Substitutions, Partial Isometries of $\mathbb{R}$, and Actions on Trees

Charles Holton Luca Q. Zamboni*

Abstract

In this paper we outline a connection between substitution dynamical systems and systems of partial isometries of $\mathbb{R}$. Let τ be a primitive substitution and (X, T) the associated minimal subshift. Associated to each real geometric realization $\mathcal{G}$ of τ is a system of partial isometries $\mathcal{I}_{\mathcal{G}}$. The subshift (X, T) generated by τ provides a symbolic coding of the dynamics of $\mathcal{I}_{\mathcal{G}}$. This allows us to deduce various dynamical properties of the system: Minimality, classification (interval exchange, homogeneous, or exotic), orbit structure, independence of generators, and self similarity. We show by example that each type of minimal system of partial isometries can arise from geometric realizations of substitutions on three letters. These systems give rise to interesting (non-simplicial) geometric actions of free groups on real trees.

1 Introduction

In [16] we exhibit a connection between substitution subshifts and isometric actions of free groups on $\mathbb{R}$-trees. Our construction begins with a minimal interval translation mapping $f : [0, 1] \to [0, 1]$ on three intervals. This transformation was originally studied by M. Boshernitzan and I. Kornfeld in [4]. As noted in [4], the dynamics of f is symbolically coded in terms of the substitution $\tau(1) = 2$, $\tau(2) = 3111$, $\tau(3) = 311$. The mapping f is "self-similar" in the sense that there exists a proper subinterval I_0 of $I = [0, 1]$ for which the first return map f_0 is an interval translation mapping

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naturally isomorphic to f (differs from f only by a linear mapping.) The f-invariant set

$$\Omega(f) = \bigcap_{n=1}^{\infty} f^n(I)$$

is a Cantor set in $[0, 1]$, and each point $x \in \Omega(f)$ has a dense forward orbit. Using a result of D. Gaboriau and G. Levitt in [14], we glue copies of the unit interval together according to the orbit of f to obtain an $\mathbb{R}$-tree with an $\mathbb{F}_3$-action by isometries. In [16] we prove the resulting action is free and exotic.

In this paper we extend and generalize this construction via the theory of geometric realizations of substitutions developed in [17]. Let τ be a primitive substitution on a finite alphabet $\mathcal{A} = \{1, 2, \dots, r\}$ and (X, T) the associated (one-sided) minimal symbolic shift space, where T denotes the (left) shift map on $\mathcal{A}^{\mathbb{N}}$. For each subword v occurring in a sequence $x \in X$, denote by X_v the cylinder in X determined by v. Suppose α is a nonzero eigenvalue of modulus less than one of the incidence matrix M_τ and $\mathbf{v}_\alpha = (v_1, v_2, \dots, v_r)$ a corresponding left eigenvector. In [17] we establish the following:

Theorem 1.1. *(Theorem 3.4 in [17]) There is a continuous map $f : X \to \mathbb{C}$ such that for each $i \in \mathcal{A}$ and for each $w \in X_i$ we have $f(\tau(w)) = \alpha f(w)$ and $f(Tw) = f(w) + v_i$. The image set $\Omega = f(X)$ is a compact perfect subset of $\mathbb{C}$.*

A triple $\mathcal{G} = \mathcal{G}(f, \alpha, \mathbf{v}_\alpha)$ satisfying the conditions of Theorem 1.1 is called a geometric realization of τ. If α and $\mathbf{v}_\alpha$ are all real, we call $\mathcal{G}$ a real geometric realization.

Associated to each real geometric realization $\mathcal{G} = \mathcal{G}(f, \alpha, \mathbf{v}_\alpha)$ is a system of partial isometries $\mathcal{I}_\mathcal{G} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ defined as follows: I is the smallest interval containing Ω, D_i is the smallest interval containing $\Omega_i = f(X_i)$, and $(x)\phi_i = x + v_i$ for each $x \in D_i$. The mapping f provides a symbolic coding of the dynamics of $\mathcal{I}_\mathcal{G}$ in the following sense: For each $w \in X$, $f(w) \in I$ is the unique point of I in the domain of the composition ϕ_v for each initial subword v of w. (See Lemma 4.1). In section 4 we study various dynamical properties of this symbolic coding (e.g., minimality, orbit structure, independence of generators) and obtain the following classification:

Theorem 1.2. *If the system of partial isometries $\mathcal{I}_\mathcal{G} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ is an interval translation mapping, i.e, the domains D_i are nonoverlapping and cover I, then the system $\mathcal{I}_\mathcal{G}$ is minimal and is either an interval exchange mapping or exotic in the sense of [15]. In each case, Ω is equal to the limit set I_∞ of the system $\mathcal{I}_\mathcal{G}$ defined by D. Gaboriau in [13]. If $\mathcal{I}_\mathcal{G}$ is an interval exchange mapping, then $\Omega = I$, while if $\mathcal{I}_\mathcal{G}$ is exotic, then Ω is a Cantor set.*

In section V we show that each type of minimal system of partial isometries (interval exchange mapping, homogeneous, and exotic) can arise as above from a real geometric realization of a primitive substitution on three letters.

In section VI we derive a condition which ensures that the system $\mathcal{I}_\mathcal{G}$ is fixed point free in the sense that no point of I is fixed by a nontrivial composition in the $\phi_i^{\pm 1}$. We specialize to the case where the domains D_i are nonoverlapping, and require that the system $\mathcal{I}_\mathcal{G}$ be self-similar. Roughly this means that the induced first return system on the subinterval αI (under forward trajectories) is isomorphic to the "parent" system $\mathcal{I}_\mathcal{G}$ (in that they differ by a scaling,) and for each $x \in D_i$, the first return of αx to αI is given by the composition $\tau(\phi_i)$. Thus, off a finite set, each point $\alpha x \in \cup \alpha D_i$ has exactly one first return to αI. A point $x \in D_i \cap D_j$ for $i \neq j$ is called a branch point of $\mathcal{I}_\mathcal{G}$. Denote by BP the finite set of branch points. Let $\mathcal{O}_0(BP)$ be the graph whose vertices are given by $BP \cup \{(x)\phi_i | x \in BP \cap D_i\}$; we put a directed edge from vertex x to vertex y if $(x)\phi_i = y$. Each connected component of $\mathcal{O}_0(BP)$ is a subgraph of the Cayley graph of the $\mathcal{I}_\mathcal{G}$-orbit of a branch point. We then show:

Theorem 1.3. *Assume the system of partial isometries $\mathcal{I}_\mathcal{G}$ is self-similar. Then $\mathcal{I}_\mathcal{G}$ is fixed point free if and only if each connected component of $\mathcal{O}_0(BP)$ is circuit free.*

Following [14], we associate to $\mathcal{I}_\mathcal{G}$ an $\mathbb{F}_r$-action by isometries on an $\mathbb{R}$-tree $Y_\mathcal{G}$. In our case, the resulting action is minimal ($Y_\mathcal{G}$ contains no proper invariant subtree) and each branch point of $Y_\mathcal{G}$ is a limit point of branch points. It follows that the action is geometric and exotic in the sense of [14]. Theorem 1.3 gives a necessary and sufficient condition for the action on $Y_\mathcal{G}$ to be free provided $\mathcal{I}_\mathcal{G}$ is self-similar. We give examples of different types of exotic geometric actions induced by primitive substitutions. The homogeneous example in section V gives rise to a nonsmall $\mathbb{F}_3$-action, the interval exchange determines a small but nonfree action, and the exotic system is self-similar and gives rise to a free action.

2 Geometric realizations of primitive substitutions

Substitution dynamical systems appear in several areas of mathematics. We quote but a few examples of recent references: Ergodic theory ([4], [22]), symbolic dynamics ([2], [7], [9], [21], [26]), geometry of fractals ([6], [17], [18], [23], [24]), number theory ([1], [3], [5], [10], [23], [24]), and the K-theory of C^*-algebras ([8], [11]).

A substitution is a pair $(\mathcal{A}, \tau)$ consisting of a finite nonempty set $\mathcal{A}$ together with a morphism τ which assigns to each letter $a \in \mathcal{A}$ a word $\tau(a)$ in the alphabet $\mathcal{A}$. Denote by $\mathcal{A}^+$ the set of all finite words in $\mathcal{A}$. The morphism τ extends by concatenation to mappings $\mathcal{A}^+ \to \mathcal{A}^+$ and $\mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$. It also extends uniquely to an endomorphism of the free group $F(\mathcal{A})$. The substitution is called primitive if there is a positive integer N such that for each pair of letters $(a, b) \in \mathcal{A} \times \mathcal{A}$, the letter b occurs in $\tau^N(a)$. Let $\omega = w_1 w_2 w_3 \ldots \in A^\mathbb{N}$ be a fixed point of τ^n for some $n > 0$. Associated to $(\mathcal{A}, \tau)$ is the symbolic subshift (X, T) where T is the (left) shift map on $\mathcal{A}^\mathbb{N}$ and X the T-orbit closure of ω in $\mathcal{A}^\mathbb{N}$. Primitivity implies that X is independent of the choice of ω.

Substitution dynamical systems are abstract objects, and it is natural to look for ways of representing them geometrically. In [23], G. Rauzy discovered a connection between the dynamics of the substitution $\tau(1) = 12$, $\tau(2) = 13$, $\tau(3) = 1$, and a Weyl automorphism on the torus $\mathbb{R}^2/\mathbb{Z}^2$. From a different point of view, E. Bombieri and J. Taylor exhibit in [3] a connection between algebraic number theory and the theory of quasicrystals by geometrically encoding the dynamics of certain substitutions on a three letter alphabet. In [6], M. Dekking develops a method of constructing fractal tilings of the plane (including the famous Penrose tiling) using substitutions.

Many others, including P. Arnoux [2], M. Boshernitzan-I. Kornfeld [4], S. Ferenczi [9], S. Ito-M. Kimura [18], and B. Solomyak [26] have constructed various types of geometric realizations of substitutions.

Most of the work in this direction has focused on the case where the Perron-Frobenius eigenvalue of the incidence matrix is a Pisot number. In [17] we give geometric realizations of a larger class of substitutions: We only require the substitution $(\mathcal{A}, \tau)$ be primitive and the incidence matrix have a nonzero eigenvalue of modulus less than one. We show in [17] that the image set Ω can also be characterized as the limit set of a Mauldin-Williams graph directed construction. This gives an upper bound on the Hausdorff dimension of Ω in terms of $|\alpha|$ and the Perron-Frobenius eigenvalue of the incidence matrix (Theorem 4.4 in [17].).

3 Systems of partial isometries

By a multi-interval we mean a finite union of disjoint closed intervals in $\mathbb{R}$. A system of partial isometries $\mathcal{I} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ is a multi-interval I together with a finite collection of partial isometries $\phi_i : D_i \to R_i$ between closed (possibly degenerate) subintervals of I. Write $(x)\phi_i$ for the image of $x \in D_i$ under ϕ_i. The isometries $\phi_i : D_i \to R_i$ are called the generators, while the intervals D_i and R_i are called the bases. Two points x and y of I are said to belong to the same $\mathcal{I}$- orbit if there is a composition $\phi = \phi_{i_1}^{\epsilon_1}\phi_{i_2}^{\epsilon_2}\dots\phi_{i_n}^{\epsilon_n}$, with $\epsilon_i \in \{1, -1\}$, defined at x and with $(x)\phi = y$. Each orbit has the structure of a Cayley graph: There is a directed edge from x to y for each $1 \leq i \leq r$ such that $(x)\phi_i = y$.

Each system of partial isometries $\mathcal{I}$ admits a canonical decomposition into a finite number of invariant subsystems (called components) such that within each component, either every orbit is finite or every orbit is dense [15]. A component in which every orbit is dense is called a minimal component, and a system of partial isometries on I is said to be minimal if every orbit is dense in I. A minimal system $\mathcal{I}$ is called homogeneous if there is an open subinterval J of I and a subgroup P of the group of isometries of $\mathbb{R}$, having dense orbits, such that for all points $x, y \in J$, x and y are in the same P-orbit if and only if they are in the same $\mathcal{I}$-orbit [15]. It is called an interval exchange mapping if every point $x \in I$ off of a finite set belongs to exactly two bases. It is called exotic if it is neither homogeneous nor an interval exchange mapping. The existence of exotic systems was first discovered by G. Levitt in [20].

The generators ϕ_i are said to be independent if the fixed point set of each nontrivial composition $\phi = \phi_{i_1}^{\epsilon_1}\phi_{i_2}^{\epsilon_2}\dots\phi_{i_n}^{\epsilon_n}$ is either empty or consists of a single point [15]. D. Gaboriau has shown in [12] that if a system of partial isometries has no homogeneous component then one can, without changing the orbits, replace each ϕ_i by its restriction to a suitably chosen subinterval of D_i in such a way that the generators become independent. Conversely, a system with independent generators has no homogeneous component [12].

Following [13], given a system of partial isometries $\mathcal{I} = \mathcal{I}_0 = (I_0, \phi_i : D_i \to R_i)_{i=1}^r$, let $L(\mathcal{I}_0)$ be the set of all points of I_0 which belong to at most one base of $\mathcal{I}_0$. Set $I_1 = I_0 - L(\mathcal{I}_0)$, and let $\mathcal{I}_1$ be the system of partial isometries obtained by restricting $\mathcal{I}_0$ to the multi-interval I_1. Thus, for $x \in I_1$, the $\mathcal{I}_1$-orbit of x is obtained from the $\mathcal{I}_0$-orbit of x by "erasing" the terminal vertices. Inductively, let $L(\mathcal{I}_j)$ be the set of all points in I_j belonging to at most one base of $\mathcal{I}_j$, and let $\mathcal{I}_{j+1}$ be the restriction of $\mathcal{I}_j$ to the multi-interval $I_{j+1} = I_j \setminus L(\mathcal{I}_j)$. In [13] D. Gaboriau defines the limit set I_∞ of $\mathcal{I}$ as

$$I_\infty = \bigcap_{j=0}^{\infty} I_j.$$

In other words, I_∞ is obtained by systematically erasing the terminal vertices of each orbit. Hence each $\mathcal{I}$-orbit having at least two (infinite) ends intersects I_∞ nontrivially. Gaboriau proves that I_∞ is a Cantor set whenever $\mathcal{I}$ is exotic [13].

4 Systems of partial isometries arising from substitutions

Let τ be a primitive substitution on the alphabet $\mathcal{A} = \{1, 2, \ldots, r\}$. The incidence matrix $M_\tau = (m_{i,j})_{r\times r}$ has entry $m_{i,j}$ equal to the number of occurrences of i in $\tau(j)$. Suppose α is a nonzero real eigenvalue of the incidence matrix with $|\alpha| < 1$, and $\mathbf{v}_\alpha = (v_1, v_2, \ldots, v_r) \in \mathbb{R}^r$ a corresponding left eigenvector. Denote by $\mathcal{G} = \mathcal{G}(f, \alpha, \mathbf{v}_\alpha)$ the associated real geometric realization of τ (Theorem 1.1) and set $\Omega = f(X) \subset \mathbb{R}$. Define $S : \mathcal{A}^+ \to \mathbb{R}$ by

$$S(w) = \sum_{i=1}^{r} |w|_i v_i,$$

where $|w|_i$ denotes the number of occurrences of i in w.

We associate to $\mathcal{G}$ a system of partial isometries $\mathcal{I}_\mathcal{G} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ where $I = [a, b]$ is the smallest closed interval containing Ω, $D_i = [a_i, b_i]$ is the smallest closed interval containing $\Omega_i = f(X_i)$, and

$$(x)\phi_i = x + S(i) = x + v_i.$$

Let $\mathcal{O}^+$ denote the set of all words in the letters $\{\phi_i\}_{i=1}^r$, $\mathcal{O}^-$ the set of all words in the letters $\{\phi_i^{-1}\}_{i=1}^r$, and $\mathcal{O}^\pm$ the set of all words in the letters $\{\phi_i^{\epsilon_i}\}_{i=1}^r$ where $\epsilon_i \in \{1, -1\}$. Given a word $w = j_1^{\epsilon_1} j_2^{\epsilon_2} \ldots j_n^{\epsilon_n}$ with $j_k \in \mathcal{A}$ and $\epsilon_k \in \{1, -1\}$, set $\phi_w = \phi_{j_1}^{\epsilon_1}\phi_{j_2}^{\epsilon_2} \ldots \phi_{j_n}^{\epsilon_n} \in \mathcal{O}^\pm$. In the next lemma, we regard the elements of $\mathcal{O}^\pm$ both as words and as compositions of functions. If x is a point in the domain of ϕ_w, define the orbit of x in ϕ_w to be the set of points $\{(x)\phi_{w(j)} : 0 \le j \le n\}$, where $w(j)$ is the initial subword of w of length j. The morphism τ acts on $\mathcal{O}^\pm$ by $\tau(\phi_w) = \phi_{\tau(w)}$.

Lemma 4.1. *Let $\mathcal{I}_\mathcal{G} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ be the system of partial isometries defined above.*

(1) If $v \in \mathcal{A}^+$ is a word which occurs in a sequence $x \in X$, then the domain of the composition $\phi_v \in \mathcal{O}^+$ is a nondegenerate closed interval containing $f(X_v)$. Moreover, for each point x in the domain of ϕ_v we have that $(x)\phi_v = x + S(v)$.

(2) If $\omega \in X$ then $f(\omega)$ is the unique point in I which is in the domain of the composition ϕ_v for all initial subwords v of ω.

(3) Each point $x \in \Omega$ has both a forward and a backward trajectory which is dense in Ω. (Compare with Proposition 1 in §5 of [4].)

(4) *For each* $x \in D_i$, $\alpha^n x$ *is in the domain of* $\phi_{\tau^n(i)}$ *and* $(\alpha^n x)\phi_{\tau^n(i)} = \alpha^n x + S(\tau^n(i)) = \alpha^n x + \alpha^n S(i) \in \alpha^n I$.

Proof. Let $v = j_1 j_2 \dots j_n$ and $w \in X_v \subset X$. It follows from Theorem 1.1 that for each $1 \le k < n$ we have $f(T^k w) = (f(w))\phi_{j_1}\phi_{j_2}\dots\phi_{j_k} \in \Omega_{j_{k+1}} \subset D_{j_{k+1}}$. Hence $f(w)$ is in the domain of ϕ_v and $(f(w))\phi_v = f(w) + S(v)$. To see that the domain of ϕ_v is nondegenerate, it suffices to show that $f(X_v)$ consists of more than one point. This follows from the fact that $\{f(T^k w)\}_{k=1}^{\infty}$ is dense in Ω (in fact $\{T^k w\}_{k=1}^{\infty}$ is dense in X), Ω is a perfect set, and v occurs in w with bounded gap.

If v is an initial subword of $\omega \in X$ then $f(\omega)$ is in the domain of ϕ_v, by (1). Because the extreme points of I are limit points of the sequence $\{(f(\omega))\phi_{\omega(n)}\}_{n=1}^{\infty}$, no other point of I can lie in the domains of all $\phi_{\omega(n)}$.

To see (3) we note that if $x \in \Omega$, then $x = f(\omega)$ for some $\omega \in X$. Since both the forward and backward orbits of ω are dense in X, it follows that $x = f(\omega)$ has both a forward and backward trajectory which is dense in Ω.

Finally, let $x \in D_i$. If $x \in \Omega_i$, then $x = f(\omega)$ for some $\omega \in X_i \subset X$. Then $\alpha^n x = f(\tau^n(\omega))$ belongs to the domain of $\phi_{\tau^n(i)}$ and $(\alpha^n x)\phi_{\tau^n(i)} = \alpha^n x + S(\tau >^n (i)) = \alpha^n x + \alpha^n S(i) \in \alpha^n I$. If x is not in Ω_i, then there are points $y < x < z$ with $y, z \in \Omega_i$. Then $\alpha^n y$ and $\alpha^n z$ are in the domain of $\phi_{\tau^n(i)}$, and hence, so is $\alpha^n x$. ∎

Let $EP = \{a_i, b_i\}_{i=1}^{q}$ be the set of all end points of the system $\mathcal{I}_{\mathcal{G}}$ and BP the set of all branch points, that is the set of all points x such that $x \in D_i \cap D_j$ for some $i \neq j$. The set BP is nonempty (Proposition 3.17 in [17].)

Proposition 4.2. *Let* $\mathcal{I}_{\mathcal{G}} = (I, \phi_i : D_i \to R_i)_{i=1}^{r}$ *be the system of partial isometries defined above. Suppose that for all* $i \neq j$, $D_i \cap D_j$ *is either empty or consists of a single point.*

(1) *If a nontrivial composition* $\phi_w \in \mathcal{O}^{\pm}$ *fixes some point* x *in its domain, then the domain of* ϕ_w *is equal to* $\{x\}$. *In other words, the generators* $\{\phi_i : D_i \to R_i\}_{i=1}^{q}$ *are independent.*

(2) *If a nontrivial composition* $\phi_w \in \mathcal{O}^{\pm}$ *fixes some point* x *in its domain, then the orbit of* x *in* ϕ_w *must meet a branch point.*

(3) *If* $w \in \mathcal{A}^+$ *occurs in a sequence* $x \in X$, *then* $S(w) \neq 0$.

(4) *The system* $\mathcal{I}_{\mathcal{G}}$ *is minimal if and only if* $I = \cup D_i$.

Proof. The hypothesis of the proposition implies that BP is a finite set. Suppose that (1) fails, i.e., that the domain of ϕ_w is a nondegenerate interval. Then there is some point y in the domain of ϕ_w whose orbit in ϕ_w misses BP. We can assume that ϕ_w (as a word in $\mathcal{O}^{\pm}$) is cyclically reduced. This then implies that $\phi_w = \phi_+\phi_-$ with ϕ_+ and ϕ_- in $\mathcal{O}^+$ and $\mathcal{O}^-$ respectively. Since ϕ_w fixes every point in its domain, each cyclic permutation of ϕ_w can be written as a product of this form. Thus, ϕ_w is either in $\mathcal{O}^+$ or in $\mathcal{O}^-$. Replacing ϕ_w by ϕ_w^{-1} if necessary, we can assume that $\phi_w \in \mathcal{O}^+$. Let $[a', b']$ denote the domain of ϕ_w. The orbit of a' in ϕ_w must meet some left end point a_i. Thus, some cyclic permutation ϕ'_w of ϕ_w fixes an end point a_i, and has a non-degenerate domain of the form $[a_i, c]$. Since $a_i \in \Omega$ is a left end point, there is a sequence $\{y_n\}_{n=1}^{\infty}$ converging to a_i with $y_n \in \Omega$ and $y_n > a_i$. Fix a point y_n

in the domain of ϕ'_w (and hence fixed by ϕ'_w) whose orbit in ϕ'_w misses the set BP. This implies that the (unique) forward trajectory of y_n with respect to the system $\mathcal{I}_\mathcal{G}$ is finite which contradicts (3) of Lemma 4.1.

We next establish (2). As in the proof of (1) if the orbit of x in ϕ_w does not meet a branch point then we deduce that ϕ_w is either in $\mathcal{O}^+$ or in $\mathcal{O}^-$. We can assume without loss of generality that $\phi_w \in \mathcal{O}^+$. By (1) the domain of ϕ_w consists only of the point x. This implies that the orbit of x in ϕ_w must meet some end point y. Thus some cyclic permutation ϕ'_w of ϕ_w fixes an end point y whose orbit in ϕ'_w does not meet a branch point. Then the (unique) forward trajectory of y with respect to $\mathcal{I}_\mathcal{G}$ is finite, contradicting that it should be dense in Ω.

To see (3) we note that if $S(w) = 0$, then by (1) the domain of ϕ_w would be degenerate. Yet by Lemma 4.1 the domain of ϕ_w is a nondegenerate closed interval.

To prove (4), suppose that $I \neq \cup D_i$. The sum of the lengths of the domains D_i is strictly less than the length of I. Since the generators are independent, it follows that the system $\mathcal{I}_\mathcal{G}$ contains a component all of whose orbits are finite. In particular, $\mathcal{I}_\mathcal{G}$ is not minimal. (See §6 of [15] or Corollary 2.5 of [20].) Next assume that $I = \cup D_i$ and suppose $\mathcal{I}_\mathcal{G}$ is not minimal. Let U be the minimal component of $\mathcal{I}_\mathcal{G}$ containing Ω. Then U is a finite union of closed intervals (Imanishi's Theorem in [15].) Let x denote the right endpoint of the connected component of U containing the left endpoint a of the interval I. By assumption $x \neq b$ for otherwise $U = I$ and $\mathcal{I}_\mathcal{G}$ would be minimal. Now, $x \notin BP$ since each branch point is a two sided limit point of Ω. It follows that x belongs to the interior of some D_i, $(x)\phi_i$ is also a right endpoint of some connected component of U, and $(x)\phi_i \neq b$. Continuing in this way we see that there must be a word $\phi_w \in \mathcal{O}^+$ and a point $y \in I$ fixed by ϕ_w whose orbit in ϕ_w does not meet the set BP. This contradicts (2). ∎

Theorem 4.3. *If the induced system of partial isometries $\mathcal{I}_\mathcal{G} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ is an interval translation mapping, (the domains D_i are nonoverlapping and cover I,) then the system $\mathcal{I}_\mathcal{G}$ is minimal and is either an interval exchange mapping or exotic. In each case, Ω is identified with the set of all points $x \in I$ having an infinite backwards trajectory, i.e., the limit set I_∞ of the system $\mathcal{I}_\mathcal{G}$. If $\mathcal{I}_\mathcal{G}$ is an interval exchange mapping, then $\Omega = I$, and if $\mathcal{I}_\mathcal{G}$ is exotic, then Ω is a Cantor set. (Compare with the example given in §5 of [4].)*

Proof. By (4) of Proposition 4.2 the system $\mathcal{I}_\mathcal{G}$ is minimal. Moreover, since the generators are independent it follows that the system is not homogeneous [12] (hence is either an interval exchange mapping or exotic). Hence Ω is identified with the limit set I_∞ of all points $x \in I$ which admit an infinite backwards trajectory, that is for which there exists an infinite sequence ϕ in $\{\phi_i^{-1}\}$ such that x is in the domain of $\phi(n)$ for all $n \geq 1$. By (3) of Lemma 4.1, we have $\Omega \subset I_\infty$. If $\mathcal{I}_\mathcal{G}$ is an interval exchange mapping, then both sets are clearly equal to I. On the other hand, suppose that the system $\mathcal{I}_\mathcal{G}$ is exotic. Let $x \in I_\infty$ and $\epsilon > 0$. There exists a composition $\phi_v \in \mathcal{O}^-$ defined at x such that for some $1 \leq i \leq r$, either $|(x)\phi_v - a_i| < \epsilon$, or $|(x)\phi_v - b_i| < \epsilon$. Otherwise, I_∞ would contain a nondegenerate interval contradicting the fact that I_∞ is a Cantor set (Proposition 5.2 in [13].) Let ϕ_v be the shortest such composition, and assume without loss of generality that $|(x)\phi_v - a_i| < \epsilon$. Then a_i is in the domain of the composition $\phi_v^{-1} \in \mathcal{O}^+$, $(a_i)\phi_v^{-1} \in \Omega$, and $|(a_i)\phi_v^{-1} - x| < \epsilon$. Thus each point $x \in I_\infty$ is a limit point of Ω and hence in Ω. ∎

5 Examples

We show by example that each type of minimal system of partial isometries (interval exchange mapping, homogeneous system, exotic) can arise from geometric realizations of substitutions on three letters. In each example the algorithm of Appendix B of [17] may be used to find the endpoints of the domains D_i.

Example 5.1: Interval exchange mapping

Define τ on $\mathcal{A} = \{1, 2, 3\}$ by

$$\begin{aligned} \tau(1) &= 13 \\ \tau(2) &= 122 \\ \tau(3) &= 12. \end{aligned}$$

Let $\mathcal{G}$ be the real geometric realization of τ associated to the eigenvalue $\alpha = 1 - \sqrt{2}$ and left eigenvector

$$\mathbf{v}_\alpha = \begin{pmatrix} 2 \\ 2\alpha \\ 2(\alpha - 1) \end{pmatrix}.$$

The induced system of partial isometries $\mathcal{I}_\mathcal{G}$ is an interval exchange mapping with

$$\begin{aligned} I &= [-1, 2 - \alpha] \\ D_1 &= [-1, -\alpha] \\ D_2 &= [-\alpha, 1 - 2\alpha] \\ D_3 &= [1 - 2\alpha, 2 - \alpha] \end{aligned}$$

Example 5.2: Exotic system

Define τ on $\mathcal{A} = \{1, 2, 3\}$, by

$$\begin{aligned} \tau(1) &= 12 \\ \tau(2) &= 13 \\ \tau(3) &= 2. \end{aligned}$$

The characteristic polynomial for M_τ is

$$p(x) = x^3 - x^2 - 2x + 1.$$

Let $\alpha \approx .4450419$ denote the unique root of $p(x)$ in the interval $[0, 1]$. Let $\mathcal{G}$ be the geometric realization of τ associated to the eigenvalue α and eigenvector

$$\mathbf{v}_\alpha = \begin{pmatrix} 1 - \alpha^2 \\ -\alpha \\ -1 \end{pmatrix}.$$

We find that $\mathcal{I}_\mathcal{G}$ is given by:

$$\begin{aligned} I &= [0, 1 + \alpha - \alpha^2] \\ D_1 &= [0, \alpha] \\ D_2 &= [\alpha, 1] \\ D_3 &= [1, 1 + \alpha - \alpha^2]. \end{aligned}$$

One readily verifies that $\mathcal{I}_{\mathcal{G}}$ is not an interval exchange mapping and hence by Theorem 4.3 is exotic. (See Figure 5.1.)

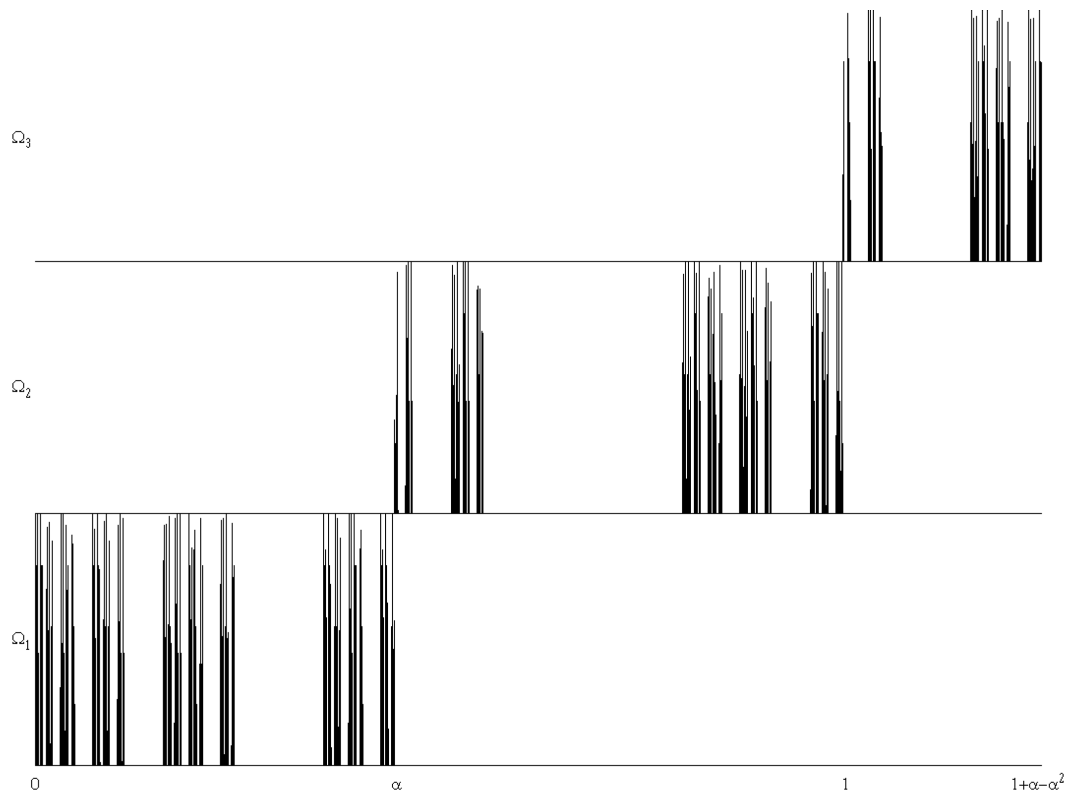


Figure 5.1

Example 5.3: Homogeneous System

A similar argument to the one used in (4) of Proposition 4.2 shows that if the domains D_i cover the interval I and the weights $\{v_1, v_2, \ldots, v_r\}$ are rationally independent, the system $\mathcal{I}_{\mathcal{G}}$ is minimal. However, if there is overlap between the domains D_i, then the generators may fail to be independent. For instance, define τ on $\{1, 2, 3\}$ by

$$\begin{aligned} \tau(1) &= 12 \\ \tau(2) &= 31 \\ \tau(3) &= 2. \end{aligned}$$

Let $\mathcal{G}$ be the geometric realization of τ associated to the eigenvalue $\alpha \approx .4450419$ and eigenvector

$$\mathbf{v}_\alpha = \begin{pmatrix} \alpha \\ \alpha^2 - \alpha \\ \alpha - 1 \end{pmatrix}.$$

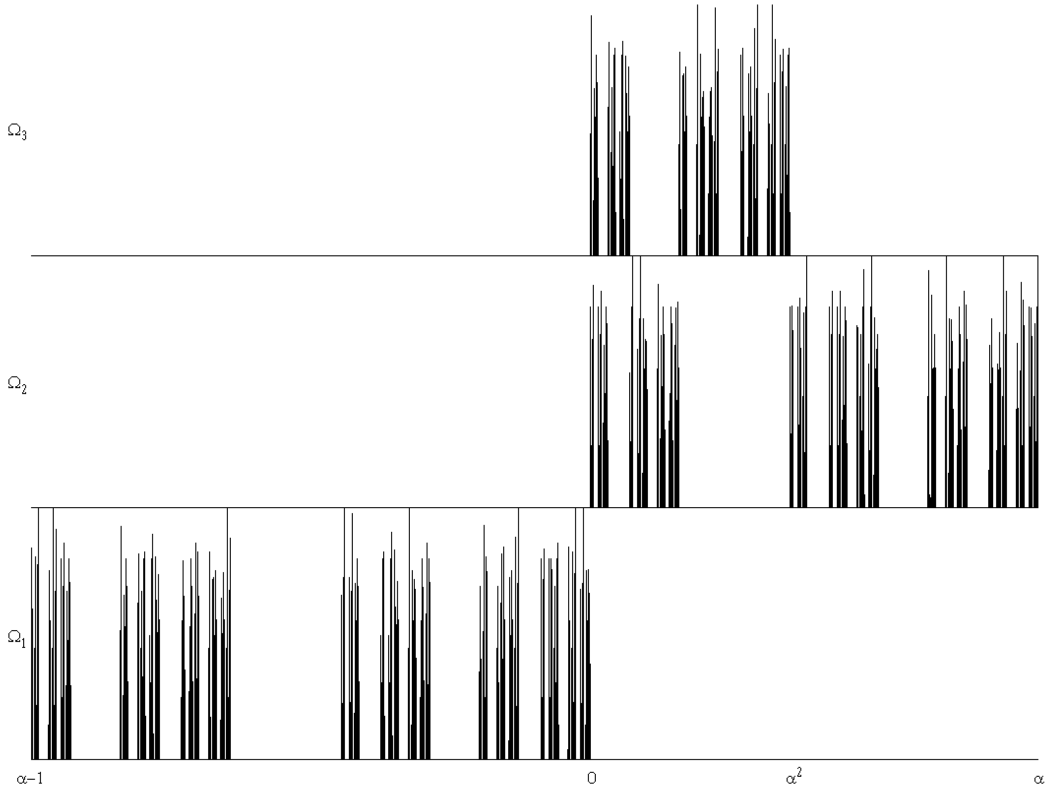


Figure 5.2

Then,

$$\begin{aligned} D_1 &= [\alpha - 1, 0], \\ D_2 &= [0, \alpha], \\ D_3 &= [0, \alpha^2]. \end{aligned}$$

(See Figure 5.2). Since $p(x)$ is irreducible, $\{\alpha, \alpha^2 - \alpha, \alpha - 1\}$ are rationally independent. It follows from the above remark that the system $\mathcal{I}_{\mathcal{G}}$ is minimal. To see that the system is homogeneous, it is enough to show that some point has an infinite $\mathcal{I}_{\mathcal{G}}(-t)$-orbit for some $t > 0$, where $\mathcal{I}_{\mathcal{G}}(-t)$ is the system obtained from $\mathcal{I}_{\mathcal{G}}$ by shrinking each base by t on both sides. (See Lemma 3.5 in [19].)
Observe that $\phi_3 \circ \phi_1$ sends the interval $[1-2\alpha, \alpha^2]$ to $[0, \alpha^3]$, and that $\phi_2^{-1} \circ \phi_1^{-1} \circ \phi_2^{-1}$ sends $[0, 3\alpha^2 - \alpha]$ to $[\alpha - 2\alpha^2, \alpha^2]$. Fix $0 < t < \alpha^4/4$ and note that $[t, \alpha^2 - t] = [t, 3\alpha^2 - \alpha - t] \cup [1 - 2\alpha + t, \alpha^2 - t]$. Each x in $[1 - 2\alpha + t, \alpha^2 - t]$ has an infinite $\mathcal{I}_{\mathcal{G}}(-t)$ trajectory given by applying $\phi_3 \circ \phi_1$ on the interval $[1 - 2\alpha + t, \alpha^2 - t]$, and applying $(\phi_2 \circ \phi_1 \circ \phi_2)^{-1}$ on the interval $[t, 1 - 2\alpha + t]$. Hence $\mathcal{I}_{\mathcal{G}}$ is homogeneous, and the generators are not independent.

6 Self-similar systems

In this section we derive a condition which ensures that $\mathcal{I}_{\mathcal{G}}$ is fixed point free, i.e., that no point of I is fixed by a nontrivial composition in $\mathcal{O}^{\pm}$. Throughout this section we assume that the domains D_i are nonoverlapping.

Using Lemma 4.1, for each $n \geq 1$ we define $\mathcal{I}_{\mathcal{G}}^{(n)} = (\alpha^n I, \phi_i^{(n)} : \alpha^n D_i \to \alpha^n R_i)_{i=1}^q$ on $\alpha^n I$ where $\phi_i^{(n)}$ is the restriction of $\phi_{\tau^n(i)}$ to $\alpha^n D_i$. For each n the system $\mathcal{I}^{(n)}$ is isomorphic to the system $\mathcal{I}_{\mathcal{G}}$ under the map $x \mapsto \alpha^n x I$.

Definition 6.1. *The system of partial isometries $\mathcal{I}_{\mathcal{G}} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ is said to be self-similar if the following conditions hold:*

(1) For each $x \in D_i$, $\phi_{\tau(i)}$ is a first (forward) return of αx to αI.

(2) Each first return of αx to αI is of the form $\phi_{\tau(i)}$ where $x \in D_i$.

(3) For $i \neq j$, $\tau(i) \neq \tau(j)$.

Although if the system is self-similar, $\phi_{\tau(i)}$ and $\phi_{\tau(j)}$ are distinct words in $\mathcal{O}^+$, it may happen that for $x \in D_i \cap D_j$, $(x)\phi_i = (x)\phi_j$ in which case $(\alpha x)\phi_{\tau(i)} = (\alpha x)\phi_{\tau(j)}$. We observe that if $\mathcal{I}_{\mathcal{G}}$ is self-similar, then for each $n \geq 1$ and for each $x \in \cup_{i=1}^r D_i$, the first returns of $\alpha^n x$ to $\alpha^n I$ are of the form $\phi_{\tau^n(i)}$ where $x \in D_i$. In particular, if x is a branch point, then $\alpha^n x$ has exactly two first returns to the interval $\alpha^n I$.

Proposition 6.2. *Suppose the system $\mathcal{I}_{\mathcal{G}} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ is self-similar. There exists a permutation $\sigma : BP \to BP$ such that for each $x \in BP$ and for each $n \geq 1$, $\sigma^n(x)$ is the unique branch point in each first return of $\alpha^n x$ to $\alpha^n I$. Thus, there is a natural number p such that for each $x \in BP$, each forward trajectory of $\alpha^p x$ visits x before returning to $\alpha^p I$.*

Proof. We begin by defining $\sigma : BP \to BP$. Let $x \in BP$. If $\alpha x \in BP$ then we set $\sigma(x)$ equal to αx. Otherwise, $\sigma(x)$ is equal to the unique branch point in the forward trajectory of αx before returning to αI. (Recall, that if $x \in D_i \cap D_j$, then αx has exactly two first returns to αI namely $\phi_{\tau(i)}$ and $\phi_{\tau(j)}$. Thus the forward trajectory of αx visits exactly one branch point prior to returning to αI).

We next show that $\alpha^n x$ visits $\sigma^n(x)$ before returning to $\alpha^n I$. By definition of σ this is true for $n = 1$. Inductively, suppose that $\alpha^{n-1}x$ visits $\sigma^{n-1}(x)$ prior to returning to $\alpha^{n-1}I$. Thus, there is a word $\phi_w \in \mathcal{O}^+$ defined at $\alpha^{n-1}x$ with $(\alpha^{n-1}x)\phi_w = \sigma^{n-1}(x)$ and with $(\alpha^{n-1}x)\phi_v \notin \alpha^{n-1}I$ for each nonempty initial subword v of w. Then, $\alpha^n x$ is in the domain of $\phi_{\tau(w)}$ and

$$\begin{aligned}
(\alpha^n x)\phi_{\tau(w)} &= \alpha^n x + S(\tau(w)) \\
&= \alpha^n x + \alpha S(w) \\
&= \alpha(\alpha^{n-1}x + S(w)) \\
&= \alpha\sigma^{n-1}(x).
\end{aligned}$$

By definition of σ, $\alpha\sigma^{n-1}(x)$ visits $\sigma^n(x)$ before returning to αI. Thus, the forward trajectory of $\alpha^n x$ visits $\sigma^n(x)$ before returning to $\alpha^n I$.

To see that σ is a permutation, we show that σ is surjective. Let $z \in BP$. If $z = 0$, then $\sigma(z) = z$. Otherwise, we choose $n \geq 1$ such that $z \notin \alpha^n I$. Let $\alpha^n x \in \cup_{i=1}^r \alpha^n D_i$ be a point having a forward trajectory which visits z before returning to $\alpha^n I$. (Recall that z has a backwards trajectory which is dense in Ω.) Since z is a branch point, $\alpha^n x$ has two first returns to $\alpha^n I$. This implies that $x \in BP$ and $\sigma^n(x) = z$. ∎

Lemma 6.3. *Suppose the system $\mathcal{I}_{\mathcal{G}} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ is self-similar. If $(x)\phi_w = 0$ for some branch point x and $\phi_w \in \mathcal{O}^+$ with $|w| \geq 1$, then $x \neq 0$ and $|w| = 1$, that is, $(x)\phi_i = 0$ for some $1 \leq i \leq r$.*

Proof. We first note that $x \neq 0$. In fact, the length of each first return of 0 to $\alpha^n I$ is increasing with n. By Proposition 6.2 there is a natural number n such that the forward trajectory of $\alpha^n x$ visits x before returning to $\alpha^n I$ and for each initial proper subword v of w, $(x)\phi_v \notin \alpha^n I$. Then for some $1 \leq i \leq r$ we have $(\alpha^n x)\phi_{\tau^n(i)} = 0$. It follows that $(x)\phi_i = x + S(i) = 0$. ∎

Lemma 6.4. *Suppose the system $\mathcal{I}_{\mathcal{G}} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ is self-similar. Let $x, y \in BP$ with $(x)\phi_w = y$ for some $\phi_w \in \mathcal{O}^+$ with $|w| \geq 1$. Then $x \neq y$, $|w| = 1$, and $y = 0$.*

Proof. By Lemma 6.3 it suffices to show that $y = 0$. Suppose to the contrary that $y \neq 0$. We claim that the forward orbit of x in ϕ_w does not visit 0 before visiting y. Otherwise, for n sufficiently large, 0 would have at least two distinct first returns to $\alpha^n I$; in fact for n large enough a forward trajectory of 0 would visit the branch point y before returning to $\alpha^n I$. But then (for n sufficiently large) 0 would have at least three distinct first returns to $\alpha^n I$. Having established that 0 does not lie in the orbit of x in ϕ_w it follows from Proposition 6.2 that there is a natural number n such that a forward trajectory of $\alpha^n x$ visits both branch points x and y before returning to $\alpha^n I$. This means that $\alpha^n x$ has at least three distinct first returns to $\alpha^n I$, a contradiction. ∎

For each branch point x of $\mathcal{I}_{\mathcal{G}}$, let $\mathcal{O}(x)$ denote the Cayley graph of the $\mathcal{I}_{\mathcal{G}}$-orbit of x. Let $\mathcal{O}_0(BP) = BP \cup \{(x)\phi_i | x \in BP\}$. Given $y, z \in \mathcal{O}_0(BP)$, put a directed edge from y to z for each i such that $(y)\phi_i = z$. Each connected component of $\mathcal{O}_0(BP)$ is a finite subgraph of the Cayley graph $\mathcal{O}(x)$ of some branch point x. Although the edges of $\mathcal{O}(x)$ for $x \in BP$ are directed, by a path in $\mathcal{O}(x)$ we shall mean a nondirected path. Thus, each path in $\mathcal{O}(x)$ from vertex y to vertex z determines an element $\phi_w \in \mathcal{O}^{\pm}$ with $(y)\phi_w = z$. Conversely, given vertices y and z in $\mathcal{O}(x)$ and $\phi_w \in \mathcal{O}^{\pm}$ with $(y)\phi_w = z$, there is a path in $\mathcal{O}(x)$ from y to z which corresponds to ϕ_w.

Proposition 6.5. *Suppose the system $\mathcal{I}_{\mathcal{G}} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ is self-similar. Let $x, y \in BP$ and suppose that $\mathcal{O}(x) = \mathcal{O}(y)$. Then x and y belong to the same connected component of $\mathcal{O}_0(BP)$.*

Proof. Let $\phi_w \in \mathcal{O}^{\pm}$ with $(x)\phi_w = y$. We can assume without loss of generality that the orbit of x in ϕ_w does not contain other branch points other than x and y. Assume also that ϕ_w is reduced. Thus, either $\phi_w \in \mathcal{O}^+$ or $\phi_w \in \mathcal{O}^-$ or $\phi_w = \phi_+\phi_-$

with $\phi_+ \in \mathcal{O}^+$ and $\phi_- \in \mathcal{O}^-$. In the first two cases it follows from Lemma 6.4 that $|w| = 1$, in which case x and y are joined by an edge in $\mathcal{O}_0(BP)$.

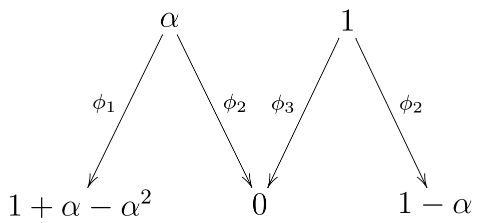
Suppose $(x)\phi_+ = (y)\phi_-^{-1}$. If 0 is not in the forward trajectories of x or y which contain the point $(x)\phi_+ = (y)\phi_-^{-1}$, then by Proposition 6.2 there is a natural number n such that $(\alpha^n x)\phi_{\tau^n(i)} = (\alpha^n y)\phi_{\tau^n(j)}$ for some i and j. This is because each of $\alpha^n x$ and $\alpha^n y$ has a forward trajectory which will visit the point $(x)\phi_+ = (y)\phi_-^{-1}$ before returning to $\alpha^n I$. But then, $(x)\phi_i = (y)\phi_j$ in which case x and y are joined by a path of length two in $\mathcal{O}_0(BP)$. (This argument holds even if $x = y$.)

If 0 is in the forward trajectory of x containing $(x)\phi_+$, then by Lemma 6.3 $(x)\phi_i = 0$ for some i. If 0 is also in the forward trajectory of y then again by Lemma 6.3 we have $(x)\phi_i = 0 = (y)\phi_j$ for some j. Again this implies that x and y are joined by a path of length two in $\mathcal{O}_0(BP)$. Finally, if 0 is not in the forward trajectory of y containing $(y)\phi_-^{-1}$, then there is a natural number n such that $\alpha^n y$ and 0 each have a forward trajectory which visits the point $(y)\phi_-^{-1}$ before returning to $\alpha^n I$. Then for some k and l we have $(0)\phi_{\tau^n(k)} = (\alpha^n y)\phi_{\tau^n(l)}$. Thus, $(0)\phi_k = (y)\phi_l$ so that x and y are joined in $\mathcal{O}_0(BP)$ by a path of length three. ∎

Theorem 6.6. *Suppose the system of partial isometries $\mathcal{I}_\mathcal{G}$ is self-similar. Then, $\mathcal{I}_\mathcal{G}$ is fixed point free if and only if each connected component of $\mathcal{O}_0(BP)$ is circuit-free.*

Proof. Clearly, if a connected component of $\mathcal{O}_0(BP)$ contains a circuit, then there is a composition $\phi_w \in \mathcal{O}^\pm$ fixing some vertex $x \in I$. Conversely, if a nontrivial composition ϕ_w fixes some point x in I, then by Proposition 4.2 some cyclic permutation ϕ'_w of ϕ_w fixes some branch point y. The result follows from Proposition 6.5. ∎

Example 6.7 It is readily verified that the interval translation mapping $\mathcal{I}_\mathcal{G}$ of Example 5.2 is self-similar and $\mathcal{O}_0(BP)$ is circuit-free. (See Figure 6.1). It follows from Theorem 6.6 that $\mathcal{I}_\mathcal{G}$ is fixed point free.



$\mathcal{O}_0(BP)$

Figure 6.1

7 Actions on $\mathbb{R}$-trees

Let $\mathcal{G}$ be a real geometric realization of a primitive substitution τ on $\mathcal{A} = \{1, 2, \dots, r\}$ and let $\mathcal{I}_\mathcal{G} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ denote the associated system of partial isometries defined in section 4. In this section we consider induced actions on $\mathbb{R}$-trees.

An $\mathbb{R}$-tree Y is a real metric space in which each pair of points is joined by a unique arc and this arc is isometric to a closed subinterval of $\mathbb{R}$ [25], [15]. Associated to a system of partial isometries $(I, \phi_i : D_i \to R_i)_{i=1}^r$ is an action by isometry of the free group $\mathbb{F}_r = F(\{g_1, g_2, \dots, g_r\})$ on an $\mathbb{R}$-tree Y (Theorem 1.1 in [14].) The tree Y is constructed by identifying points in $I \times \mathbb{F}_r$ along the domains and ranges of the pseudogroup of partial isometries generated by $\{\phi_1, \phi_2, \dots, \phi_r\}$ [14]. More precisely, one identifies points (x, g) and (y, h) if and only if $y = (x)\phi_{i_1}^{\epsilon_1} \circ \phi_{i_2}^{\epsilon_2} \circ \dots \circ \phi_{i_n}^{\epsilon_n}$, where $\epsilon_i = \pm 1$ and $g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_n}^{\epsilon_n} = h^{-1}g$. Alternatively, Y is the metric space associated to $I \times \mathbb{F}_r$ with respect to the pseudometric d' defined by:

$$d'((x, g), (y, h)) = d_I(x, x_1) + d_I((x_1)\phi_{i_1}^{\epsilon_1}, x_2) + \dots + d_I((x_n)\phi_{i_n}^{\epsilon_n}, y)$$

where $g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_n}^{\epsilon_n}$ is the reduced word representing the element $h^{-1}g$ in $\mathbb{F}_r$, and where x_1 is the point in the domain of $\phi_{i_1}^{\epsilon_1}$ closest to x, x_2 is the point in the domain of $\phi_{i_2}^{\epsilon_2}$ closest to $(x_1)\phi_{i_1}^{\epsilon_1}$, and so on. There is a natural $\mathbb{F}_r$-action on Y induced by $(x, h)g = (x, hg)$.

It is shown in [14] that the resulting $\mathbb{F}_r$-action on Y satisfies the following "geometric" properties:

- Y contains I as an isometrically embedded subtree.
- For all $x \in D_i$, $(x)g_i = (x)\phi_i$.
- For all points $x \in Y$, the orbit of x meets I.
- For all points $x, y \in I$ and for all $g \in \mathbb{F}_r$ with $y = (x)g$, we can write $y = (x)\phi_{i_1}^{\epsilon_1} \circ \phi_{i_2}^{\epsilon_2} \circ \dots \circ \phi_{i_n}^{\epsilon_n}$ where $g = g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_n}^{\epsilon_n}$ and $\epsilon_i = \pm 1$.

If the $\mathbb{F}_r$-action on Y is minimal in the sense that there is no proper $\mathbb{F}_r$-invariant subtree contained in Y, then the action is called geometric. In general, the above construction does not guarantee minimality. However, in our case we have:

Proposition 7.1. *Let $\mathcal{I}_\mathcal{G} = (I, \phi_i : D_i \to R_i)_{i=1}^r$ be the system of partial isometries associated to a real geometric realization of τ. Let $Y_\mathcal{G} \times \mathbb{F}_r \to Y_\mathcal{G}$ be the induced $\mathbb{F}_r$-action on the $\mathbb{R}$-tree $Y_\mathcal{G}$ described above. Then:*

(1) The action of $\mathbb{F}_r$ on $Y_\mathcal{G}$ is minimal.
(2) Each branch point of $Y_\mathcal{G}$ is a limit point of branch points of $Y_\mathcal{G}$.
(3) The tree $Y_\mathcal{G}$ is not a complete metric space.
(4) $\mathcal{I}_\mathcal{G}$ is minimal if and only if the $\mathbb{F}_r$-orbit of each point $x \in Y_\mathcal{G}$ is dense in $Y_\mathcal{G}$.
(5) The generators $\phi_i : D_i \to R_i$ are independent if and only if each edge stabilizer is trivial. Moreover, if $\mathcal{I}_\mathcal{G}$ is minimal, then this latter condition is equivalent to the action being small, i.e., no edge stabilizer contains a free group of rank 2.
(6) The action of $\mathbb{F}_r$ on $Y_\mathcal{G}$ is free if and only if $\mathcal{I}_\mathcal{G}$ is fixed point free.

Proof. Items (4),(5), and (6) are true in general (see [14]). To prove (1), let Y be an invariant subtree of $Y_{\mathcal{G}}$. Then, Y must meet $I \subset Y_{\mathcal{G}}$. Writing $[a', b'] = Y \cap I$ it follows that both $a', b' \in \Omega$ since each branch point of $Y_{\mathcal{G}}$ belongs to the orbit of an endpoint of a base of the system $\mathcal{I}_{\mathcal{G}}$. It follows then that $[a', b'] = I$. Otherwise, there is a group element $g \in \mathbb{F}_r$ with $(a')g \in I \setminus [a', b']$. This is because the endpoints of I are in the closure of the orbit of each point in Ω. But this is a contradiction since $(a')g \in Y$. Thus, $I \subset Y$ and therefore $Y = Y_{\mathcal{G}}$. (2) follows from the fact that each branch point of $Y_{\mathcal{G}}$ is in the orbit of a point in Ω. Finally, (3) follows immediately from (2). ∎

Proposition 7.1 implies that the resulting F_r-action on $Y_{\mathcal{G}}$ is always geometric and exotic, i.e., nonsimplicial. Theorem 6.6 gives a necessary and sufficient condition for the resulting action to be free provided $\mathcal{I}_{\mathcal{G}}$ is self-similar.

Theorem 7.2. *Let τ be a primitive substitution on the alphabet $\mathcal{A} = \{1, 2, \ldots, r\}$. Let $\mathcal{G}$ be a real geometric realization of τ. Suppose that the associated system of partial isometries $\mathcal{I}_{\mathcal{G}} = (I, \phi_i : D_i \to R_i)_{i=1}^{r}$ is self-similar. Then, the $\mathbb{F}_r$-action on the $\mathbb{R}$-tree $Y_{\mathcal{G}}$ defined above is free if and only if each connected component of $\mathcal{O}_0(BP)$ is circuit-free.*

The homogeneous system of Example 5.3 gives rise to a non-small $\mathbb{F}_3$-action. In fact, since the generators are not independent, there exist nontrivial edge stabilizers. Also, since the orbit of each point $x \in Y_{\mathcal{G}}$ is dense, no edge stabilizer is equal to $\mathbb{Z}$ (see [14]). Hence if the action were small, each edge stabilizer would be trivial. The interval exchange of Example 5.1 gives rise to a small but non-free $\mathbb{F}_3$-action, while the exotic system of Example 5.2 satisfies the hypothesis of Theorem 6.6 and gives rise to a free $\mathbb{F}_3$-action.

References

[1] J.-P. Allouche, L.Q. Zamboni, *Algebraic irrational binary numbers cannot be fixed points of non-trivial constant length or primitive morphisms,* J. Number Theory **69** (1998), 119–124.

[2] P. Arnoux, *Représentations géométrique des systèmes dynamiques symboliques,* Habilitation à diriger des recherches, Université d'Aix-Marseille II, 1991.

[3] E. Bombieri, J.E. Taylor, *Quasicrystals, tilings, and algebraic number theory: Some preliminary connections,* Contemp. Math. **64** (1987), 241–264.

[4] M. Boshernitzan, I. Kornfeld, *Interval translation mappings,* Ergod. Th. & Dynam. Sys. **15** (1995), 821–832.

[5] G. Christol, T. Kamae, M. Mendes-France, G. Rauzy, *Suites algébriques et substitutions,* Bull. Soc. Math. France **108** (1980), 401–419.

[6] M. Dekking, *Recurrent sets,* Adv. Math. **44** (1982), 78–104.

[7] F. Durand, *Contributions à l'étude des suites et systèmes dynamiques substitutifs,* Thèse de doctorat, Université de la Méditerranée, Aix-Marseille II, Faculté de Sciences de Luminy, 1996.

[8] F. Durand, B. Host, C. F. Skau, *Substitution dynamical systems, Bratteli diagrams and dimension groups,* Ergod. Th. & Dynam. Sys. (1996), in press.

[9] S. Ferenczi, *Les transformations de Chacon: combinatoire, structure géométrique, lien avec les systèmes de complexité* $2n + 1$, Bull. Soc. Math. France, (1995), 271–292.

[10] S. Ferenczi, C. Mauduit, *Transcendence of numbers with a low complexity expansion,* J. Number Theory, **67** (1997), 146–161.

[11] A. H. Forrest, *K-groups associated with substitution minimal systems,* Israel J. Math. **98** (1997), 101–139.

[12] D. Gaboriau, *Générateurs indépendants pour les systèmes d'isométries de dimension un,* Ann. Inst. Fourier **47** (1997), 101–122.

[13] D. Gaboriau, *Dynamique des systèmes d'isométries: Sur les bouts des orbites,* Invent. Math. **126** (1996), 297–318.

[14] D. Gaboriau, G. Levitt, *The rank of actions on* $\mathbb{R}$*-trees,* Annales Scientifiques de L'École Normale Supérieure **28**, no. 4 (1995), 549–570.

[15] D. Gaboriau, G. Levitt, F. Paulin, *Pseudogroups of isometries of* $\mathbb{R}$ *and Rips' theorem on free actions on* $\mathbb{R}$*-trees,* Israel J. Math. **87** (1994), 403–428.

[16] C. Holton, L.Q. Zamboni, *Exotic actions on trees,* Bull. London Math. Soc. **29** (1997), 309–313.

[17] C. Holton, L.Q. Zamboni, *Geometric realizations of substitutions,* Bull. Soc. Math. France, **126** (1998), 149–179.

[18] S. Ito, M. Kimura, *On Rauzy fractal,* Japan J. Indust. Appl. Math. **8** (1991), 461–486.

[19] G. Levitt, 1*-formes fermées singulières et groupe fondamental,* Invent. Math. **88** (1987), 635–667.

[20] G. Levitt, *La dynamique des pseudogroupes de rotations,* Invent. Math. **113** (1993), 633–670.

[21] M. Morse, G.A. Hedlund, *Sturmian sequences,* Amer. J. Math. **62** (1940), 1–42.

[22] M. Queffélec, *Substitution Dynamical Systems-Spectral Analysis,* Lecture Notes in Math., #1294, 1987.

[23] G. Rauzy, *Nombres algébriques et substitutions,* Bull. Soc. Math. France **110** (1982), 147–178.

[24] G. Rauzy, *Rotations sur les groupes, nombres algébriques et substitutions,* Séminaire de Théorie des Nombres, Bordeaux, Exp. No. 21, Univ. Bordeaux, Talence, 1987-1988.

[25] P. Shalen, *Dendrology and its applications,* in "Group theory from a geometrical viewpoint (E. Ghys, A. Haeflinger, A. Verjovsky, eds.)," (1991), 543–616.

[26] B. Solomyak, *Substitutions, adic transformations and beta-expansions,* Contemp. Math. **135** (1992), 361–372.

Charles Holton
Department of Mathematics
University of California
Berkeley, CA 94720-3840
`cholton@math.berkeley.edu`

Luca Q. Zamboni
Department of Mathematics
University of North Texas
Denton, Texas, TX 76203-5116
`luca@unt.edu`