m-convex properties in locally convex algebras

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Abstract

We consider m-convex properties and compare different radii in locally convex algebras.

Introduction

We say that the continuity of the inverse map $x \mapsto x^{-1}$ and the equicontinuity of the power maps $(x \mapsto x^n)_n$, are m-convex properties. They are essential in m-convex algebras. It is also known that several radii ρ, β and R_1 to R_7 are equal in such complete algebras ([5]). We prove the same in a more general situation and under the weakest completion notion (pseudo-completeness).

In section 2, we show, in particular, that in a unitary pseudo-complete locally convex algebra with continuous inverse, the equality of ρ, β and R_1 to R_7 is still true (Proposition 2.2). We also obtain $\rho = \beta$ in any unitary pseudo-complete locally convex algebra for which the power maps $(x \mapsto x^n)_n$ are equicontinuous at zero. In section 3, we examine links between the two m-convex properties above. In the general context of unitary locally convex algebra's, the equicontinuity of power maps implies the continuity of the inverse map under the additional condition that the map $x \mapsto R_7(x)$ be upper semi-continuous (Proposition 3.1). It appears that in a unitary pseudo-complete locally convex algebra (A, τ) which is a Q-algebra such that the power maps are equicontinuous at zero, the continuity of the inverse is equivalent to the boundedness of every element of (A, τ) (Proposition 3.2). In section 4, we generalize a result of A. Arosio ([3]) on B_0-algebras. We obtain that the power maps are equicontinuous at zero in any Baire locally convex algebra (A, τ) with a

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continuous product such that the map $x \mapsto R_7(x)$ is upper semi-continuous ; whence (A, τ) is m-convex in the commutative case.

1 Preliminaries

Let (A, τ) be a locally convex algebra (l. c. a.) the topology of which is given by a family $(p_i)_{i \in I}$ of seminorms. The set of non zero continuous characters of (A, τ) is denoted by $m(A)$. In the sequel ρ and β will designate respectively the spectral radius and radius of boundedness. The set of invertible elements in A is denoted by $G(A)$.
We consider the following radii

$R_1(x) = \sup\left\{ \limsup\limits_{n \to +\infty} \sqrt[n]{p_i(x^n)} : i \in I \right\}$,

$R_2(x) = \sup\left\{ \limsup\limits_{n \to +\infty} \sqrt[n]{p(x^n)} : p \text{ is a continuous seminorm on } A \right\}$,

$R_3(x) = \sup\left\{ \limsup\limits_{n \to +\infty} \sqrt[n]{|f(x^n)|} : f \in A' \right\}$,

$R_4(x) = \sup\{|\chi(x)| : \chi \in m(A)\}$,

$R_5(x) = \inf\{r > 0 : x - \lambda e \in G(A) \text{ for all } |\lambda| > r\}$,

$R_6(x) = \inf\{0 < r \leq +\infty$: there is a sequence of complex numbers $(\alpha_n)_n$ such that the radius of convergence of $\sum_{n=0}^{+\infty} \alpha_n z^n$ is r and $\sum_{n=0}^{+\infty} \alpha_n x^n$ converges in $A\}$,

$R_7(x) = \inf\{0 < r \leq +\infty$: for any sequence of complex numbers $(\alpha_n)_n$ such that the radius of convergence of $\sum_{n=0}^{+\infty} \alpha_n z^n$ is r, $\sum_{n=0}^{+\infty} \alpha_n x^n$ converges in $A\}$.

In a general l. c. a. (A, τ), G. R. Allan showed that $R_1 = R_2 = R_3 = \beta$ ([1], proposition 2.18). Thus by ([1], theorem 3.12 and theorem 4.1), $R_1 = R_2 = R_3 = \beta = \rho$ if (A, τ) is pseudo-complete (i.e. ; if every bounded and closed idempotent disk is Banach) with continuous inverse. W. Zelazko obtained the equality of R_1 to R_7 in any unitary commutative and complete m-convex algebra ([5]). H. Arizmendi and K. Jarosz showed that $\beta = R_7 \geq R_6 \geq R_4$ in any unitary commutative complete locally convex algebra with continuous product ([2], theorem 1.8). They also provide examples where the inequalities can be strict ([2], theorem 3.2 and theorem 4.1).

For general notions in l. c. a.'s, especially boundedness of an element and pseudo-completeness, the reader is refereed to [1]. Concerning locally m-convex algebras (l. m. c. a.) see ([5]) and for B_0-algebras see ([5], [6]).

2 Comparison of radii

The radii ρ and β are not always comparable. Consider A the field $C(X)$ of rational fractions of the ring of complex polynomials endowed with its strongest locally convex topology. One has $\rho(X) = 0$ and $\beta(X) = +\infty$, hence ρ and β are different. Now let B be a normed non Q-algebra. In B, we have $\beta \ngeq \rho$. In the product algebra $A \times B$, ρ and β are not comparable. We have the following.

Proposition 2.1. *Let (A, τ) be a unitary l. c. a. and $x \in A$. Then*
(i) *Max$(\rho(x), \beta(x)) \leq R_7(x)$.*
(ii) *If (A, τ) is pseudo-complete, then $\beta(x) = R_7(x)$.*

(iii) *If (A, τ) is pseudo-complete and with continuous inverse, then $\beta(x) = \rho(x) = R_7(x)$.*

Proof. **(i)** Suppose $R_7(x) < +\infty$ and let $\lambda \in C$ with $|\lambda| > R_7(x)$. Since the radius of convergence of the series $\sum_{n=}^{+\infty} \lambda^{-n} z^n$ is $|\lambda|$, the series $\sum_{n=0}^{+\infty} \lambda^{-n} x^n$ converges in A. Hence $\lambda^{-n} x^n \underset{n}{\longrightarrow} 0$. Whence $\beta(x) \leq R_7(x)$. We also have $e - \lambda^{-1} x$ invertible whence $\rho(x) \leq R_7(x)$.
(ii) It remains to show that $R_7(x) \leq \beta(x)$. Let $\lambda \in C$ such that $\beta(x) < |\lambda|$. For $\delta > 0$, $\beta(x) < \delta < |\lambda|$, consider the closed absolutely convex hull of $B = \{(\delta^{-1}x)^n : n = 1, 2, ...\}$. It is an idempotent completant disc such that $\|x^n\|_B < \delta^n$ for every $n = 1, 2, \ldots$. Now, if a series $\sum_{n=}^{+\infty} \alpha_n z^n$ is with radius of convergence $|\lambda|$, then

$$\limsup_{n \to +\infty} \|\alpha_n x^n\|_B^{\frac{1}{n}} = \limsup_{n \to +\infty} |\alpha_n|^{\frac{1}{n}} \|x^n\|_B^{\frac{1}{n}} \leq \frac{\delta}{|\lambda|} < 1.$$

Hence the series $\sum_{n=}^{+\infty} \alpha_n x^n$ converges in A. Whence $R_7(x) \leq |\lambda|$.
(iii) In this case we also have $\beta(x) \leq \rho(x)$ ([1], Theorem 4.1).

Remark 2.2. Pseudo-complete normed algebras are Banach. In non Banach algebras, the assertion ii) is not valid in general. Otherwise, by i), we should have $\rho \leq \beta$; and this is not true in the algebra of real polynomial functions endowed with the norm given by $\|P\| = \sup\{|P(t)| : t \in [0, 1]\}$.

It is known that in a unitary commutative and complete *m*-convex algebra the radii R_1 to R_7 are equal ([5]). We obtain the same in a more general context and under the weakest completion notion.

Proposition 2.3. *Let (A, τ) be a unitary and pseudo-complete l. c. a. with continuous inverse and $x \in A$.*
(i) *We have $R_6(x) \leq R_7(x) = \rho(x) = \beta(x) = R_1(x) = R_2(x) = R_3(x) = R_5(x)$.*
(ii) *If A is commutative, $m(A) \neq \varnothing$ and $\rho(x) = R_4(x)$, then the radii ρ, β and R_1 to R_7 are all equal.*

Proof. **(i)** By the previous proposition and proposition 2.18 of [1], we have $R_7(x) = \rho(x) = \beta(x) = R_1(x) = R_2(x) = R_3(x)$. Finally, a general fact is $R_5(x) = \rho(x)$.
(ii) We show, as in the *m*-convex commutative case ([5]), that $R_4(x) \leq R_6(x)$.

Remark 2.4. As noticed by the referee R_4 should not be included in (i). Indeed it is zero in the algebra of 2×2-matrices.

Corollary 2.5. *The radii ρ, β and R_1 to R_7 are all equal in any unitary commutative and pseudo-complete m-convex algebra A.*

Proof : If $\hat{\rho}$ and $\hat{\beta}$ designate respectively the spectral radius and the radius of boundedness in the completion $\hat{A}$ of A, we have $R_4(x) = \hat{\rho}(x) = \hat{\beta}(x) = \beta(x)$ and it is known that, in this case $\beta(x) = \rho(x)$.

Instead of the continuity of the inverse map $x \mapsto x^{-1}$, we now consider the equicontinuity of the power maps $x \mapsto x^n$, $n \in N^*$.

Proposition 2.6. *Let (A, τ) be a unitary and pseudo-complete l. c. a. such that the sequence $(x \mapsto x^n)_n$, of the power maps, is equicontinuous at zero. Then $\rho(a) = \beta(a)$, for every element $a \in A$.*

Proof : For $a \in A$, let M be the maximal commutative and unitary subalgebra, of A, containing a. We then have $\rho(a) = \rho_M(a)$ and $\beta(a) = \beta_M(a)$. Now $(x \mapsto x^n)_n$ being equicontinuous at zero, the algebra $(M, \tau/M)$ is an m-convex algebra by a result of Turpin ([4]). Then so is its closure $\overline{M}$ in A. It is also pseudo-complete, hence $\rho_M(a) = \rho_{\overline{M}}(a) = \beta_{\overline{M}}(a) = \beta_M(a)$.

3 Continuity of the inverse

By a result of Turpin ([4], Proposition 2), the sequence $(x \mapsto x^n)_n$ is equicontinuous at zero in any l. c. a. which is a Q-algebra and in which the inverse map $x \mapsto x^{-1}$ is continuous. The converse is not always true; just take any l. m. c. a. which is not a Q-algebra. However in this case the inverse map $x \mapsto x^{-1}$ is continuous. This fact remains true in a l. c. a. under an additional condition.

Proposition 3.1. *Let (A, τ) be a unitary and l. c. a. If the sequence $(x \mapsto x^n)_n$ is equicontinuous at zero and $R_7 : (A, \tau) \to R_+ \cup \{+\infty\}$ is upper semi-continuous, then the inverse map $x \mapsto x^{-1}$ is continuous.*

Proof : The set $A(R_7) = \{x \in A : R_7(x) < 1\}$ is an open neighbourhood of zero for every element x of which

$$(e - x)^{-1} = e + \sum_{n=1}^{+\infty} x^n.$$

Let U be an absolutely convex neighbourhood of zero. Then for every $x \in A(R_7)$ there is an integer $n(x)$ such that

$$\sum_{n=1}^{+\infty} x^n - \sum_{n=1}^{n(x)} x^n \in \frac{1}{2}U.$$

Since the sequence $(x \mapsto x^n)_n$ is equicontinuous at zero, there is a neighbourhood V of zero, $V \subset A(R_7)$, such that $(2x)^n \in \frac{1}{2}U$ for every $x \in V$ and $n \in N^*$.
Finally

$$\begin{aligned}(e - x)^{-1} &\in e + \frac{1}{2}U + \frac{1}{2}\sum_{n=1}^{n(x)} \frac{1}{2^n}U \\ &\subset e + U.\end{aligned}$$

In a pseudo-complete l. m. c. a. which is a Q-algebra, every element is bounded. In this case the sequence $(x \mapsto x^n)_n$ is equicontinuous at zero. In general l. c. a.'s we have the following

Proposition 3.2. *Let (A, τ) be a unitary and pseudo-complete l. c. a. which is a Q-algebra such that the sequence $(x \mapsto x^n)_n$ is equicontinuous at zero. The following assertions are equivalent*

(i) *Every element of* (A, τ) *is bounded.*
(ii) *The inverse map* $x \mapsto x^{-1}$ *is continuous.*

Proof : **(i)** $\Rightarrow$ **(ii)** It is enough to show that $x \mapsto x^{-1}$ is continuous at the unit e ([4]). By ([1], proposition 3.13), $\rho = \beta$. So $U = \{x \in A : \beta(x) < 1\}$ is a neighbourhood of zero such that $e - U \subset G(A)$. Moreover $(e-x)^{-1} = \sum_{n=0}^{+\infty} x^n$, for every $x \in U$. The proof is finished as in proposition 3.1.
(ii) $\Rightarrow$ **(i)** Follows from ([1], corollary 4.2).

4 Equicontinuity of power maps

A. Arosio showed ([3]) that a commutative B_0-algebra every element of which is bounded is locally m-convex. In the non commutative case, the result is not true as the example of Zelazko ([6]) shows. In the context of l. c. a.'s we have the following

Proposition 4.1. *Let* (A, τ) *be a Baire l. c. a. with a continuous product. If the map* $R_7 : (A, \tau) \to R_+ \cup \{+\infty\}$ *is upper semi-continuous, then the sequence* $(x \mapsto x^n)_n$ *of power maps is equicontinuous at zero. In particular, if* A *is commutative, it is* m*-convex.*

Proof : Let U be an absolutely convex and closed neighbourhood of zero in A and put $B = \{x \in A : x^n \in U : n = 1, 2, ...\}$. Since $x^n \underset{n}{\longrightarrow} 0$, for every $x \in A(R_7)$, where $A(R_7) = \{x \in A : R_7(x) < 1\}$, we have $A(R_7) \subset \bigcup\{pB; p = 1, 2, ...\}$. But $A(R_7)$ being open is also a Baire space. Hence B is with a non void interior. Let $x_0 \in B$ and a balanced neighbourhood V of zero such that $x_0 + V \subset B$. So

$$(x_0 + x)^n \in U; x \in V, n \in N^*.$$

And by Mazur-Orlicz formula,

$$\left(\frac{x}{n}\right)^n = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} C_n^k \left(x_0 + \frac{k}{n} x\right)^n \quad x \in V, n \in N^*.$$

Then

$$\begin{aligned} x^n &\in \frac{n^n}{n!} \sum_{k=0}^{k} C_n^k U \quad \text{for } U \text{ is balanced} \\ &\subset (2e)^n U \qquad x \in V, n \in N^*. \end{aligned}$$

Whence

$$x^n \in U; x \in \frac{1}{2e} V, n \in N^*.$$

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