Weak convergence in spaces of measures and operators

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Abstract

J. K. Brooks and P. W. Lewis have established that if E and E^* have RNP, then in $M(\Sigma, E)$, m_n converges weakly to m if and only if $m_n(A)$ converges weakly to $m(A)$ for each $A \in \Sigma$. Assuming the existence of a special kind of lifting, N. Randrianantoanina and E. Saab have shown an analogous result if E is a dual space. Here we show that for the space $M(\mathcal{P}(\mathbb{N}), E)$ where E^* is a Grothendieck space or E is a Mazur space, this kind of weak convergence is valid. Also some applications for subspaces of $L(E, F)$ similar to the results of N. Kalton and W. Ruess are given.

1 Introduction

Let E and F be two infinite dimensional Banach spaces. By $L(E, F)$ (resp. $K(E, F)$) we denote the Banach space of all bounded linear (resp. compact linear) operators from E to F. The ϵ-product $E\epsilon F$ is the operator space $K_{w^*}(E^*, F)$ of compact and weak*-weak continuous linear operators from E^* to F, endowed with the usual operator norm. Let Σ be a σ-algebra on a non-empty set S, then $M(\Sigma, E)$ (resp. $ca(\Sigma, E)$) denotes the Banach space of all bounded countably additive vector measures endowed with the variation norm (resp. semivariation norm). The space E is said to be Grothendieck if weak* and weak sequential convergence in E^* coincide; E is called Mazur if any weak*-sequentially continuous linear functional on E^* lies in E. For unexplained notations we refer the reader to [4], [5], [6].

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2 Weak convergence in measure spaces

In the following we use the techniques due to M. Talagrand [18] for the space $M(\Sigma, E)$. For the sake of simplicity, we say that $M(\Sigma, E)$ has the $\mathcal{K}$-property if a sequence (m_n) converges weakly to m in $M(\Sigma, E)$ if and only if $m_n(A)$ converges weakly to $m(A)$ in E for all $A \in \Sigma$.

Theorem 2.1. *If E^* is a Grothendieck space, then $M(\mathcal{P}(\mathbb{N}), E)$ has the $\mathcal{K}$-property.*

Proof. Let (m_n) be a sequence in $M(\mathcal{P}(\mathbb{N}), E)$ such that for all $A \in \Sigma$, $(m_n(A))$ converges weakly. We denote its limit by $m(A)$. The proof of Corollary 1 in [2] implies that $m \in M(\mathcal{P}(\mathbb{N}), E)$, so we can assume that $m_n(A) \to 0$ (weakly). By the proof of Theorem 17 of [18], there is a probability measure λ' on $\mathcal{P}(\mathbb{N})$ such that $V(m_n) \leq \lambda'$ for each n where $V(m)$ is the variation norm of m. Moreover, the measure λ defined by $\lambda = \sum \frac{\delta_n}{2^n} + \lambda'$ is also a bounded positive measure which vanishes only on $\emptyset$ and clearly $V(m_n) \leq \lambda$ for each n. Let ρ be a lifting on $L^\infty(\lambda)$, then $x^*(m(A)) = \int_{\mathbb{N}} \rho(m)(\omega)(x^*) d\lambda$, where $\rho(m)(\omega) \in E^{**}$ and $\rho(m)(\omega)(x^*) = \rho(\frac{d(x^*.m)}{d\lambda})$. But because of our choice of λ, $\rho(m_n)(\omega)(x^*) \to 0$ for all $\omega \in \mathbb{N}$, and for all $x^* \in E^*$. Now since E^* is a Grothendieck space, then $\rho(m_n)(\omega)$ converges weakly to zero. Now an appeal to Theorem 15 of [18] completes the proof. ∎

For the case when E is a Mazur space a similar result is obtained.

Theorem 2.2. *If E is a Mazur space, then $M(\mathcal{P}(\mathbb{N}), E)$ has the $\mathcal{K}$-property.*

Proof. We use the same line of proof as the one of Theorem 2.1. One has just to note that since E is a Mazur space, the elements $\rho(m)(\omega)$ and $\rho(m_n)(\omega)$ are in fact in E. ∎

Remarks. (a) A measure theoretical version of Batt's example in [6, page 103] shows that the condition $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ on σ-algebra is essential.

(b) The following example shows that the hypothesis on E is also essential. Let $m_n \in M(\mathcal{P}(\mathbb{N}), \ell_\infty)$ be defined by $m_n(A) = \chi_{\{1,\ldots,n\} \cap A}$. The sequence $(m_n(A))$ converges in norm to χ_A for each $A \in \mathcal{P}(\mathbb{N})$ and so converges to χ weakly. But (m_n) does not converge weakly to χ. In fact if (m_n) converges weakly to χ, then there is a convex combinations of (m_n) say (m'_n) such that $m'_n \in co(m_n, m_{n+1}, \ldots)$ and (m'_n) converges in norm to χ. So for $\epsilon = \frac{1}{2}$ there is $N_0 > 0$ such that for $n \geq N_0$, $\| m'_n - \chi \| < \frac{1}{2}$. Therefore for sufficiently large N we have $\| (m'_n - \chi)\{N\} \| < \frac{1}{2}$, which is a contradiction.

L. Drewnownski (Lemma 3 of [7]) has shown that, for a sequence (m_n) in $ca(\mathcal{P}(\mathbb{N}), E)$, $m_n(A) \to 0$ in norm for all $A \in \mathcal{P}(\mathbb{N})$, implies $\| m_n \| \to 0$. Using this result, we have the following Schur characterization of $ca(\mathcal{P}(\mathbb{N}), E)$.

Theorem 2.3. *E is a Schur space if and only if $ca(\mathcal{P}(\mathbb{N}), E)$ is a Schur space.*

Proof. Let E be a Schur space and (m_n) be a weak-null sequence in $ca(\mathcal{P}(\mathbb{N}), E)$. Therefore $(m_n(A))$ is norm null in E. Then the Lemma 3 of [7] completes the proof. The other direction is straightforward. ∎

Remark. Theorem 2.3 also shows that $ca(\mathcal{P}(\mathbb{N}), E)$ has the $\mathcal{K}$-property. We say that a subspace $S(E, F)$ of $L(E, F)$ has the $\mathcal{K}$-property if the convergence of sequences coincides under the weak operator topology and weak topology. N. Kalton [10] has proved that the $K(E, F)$ has the $\mathcal{K}$-property if and only if E is a Grothendieck space. The analogous result for $K_{w^*}(E^*, F)$ is given in [16]. It is well known that $L_{w^*}(E^*, ca(\Sigma))$ the space of all weak*-weak continuous bounded operators from E^* to $ca(\Sigma)$ is isometrically isomorphic to $ca(\Sigma, E)$ (cf. [4], [16]). By using this identification, $L_{w^*}(E^*, ca(\mathcal{P}(\mathbb{N}))$ has the $\mathcal{K}$-property. Moreover the compactness condition on these subspaces of operators given in [10] and [16] is not neccessary.

3 Weak convergence in spaces of operators

G. Emmanuele [8] and R. Ryan [17] have studied the complemented copies of c_0 in some spaces of operators on Banach spaces. Recently J. Zafarani [20] has extended these results to some spaces of operators between locally convex spaces. Here we give necessary and sufficient conditions for the inclusion of c_0 in the space of operators with the $\mathcal{K}$-property.

Theorem 3.1. *Let $S(E, F)$ be any closed subspace of $L(E, F)$ that has the $\mathcal{K}$-property, then F contains a copy of c_0 if and only if $S(E, F)$ contains a copy of c_0.*

Proof. We follow the elegant techniques due to H. Rosenthal [15]. Suppose F does not contain a copy of c_0 and (T_n) is a non trivially weak Cauchy sequence. Since $S(E, F)$ has the $\mathcal{K}$-property, there are elements $x \in E$, $y^* \in F^*$ such that $(y^* T_n x)$ is non trivial, so $(T_n x)$ is a non trivial weak Cauchy sequence and therefore has a strongly summing subsequence, hence (T_n) has a strongly summing subsequence. An appeal to Theorem 1 of [15] shows that $S(E, F)$ can not have a copy of c_0. The other direction is trivial. ∎

Corollary 3.2. *Let E be a Grothendieck space and $K(E, F)$ be closed under the weak operator topology in $L(E, F)$, then F contains a copy of c_0 if and only if $K(E, F)$ contains a copy of c_0.*

Proof. By [10], $K(E, F)$ has the $\mathcal{K}$-property and if it contains a copy of c_0, the proof will follow from Theorem 3.1. The converse is trivial. ∎

Theorem 3.3. *Let $M(\Sigma, E)$ have the $\mathcal{K}$-property, then E contains a copy of c_0 if and only if $M(\Sigma, E)$ contains a copy of c_0 .*

Proof. Again we use the technique of Rosenthal [15]. Let (m_n) be a non trivial weak Cauchy sequence in $M(\Sigma, E)$ and suppose E does not contain a copy of c_0. From the $\mathcal{K}$-property of $M(\Sigma, E)$, there is $A \in \Sigma$ such that $(m_n(A))$ is a non trivial weak Cauchy sequence. So it has a strongly summing subsequence and by our hypothesis on $M(\Sigma, E)$, (m_n) will also have a strongly summing subsequence. The Theorem 1 of [15] completes the proof. The converse is direct. ∎

H. S. Collins and W. Ruess [4] have shown that if E^* has RNP and F does not contain a copy of ℓ_1, $K_{w^*}(E^*, F)$ will not contain either. The following theorem is a refinement of this result.

Theorem 3.4. *Let E be a separable Banach space. Suppose that $S(E, F)$ is any closed subspace of $L(E, F)$ with the $\mathcal{K}$-property. Then $S(E, F)$ contains a copy of ℓ_1 if and only if F contains a copy of ℓ_1.*

Proof. Let $T \in S(E, F)$ and set $T' : B_E \to F$ where T' is the restriction of T to B_E and is a Borel measurable map by the Pettis Measurability Theorem [5, page 25]. Let $\lambda = \sum_{n=1}^{\infty} \frac{\delta_{x_n}}{2^n}$ where $\{x_n\}$ is a dense subset of B_E. By Theorem 1 of [18], there exists a subset C of B_E and for each n a function g_n in $co(T'_n, T'_{n+1}, \dots)$ such that $(g_n(x))$ is a weakly Cauchy sequence for each $x \in C$ and $\lambda(C) = \lambda(B_E)$. By this assumption $(g_n(x))$ is weakly Cauchy for each $x \in B_E$. Therefore $g_n(x)$ represents an element of $S(E, F)$ such that $(g_n(x))$ is weakly Cauchy for all $x \in E$. Hence by the assumption (g_n) is a weakly Cauchy sequence but by a result of A. Ülger $B_{S(E,F)}$ is weakly precompact [19]. The ℓ_1-Rosenthal theorem shows that $S(E, F)$ does not contain a copy of ℓ_1. The other direction is easy. ∎

When $S(E, F)$ does not contain a copy of ℓ_1, we have a representation of its dual.

Theorem 3.5. *If $S(E, F)$ is any separable subspace of $L(E, F)$ that does not contain a copy of ℓ_1 and has the $\mathcal{K}$-property, then $S(E, F)^* = E\hat{\otimes}F^*$.*

Proof. It is clear that $E\hat{\otimes}F^* \subseteq S(E, F)^*$. Now if $S(E, F)^* \neq E\hat{\otimes}F^*$, then there is $x^{**} \in S(E, F)^{**}$ and $x^* \in S(E, F)^*$ such that, for every $y^* \in E\hat{\otimes}F^*$, $x^{**}(y^*) = 0$ and $x^{**}(x^*) = 1$. Now by the Odell-Rosenthal Theorem [5, page 215] there exists a sequence (x_n) in $S(E, F)$ such that $x^{**} = w^* - \lim x_n$. Therefore $\lim_n y^* x_n = x^{**} y^* = 0$ for all $y^* \in E\hat{\otimes}F^*$. But $S(E, F)$ has the $\mathcal{K}$-property, hence $x_n \to 0$ (weakly). This shows that $x^* x_n \to 0$ and then $x^{**} x^* = 0$, which is a contradiction. ∎

N. Randrianantoanina [13] posed the following question: Is it true that, if E has the (V^*)-property, then $M(\Sigma, E)$ has the (V^*)-property ? A partial answer to this question is given here under. We need the following lemma first.

Lemma 3.6. *Let $\mathcal{A}$ be a countable set and Σ be the σ-algebra generated by $\mathcal{A}$. Suppose that (m_n) is a sequence of uniformly countably additive of vector valued measures for which $\lim_n m_n(A)$ exits for each $A \in \mathcal{A}$. Then $(m_n(A))$ is weakly*

Cauchy for all $A \in \Sigma$.

Proof. Much of our inspiration here comes from the Vitali-Hahn-Sacks theorem [5, page 89]. Set $\Lambda = \{A \in \Sigma : (m_n(A)) \text{ is weak Cauchy}\}$, by hypothesis $\mathcal{A} \subseteq \Lambda$. We claim that Λ is a monotone class which implies that $\Lambda = \Sigma$. Let (A_j) be a monotone sequence of members of Λ with $A_m \to A$. By the uniform countable additivity of the (m_n), $m_n(A) = \lim_m m_n(A_m)$ for all n, so $(m_p(A_m) - m_q(A_m))$ is norm null. These results show that $(m_n(A))$ is a weak Cauchy sequence for all $A \in \Sigma$. ∎

Definition 3.7. [1], [12] *A subset H of E is a (V^*)-set if for any sequence (x_n^*) that is w.u.C, $\lim_n sup_{x \in H} \mid x_n^* x \mid = 0$. The space E has the (V^*)-property, if its (V^*)-subsets are relatively weakly compact.*

Theorem 3.8. *Let $M(\Sigma, E)$ have the $\mathcal{K}$-property then E has the (V^*)-property if and only if $M(\Sigma, E)$ has the (V^*)-property.*

Proof. Let E have the (V^*)-property and (m_n) be a (V^*)-subset of $M(\Sigma, E)$. By the diagonal method there is a subsequence (m_n') of (m_n) such that $(m_n'(A_k))$ is weakly Cauchy for each k. The last lemma implies that $(m_n'(A))$ is weakly Cauchy for each $A \in \Sigma$. But E is weakly sequentially complete so $m_n'(A) \to m(A)$ (weakly) and the proof of theorem 1 [2] shows that m lies in $M(\Sigma, E)$. Therefore by our hypothesis on $M(\Sigma, E)$, m_n' converges weakly to m. The other direction is trivial. ∎

Definition 3.9. *A Banach space E is said to have the BD-property if every limited subset is relatively weakly compact.*

G. Emmanuele [8] has shown that $L^1(E)$ has the BD-property if and only if E has it. Here we give a similar result for $M(\Sigma, E)$.

Theorem 3.10. *If E is a weakly sequentially complete Banach space and $M(\Sigma, E)$ has the $\mathcal{K}$-property, then $M(\Sigma, E)$ has the BD-property if and only if E has it.*

Proof. Let E have the BD-property and (m_n) be a limited set. First we show that (m_n) must be uniformly countably additive. Since if not, we can find a sequence (A_j) of pairwise disjoint elements of Σ such that $\| m_j(A_j) \| \geq \epsilon$ and there exist $x_j^* \in B_{E^*}$ such that $x_j^* m_j A_j \geq \epsilon$, but $x_j^* \otimes A_j \to 0$ (weak*), which contradicts our assumption. It is easily deduced from a brief outline of the proof of Theorem 1 given in [2] that we can assume there is a countable algebra (A_j) of Σ such that $\Sigma = \sigma(\{A_j\})$. By our assumption there is a subsequence which again is denoted by (m_n) such that $m_n(A_j)$ converges weakly for all i. But by Lemma 3.6, $(m_n(A))$ is weak Cauchy for each $A \in \Sigma$. Since E is weakly sequentially complete, $(m_n(A))$ will converge weakly to $m(A)$ which by the proof of Theorem 1 of [2] implies $m \in M(\Sigma, E)$, and by the $\mathcal{K}$-property of $M(\Sigma, E)$, m_n converges weakly to m. The converse of the theorem is straightforward. ∎

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