a contradiction $(\xi \in V \subset W)$ . ■

## 4 The equation $(\Gamma + \alpha)f = g$

We will first determine the global solutions of $(\Gamma + \alpha)f = g,\ \alpha \in \mathbb{C}$, $g$ belonging to $C^\infty(S^{m-1})$ or $\mathcal{E}'(S^{m-1})$. Of course the situation is quite different when $\alpha \in \mathbb{N} \cup (-m + 1 - \mathbb{N})$ because in this case the kernel of the operator $\Gamma + \alpha$ consists precisely of the classical inner and outer spherical monogenics. Next we will consider the equation $(\Gamma + \alpha)f = g,\ g \in C^\infty(\Omega),\ \Omega \subset S^{m-1}$ open. In case $g$ has compact support contained in $\Omega$, the problem is reduced to the global case by extending $g$ to a $C^\infty$-function on $S^{m-1}$ equal to zero in $S^{m-1}\backslash\Omega$. However in case $g \in C^\infty(\Omega)$ the problem is not so straightforward and requires Runge's Theorem.

### 4.1 The case $\Omega = S^{m-1}$

Let us repeat the following expansions in terms of spherical monogenics given in [17].

(i)

$$\delta(\omega - \xi) = \frac{1}{A_m} \sum_{k=0}^{\infty} [K_k(\omega, \xi) - \omega K_k(\omega, \xi)\xi]$$

(ii)

$$\frac{\pi}{A_m \sin \pi\alpha} K_\alpha(-\omega,\xi) = \frac{1}{A_m}\sum_{k=0}^{\infty}[\frac{K_k(\omega,\xi)}{\alpha-k} - \frac{\omega K_k(\omega,\xi)\xi}{\alpha+k+m-1}] ,$$
$$\alpha \in \mathbb{C}\setminus(\mathbb{N}\cup(-m+1-\mathbb{N}))$$

(Mittag-Leffler expansion)

(iii)

$$PV[\frac{\pi}{A_m \sin\pi\alpha}K_\alpha(-\omega,\xi),\alpha=k] = \frac{1}{A_m}\sum_{\substack{l=0\\ l\neq k}}^{\infty}\frac{K_l(\omega,\xi)}{k-l} - \frac{1}{A_m}\sum_{l=0}^{\infty}\frac{\omega K_l(\omega,\xi)\xi}{k+l+m-1}] ,$$
$$k\in\mathbb{N}$$

where the series converge in $\mathcal{E}'(S^{m-1})$. The principal value $PV[f, z = z_0]$ of $f$ in $z_0$ is defined to be the value in $z_0$ of the regular part of $f$. In [13] the following Theorem was proved.

**Theorem 14.** *(Characterisation of the spaces $C^\infty(S^{m-1})$ and $\mathcal{E}'(S^{m-1})$ in terms of spherical monogenics)*

*(i)* *$g(\omega) = \sum_{k=0}^{\infty} P_k(\omega) + Q_k(\omega) \in C^\infty(S^{m-1})$ implies for all $s \in \mathbb{N}$ the existence of a constant $c_s > 0$ such that $\sup_{\omega\in S^{m-1}}\{|P_k(\omega)|_0, |Q_k(\omega)|_0\} \leq c_s(1+k)^{-s}$ for each $k \in \mathbb{N}$.*
*Conversely, let $(P_k, Q_k)_{k\in\mathbb{N}}$ be a sequence of spherical monogenics which satisfies the above estimate, then $g(\omega) = \sum_{k=0}^{\infty} P_k(\omega) + Q_k(\omega) \in C^\infty(S^{m-1})$.*

*(ii)* *$S(\omega) = \sum_{k=0}^{\infty} P_k(\omega) + Q_k(\omega) \in \mathcal{E}'(S^{m-1})$ implies for all $s \in \mathbb{N}$ the existence of a constant $d_s > 0$ such that $\sup_{\omega\in S^{m-1}}\{|P_k(\omega)|_0, |Q_k(\omega)|_0\} \leq d_s(1+k)^{s}$ for each $k \in \mathbb{N}$.*
*Conversely, let $(P_k, Q_k)_{k\in\mathbb{N}}$ be a sequence of spherical monogenics which satisfies the above estimate, then $S(\omega) = \sum_{k=0}^{\infty} P_k(\omega) + Q_k(\omega) \in \mathcal{E}'(S^{m-1})$.*

By means of this Theorem and the decompositions above one can easily prove the following

**Theorem 15.**
*Let $g = \sum_{k=0}^{\infty} P_k + Q_k \in C^\infty(S^{m-1})$ and $S = \sum_{k=0}^{\infty} P_k + Q_k \in \mathcal{E}'(S^{m-1})$. Then:*

*(i)* *In case $\alpha \in \mathbb{C}\setminus(\mathbb{N}\cup(-m+1-\mathbb{N}))$*

*1. the equation $(\Gamma+\alpha)f = g$ has a unique solution $f \in C^\infty(S^{m-1})$ given by*

$$\begin{aligned} f(\omega) &= \sum_{k=0}^{\infty}\frac{P_k(\omega)}{\alpha-k} + \frac{Q_k(\omega)}{\alpha+k+m-1} \\ &= \int_{S^{m-1}}\frac{\pi}{A_m\sin\pi\alpha}K_\alpha(-\omega,\xi)g(\xi)dS(\xi) \end{aligned}$$

*where the series converges in $C^\infty(S^{m-1})$,*

*2. the equation $(\Gamma+\alpha)T = S$ has a unique solution $T \in \mathcal{E}'(S^{m-1})$ given by*

$$T(\omega) = \sum_{k=0}^{\infty}\frac{P_k(\omega)}{\alpha-k} + \frac{Q_k(\omega)}{\alpha+k+m-1}$$

*where the series converges in $\mathcal{E}'(S^{m-1})$.*

*(ii) In case* $\alpha = k \in \mathbb{N}$
*1. the equation* $(\Gamma + k)f = g$ *has a solution iff* $P_k(\omega) = 0$. *In this case the unique solution* $f \in C^\infty(S^{m-1})$ *such that* $P(k)(f) = 0$ *is given by*

$$\begin{aligned} f(\omega) &= \sum_{\substack{l=0\\ l\neq k}}^{\infty} \frac{P_l(\omega)}{k-l} + \sum_{l=0}^{\infty} \frac{Q_l(\omega)}{k+l+m-1} \\ &= \int_{S^{m-1}} PV[\frac{\pi}{A_m \sin \pi\alpha} K_\alpha(-\omega,\xi), \alpha = k] g(\xi) dS(\xi) \end{aligned}$$

*where the series converges in* $C^\infty(S^{m-1})$. *The total solution space is then given by* $\{f(\omega) + P_k(\omega), P_k \in M^+(k)\}$,
*2. the equation* $(\Gamma + k)T = S$ *has a solution iff* $P_k(\omega) = 0$. *In this case the unique solution* $f \in \mathcal{E}'(S^{m-1})$ *such that* $P(k)(f) = 0$ *is given by*

$$T(\omega) = \sum_{\substack{l=0\\ l\neq k}}^{\infty} \frac{P_l(\omega)}{k-l} + \sum_{l=0}^{\infty} \frac{Q_l(\omega)}{k+l+m-1}$$

*where the series converges in* $\mathcal{E}'(S^{m-1})$. *The total solution space is then given by* $\{T(\omega) + P_k(\omega), P_k \in M^+(k)\}$.

*Proof.*
Follows from applying $\Gamma$ under the summation symbol and the decompositions above. The uniqueness in (i) follows from the fact that a global $C^\infty$-or distributional null solution of $\Gamma + \alpha,\ \alpha \in \mathbb{C} \setminus (\mathbb{N} \cup (-m+1-\mathbb{N}))$ must be identical zero. ∎

## 4.2 The general case $g \in C^\infty(\Omega)$

Consider the locally finite cover $(H_j)_{j\in\mathbb{N}_0}$ of $\Omega$ (see Lemma 5). Let $(\phi_j)_{j\in\mathbb{N}_0}$ be a partition of unity subordinate to this cover; then $\phi_j \in \mathcal{D}(H_j, \mathbb{R})$ and

$$g(\omega) = \sum_{j=1}^{\infty} \phi_j g(\omega) \text{ in } C^\infty(\Omega)$$

Since $\phi_j g$ is $C^\infty$ on $S^{m-1}$ and has support contained in $H_j$, the function

$$g_j(\omega) = \int_{S^{m-1}} \frac{\pi}{A_m \sin \pi\alpha} K_\alpha(-\omega,\xi)(\phi_j g)(\xi) dS(\xi)$$

is $C^\infty$ on $S^{m-1}$ and satisfies $(\Gamma + \alpha)g_j = \phi_j g$. Hence $(\Gamma + \alpha)g_j = 0$ in $G_{j-1}$ $(G_{j-1} \cap H_j = \emptyset)$, thus $g_j \in M^\alpha_{(r)}(K_{j-2})$. In view of Lemma 5 we can apply Runge's Theorem 12 which ensures the existence of a sequence $(h_j)_{j\geq 3}$ in $M^\alpha_{(r)}(\Omega)$ such that

$$\sup_{\omega \in K_{j-2}} |(g_j - h_j)(\omega)|_0 < 2^{-j}.$$

Therefore the series $g_1 + g_2 + \sum_{j=3}^{\infty}(g_j - h_j)$ converges in the compact open topology on $C^0(\Omega)$ to an element $f \in C^0(\Omega)$. Moreover $f \in C^\infty(\Omega)$; this can be seen as follows: consider an arbitrary ball $\overline{B}_S(u,\delta) \subset \Omega$; for $l$ sufficiently large $\overline{B}_S(u,\delta) \subset \overset{\circ}{K}_l$ and the finite sum $g_1 + g_2 + \sum_{j=3}^{l+1}(g_j - h_j) \in C^\infty(B_S(u,\delta))$ while by Weierstrass' Theorem for spherical monogenics of complex degree:

$$\sum_{j=l+2}^{\infty}(g_j - h_j) \in M^\alpha_{(r)}(B_S(u,\delta)) .$$

Hence $f \in C^\infty(\Omega)$ and for each $\omega \in \Omega$:

$$(\Gamma_\omega + \alpha)f(\omega) = \sum_{j=1}^{\infty}(\Gamma_\omega + \alpha)g_j(\omega) = \sum_{j=1}^{\infty}\phi_j g(\omega) = g(\omega) .$$

■

We thus proved the following

**Theorem 16.** *Let $\alpha \in \mathbb{C}\setminus(\mathbb{N}\cup(-m+1-\mathbb{N}))$ and let $g \in C^\infty(\Omega)$, $\Omega \subset S^{m-1}$ open. Then the equation $(\Gamma+\alpha)f = g$ has a solution $f \in C^\infty(\Omega)$.*

We can now prove the following important Theorem.

**Theorem 17.** *(Mittag-Leffler's Theorem for the operator $\Gamma + \alpha$)*

*Let $\alpha \in \mathbb{C}\setminus(\mathbb{N}\cup(-m+1-\mathbb{N}))$. Let $\Omega \subset S^{m-1}$ be open and let $(V_i)_{i\in I}$ be an open cover of $\Omega$. Suppose that for each $j,k \in I$ such that $V_j \cap V_k \neq \emptyset$ there is a $f_{jk} \in M^\alpha_{(r)}(V_j \cap V_k)$ satisfying the cocycle condition:*

*(i)* $f_{jk} = -f_{kj}$ *in* $V_k \cap V_j$

*(ii)* $f_{jk} + f_{kl} + f_{lj} = 0$ *in* $V_k \cap V_j \cap V_l$ .

*Then there exists a family of functions $(f_i)_{i\in I}$, $f_i \in M^\alpha_{(r)}(V_i)$ such that $f_k - f_j = f_{kj}$ in $V_k \cap V_j$.*

*Proof.*
First of all remark that the following $C^\infty$-equivalent of this problem always has a solution. Let $\phi_{jk} \in C^\infty(V_j \cap V_k)$ satisfy:

$$(i) \qquad \phi_{jk} = -\phi_{kj} \text{ in } V_k \cap V_j$$
$$(ii) \qquad \phi_{jk} + \phi_{kl} + \phi_{lj} = 0 \text{ in } V_j \cap V_k \cap V_l\,;$$

then there exists a family of functions $(\phi_i)_{i\in I}$, $\phi_i \in C^\infty(V_i)$ such that $\phi_k - \phi_j = \phi_{kj}$ in $V_k \cap V_j$ for all $k, j \in I$.
To see this, let $(\psi_i)_{i\in I}$ be a partition of unity subordinate to the cover $(V_i)_{i\in I}$ and define

$$\phi_j(\omega) = \sum_k \psi_k(\omega)\phi_{kj}(\omega), \quad \omega \in V_i\,;$$

then $\phi_j \in C^\infty(V_j)$ and in $V_j \cap V_k$:

$$\begin{aligned}\phi_j(\omega) - \phi_k(\omega) &= \sum_l \psi_l(\omega)[\phi_{lj}(\omega) - \phi_{lk}(\omega)] \\ &= \sum_l \psi_l(\omega)\phi_{kj}(\omega) \\ &= \phi_{kj} .\end{aligned}$$

Therefore one can always find functions $h_j \in C^\infty(V_j)$ such that $h_j - h_k = f_{jk}$ in $V_j \cap V_k$ for all $j,k \in I$. Define $h|_{V_i} = (\Gamma+\alpha)h_i$, $i \in I$; then $h$ is well defined ($(\Gamma+\alpha)h_j = (\Gamma+\alpha)h_k$ in $V_j \cap V_k$) and $h \in C^\infty(\Omega)$. By the previous Theorem 16 there is a $g \in C^\infty(\Omega)$ such that $(\Gamma+\alpha)g = h$ in $\Omega$. Put $f_i = h_i - g$ in $V_i$, then $(\Gamma+\alpha)f_i = 0$ in $V_i$ and $f_j - f_k = (h_j - g) - (h_k - g) = h_j - h_k = f_{jk}$ in $V_j \cap V_k$. ■

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