Transitivity, dense orbit and discontinuous functions*

Alfredo Peris

The main “ingredient” in Devaney’s definition of chaos is transitivity (see [3]). Banks, Brooks, Cairns, Davis and Stacey [1] demonstrated the redundancy of sensitive dependence on initial conditions (the most popularly understood hypothesis of Devaney). They showed as a consequence that chaos is a topological property (not metric, as one could think about the original definition). Moreover, transitivity implies chaos for continuous functions on intervals (we refer to [4] for a simple proof). We recall that a map $f : M \to M$ is *transitive* if for any pair of non-empty open sets U and V in M, there is some $k > 0$ with $f^k(U) \cap V \neq \emptyset$. Here M denotes a metric space and f^k is f composed with itself k times. Transitivity can be seen, in words of Banks et al., as an irreducibility condition. It is worth mentioning that other alternatives to transitivity had been provided by A. Crannell [2], which can be regarded as more intuitive properties.

For Baire separable metric spaces M (which is usually the case) and continuous f, transitivity implies the existence of a point $x \in M$ whose orbit is dense in M. Let us notice that separability of M is obviously necessary to get the existence of a dense orbit. The phenomenon of dense orbit is much more intuitive than transitivity. On the other hand there are discontinuous maps which are interesting from the point of view of discrete dynamical systems (e.g., Baker’s function $B(x) := 2x$ for $0 \leq x \leq 1/2$, $B(x) := 2x-1$ for $1/2 < x \leq 1$). But Baire’s Theorem is not applicable in general for discontinuous functions. This leads us to the following question: For which type of discontinuous functions are transitivity and existence of dense orbits equivalent?

*Supported in part by DGICYT Proyecto PB94-0541
Received by the editors December 1997.
Communicated by Fr. Bastin.
1991 *Mathematics Subject Classification* : 26A18, 54H20, 58F03, 58F13.
Key words and phrases : Transitive map, dense orbit, chaos.

We will show that this is the case for functions $f : M \to M$ on a Baire separable metric space M which have at most one point of discontinuity, but it is false in general if f has more than one point of discontinuity.

Proposition 1. *Let M be a Baire separable metric space and let $f : M \to M$ be a transitive map with (only) one point of discontinuity $a \in M$. Then there is $x \in M$ such that $\{f^n(x)\}_{n\in\mathbb{N}}$ is dense in M.*

Proof. If $\{f^n(a)\}_{n\in\mathbb{N}}$ is dense in M, there is nothing to prove. Thus we assume that the orbit of a is not dense in M.

We first show that $\{f^n(a)\}_{n\in\mathbb{N}}$ is rare. If this is not the case, we can find open sets U and V in M such that

$$U \subset \overline{\{f^n(a)\}_{n\in\mathbb{N}}} \text{ and } V \cap \overline{\{f^n(a)\}_{n\in\mathbb{N}}} = \emptyset.$$

By the transitivity of f we select $m \in \mathbb{N}$ and $u \in U$ such that $f^m(u) \in V$. Now f^m is continuous at u, since $f^m(u) \neq f^n(a)$ for each $n \in \mathbb{N}$. Then there is an open subset $\tilde{U}$ of U with $f^m(\tilde{U}) \subset V$. But, taking $k \in \mathbb{N}$ with $f^k(a) \in \tilde{U}$, we get $f^{m+k}(a) \in V$, which is a contradiction.

Now we select a countable basis $\{V_n\}_{n\in\mathbb{N}}$ of open sets in $M \setminus \overline{\{f^n(a)\}_{n\in\mathbb{N}}}$ and define

$$G_n := \{x \in M \ \ / \ \exists m : \ f^m(x) \in V_n\}, \qquad n \in \mathbb{N}.$$

We will show that $\{G_n\}_{n\in\mathbb{N}}$ is a sequence of dense open subsets of M. To see this, we fix $n \in \mathbb{N}$. If $x \in G_n$, there is $m \in \mathbb{N}$ such that $f^m(x) \in V_n$ and f^m is continuous at x, by the selection of V_n. This implies the existence of an open neighbourhood U of x with $f^m(U) \subset V_n$. Therefore G_n is open. On the other hand, given an arbitrary open set W in M, by the transitivity of f we find $m \in \mathbb{N}$ and $w \in W$ satisfying $f^m(w) \in V_n$. Thus $w \in G_n$ and G_n is dense.

Finally, since M is a Baire space, $G := \bigcap_{n\in\mathbb{N}} G_n$ is dense in M, and we conclude that $\{f^m(x)\}_{m\in\mathbb{N}}$ is dense in M for every $x \in G$. ∎

The following example shows that the proposition above is optimal in the sense that a transitive map with two points of discontinuity does not have, in general, a dense orbit.

Example: Let $T : [0,1] \longrightarrow [0,1]$ be the tent map

$$T(x) := \begin{cases} 2x, & x \in [0,1/2[\\ 2-2x, & x \in [1/2,1] \end{cases}$$

which is a continuous transitive map (thus has a dense orbit). Fix $y_1 \in]0,1[$ such that $\{T^n(y_1)\}_{n\in\mathbb{N}}$ is dense in $[0,1]$, set $y_0 := y_1 + 1$ and define $f : [0,2] \longrightarrow [0,2]$ by

$$f(x) := \begin{cases} T(x), & x \in [0,1[\\ y_0, & x = 1 \\ 1 + T(x-1), & x \in]1,2[\\ y_1, & x = 2 \end{cases}$$

(see figure).

We are going to prove that f is a transitive map without a dense orbit.

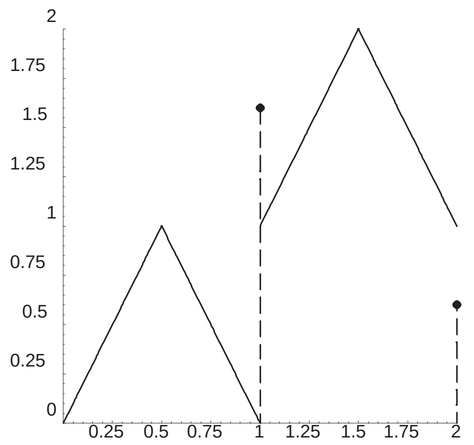

Figure 1:

Proof.
(i) $\{f^n(y_1)\}_{n\in\mathbb{N}}$ and $\{f^n(y_0)\}_{n\in\mathbb{N}}$ are dense in $]0,1[$ and $]1,2[$, respectively:

First of all $\{f^n(y_1)\}_{n\in\mathbb{N}} = \{T^n(y_1)\}_{n\in\mathbb{N}}$ is dense in $]0,1[$. On the other hand this sequence coincides with $\{T^n(y_0-1)\}_{n\in\mathbb{N}}$. Hence $\{x_n\}_{n\in\mathbb{N}} := \{1+T^n(y_0-1)\}_{n\in\mathbb{N}}$ is dense in $]1,2[$. Moreover,

$$f(y_0) = 1 + T(y_0-1) = x_1, \quad f(x_1) = 1 + T(x_1-1) = 1 + T^2(y_0-1) = x_2.$$

And, proceeding by induction, we get $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$, and we conclude that $\{f^n(y_0)\}_{n\in\mathbb{N}}$ is dense in $]1,2[$.

(ii) If $x = k/2^n$, $k,n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that $f^m(x) = y_0$ if $x \in]0,1]$ and $f^m(x) = y_1$ if $x \in]1,2]$. Indeed, by the definition of T we easily have $T(1/2) = 1$, $T^2(1/4) = T^2(3/4) = 1, \ldots, T^n(k/2^n) = 1$, for every irreducible fraction $k/2^n \in]0,1[$. Hence, if $x = k/2^n \in]0,1[$, there is $m \in \mathbb{N}$ such that $f^m(x) = f(T^{m-1}(x)) = f(1) = y_0$.

If $x \in]1,2[$, $f(x) > 1$, then either $f(x) = 2$ and we conclude $f^2(x) = y_1$, or $f(x) \in]1,2[$. In the second case we continue with the iteration of f until we reach $m \in \mathbb{N}$ with $T^m(x-1) = 1$. On account that $f^l(x) = 1 + T^l(x-1)$ if $1 \le l \le m$, we would have $f^{m+1}(x) = y_1$.

(iii) f is transitive. To see this, we take open subsets U and V of $]0,2[$. We consider four cases:

Case 1: $U \cap]0,1[\neq \emptyset$ and $V \cap]0,1[\neq \emptyset$. Since $\{f^n(y_1)\}_{n\in\mathbb{N}}$ is dense in $]0,1[$, there are $n_1 < n_2$ such that $u := f^{n_1}(y_1) \in U$ and $f^{n_2}(y_1) \in V$. For $m := n_2 - n_1$ it follows $f^m(u) \in f^m(U) \cap V$.

Case 2: $U \cap]1,2[\neq \emptyset$ and $V \cap]1,2[\neq \emptyset$. This case is analogous because $\{f^n(y_0)\}_{n\in\mathbb{N}}$ is dense in $]1,2[$.

Case 3: $U \cap]0,1[\neq \emptyset$ and $V \cap]1,2[\neq \emptyset$. We find $k,n,m \in \mathbb{N}$ such that $u := k/2^n \in U \cap]0,1[$ and $f^m(u) = y_0$. The conclusion follows if we take $m' > m$ with $f^{m'}(u) \in V$.

Case 4: $U \cap]1,2[\neq \emptyset$ and $V \cap]0,1[\neq \emptyset$. Similarly as in case 3, there are $k,n,m,m' \in \mathbb{N}$ such that $u := k/2^n \in U \cap]1,2[$, $f^m(u) = y_1$ and $f^{m'}(u) \in V$.

(iv) If $x \in [0,2]$, then $\{f^n(x)\}_{n\in\mathbb{N}}$ is not dense in $[0,2]$:

If $x \in [0,1]$, either $x \neq k/2^n$, $\forall k,n \in \mathbb{N}$, which implies $f^n(x) = T^n(x) \in [0,1[$, for all $n \in \mathbb{N}$; or $x = k/2^n$ for some $n \in \mathbb{N}$, $1 \leq k \leq 2^n$, which implies the existence of $m \in \mathbb{N}$ with $f^m(x) = y_0$ and then $f^{m'}(x) \in]1,2[$, $\forall m' \geq m$.

If $x \in]1,2]$, we have either $x \neq k/2^n$, for all $k,n \in \mathbb{N}$, which implies $f^n(x) \in]1,2[$, $\forall n \in \mathbb{N}$; or there is m with $f^m(x) = y_1$ and we thus get $f^{m'}(x) \in]0,1[$, $\forall m' \geq m$. ■

Acknowledgements: The author wishes to express his thanks to J. Bonet for several helpful comments concerning the final version of the paper.

References

[1] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, *On Devaney's definition of chaos*, American Mathematical Monthly **99** (1992), 332-334.

[2] A. Crannell, *The role of transitivity in Devaney's definition of chaos*, American Mathematical Monthly **102** (1995), 788-793.

[3] R.L. Devaney, *An introduction to Chaotic Dynamical Systems*, Addison-Wesley, 1989.

[4] M. Vellekoop and R. Berglund, *On intervals, transitivity = chaos*, American Mathematical Monthly **101** (1994), 353-355.

E.T.S. Arquitectura
Universidad Politécnica de Valencia
Departamento de Matemática Aplicada
E-46071 Valencia
Spain
aperis@pleiades.upv.es