Solutions to quasilinear equations by an iterative method

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Abstract

We apply an iterative method in order to construct a solution to the mean curvature equation for nonparametric surfaces.

1 Introduction

The prescribed mean curvature equation with Dirichlet condition for a nonparametric surface $X : \overline{\Omega} \longrightarrow \mathbb{R}^3$, $U(x,y) = (x,y,u(x,y))$ is the quasilinear partial differential equation

$$(1) \begin{cases} (1+u_y^2)u_{xx} + (1+u_x^2)u_{yy} - 2u_x u_y u_{xy} = 2h(u)\left(1+|\nabla u|^2\right)^{\frac{3}{2}} & \text{in } \Omega \\ u = g \quad \text{in } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function. This problem and the general parametric case have been studied by several authors, see e.g. [2-5,6,7,9-13].

Received by the editors May 1999.
Communicated by J. Mawhin.

2 Solutions by an iterative method

We'll apply an iterative method inspired in the Newton Imbedding procedure [8].

For this purpose, let us define for each $v \in C^1(\overline{\Omega})$ the bounded linear operator $Q_v : W^{2,p}(\Omega) \to L^p(\Omega)$ given by

$$Q_v u = \frac{1}{2(1+\nabla v^2)^{\frac{3}{2}}}((1+v_y^2)u_{xx} + (1+v_x^2)u_{yy} - 2v_x v_y u_{xy})$$

Remark: $u \in W^{2,p}(\Omega)$ is a solution of (1) if and only if

$$\begin{cases} Q_u u = h(u) \text{ in } \Omega \\ u = g \quad \text{ in } \partial\Omega \end{cases}$$

We'll assume that $h \in C^2(\mathbb{R})$, $h' \geq 0$, $g \in C^{2,\gamma}(\overline{\Omega})$ for $0 < \gamma < 1$, and $\partial\Omega \in C^{2,\gamma}$. The aim of the method is to start with u_0 solution of

$$(2_t) \begin{cases} Q_{u_0} u_0 = t h(u_0) \text{ in } \Omega \\ u_0 = g \quad \text{ in } \partial\Omega \end{cases}$$

and then find a step ε such that a solution of the problem

$$(2_{t+\varepsilon}) \begin{cases} Q_u u = (t+\varepsilon) h(u) \text{ in } \Omega \\ u = g \quad \text{ in } \partial\Omega \end{cases}$$

may be obtained as a limit of a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{2,p}(\Omega)$ for some p such that $\gamma < 1 - \frac{2}{p}$.

Remark: If the curvature of $\partial\Omega$ is positive then (1) is solvable for $h = 0$ [5]. Thus, by this method it's possible to find a sequence $0 = t_0 < t_1 < t_2 < ....$ such that (2_{t_j}) admits a solution for every t_j.

In order to define the sequence $\{u_n\}_{n\in\mathbb{N}}$ we'll use the following results:

Lemma 1

Let $u, u_0 \in C^1(\overline{\Omega})$. Then

$$\|(Q_u - Q_{u_0})v\|_p \leq \sqrt{3}\|u - u_0\|_{1,\infty}\|v\|_{2,p}$$

for any $v \in W^{2,p}(\Omega)$ (i.e. $Q : C^1(\overline{\Omega}) \to \mathcal{L}(W^{2,p}(\Omega), L^p(\Omega))$ is Lipschitz continuous with constant $k \leq \sqrt{3}$).

<u>*Proof*</u>

Let $F_i(a_1, a_2) = \frac{1+a_i^2}{2(1+a_1^2+a_2^2)^{3/2}}$, $G(a_1, a_2) = \frac{a_1 a_2}{(1+a_1^2+a_2^2)^{3/2}}$. By simple computation we obtain:

$$|\frac{\partial F_i}{\partial a_j}| \leq \begin{cases} \frac{2}{3\sqrt{3}} & \text{if } i = j \\ \frac{1}{\sqrt{3}} & \text{if } i \neq j \end{cases}$$

and

$$|\frac{\partial G}{\partial a_j}| \leq \frac{4}{3\sqrt{3}}$$

Thus,

$$\|(Q_u - Q_{u_0})v\|_p =$$
$$\|(F_2(\nabla u) - F_2(\nabla u_0))v_{xx} + (F_1(\nabla u) - F_1(\nabla u_0))v_{yy} - (G(\nabla u) - G(\nabla u_0))v_{xy}\|_p$$
$$= \|\nabla F_2(\xi_1)\nabla(u-u_0)v_{xx} + \nabla F_1(\xi_2)\nabla(u-u_0)v_{yy} - \nabla G(\xi_3)\nabla(u-u_0)v_{xy}\|_p$$

and the result follows.

We recall now the following apriori bound (see e.g. [5], lemma 9.17): let $u \in C^1(\overline{\Omega})$ and $L : W^{2,p}(\Omega) \to L^p(\Omega)$ the operator given by $Lv = Q_u v + \alpha\nabla v + \beta v$, where $\alpha \in L^\infty(\Omega, \mathrm{I\!R}^2)$ and $\beta \in L^\infty(\Omega)$ is nonnegative. Then $L|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)}$ is bounded by below, i.e. there exists a constant $c = c(u)$ such that

$$(3) \qquad \|v\|_{2,p} \le c\|Lv\|_p$$

for any $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. We'll see that c may be choosen uniformly in a neighborhood of any (u, α, β). In other words, if $E = C^1(\overline{\Omega}) \times L^\infty(\Omega, \mathrm{I\!R}^2) \times L^\infty_{\ge 0}(\Omega)$ with the norm $\|(u,\alpha,\beta)\| = max\{\|u\|_{1,\infty}, \|\alpha\|_p, \|\beta\|_p\}$, then:

Lemma 2

Let $c(u,\alpha,\beta)$ be the minimum such that (3) holds. Then $c : E \to \mathrm{I\!R}$ is upper semicontinuous.

<u>*Proof*</u>

Let $(u_0,\alpha_0,\beta_0), (u,\alpha,\beta) \in E$ and $t > c(u_0,\alpha_0,\beta_0)$. Then, for $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$,

$$\|L_{(u,\alpha,\beta)}v\|_p \ge \|L_{(u_0,\alpha_0,\beta_0)}v\|_p - \|(Q_u - Q_{u_0})v\|_p - \|(\alpha-\alpha_0)\nabla v\|_p - \|(\beta-\beta_0)v\|_p \ge$$

$$\frac{1}{c(u_0)}\|v\|_{2,p} - \sqrt{3}\|u-u_0\|_{1,\infty}\|v\|_{2,p} - c_1\|\alpha-\alpha_0\|_p\|v\|_{2,p} - c_0\|\beta-\beta_0\|_p\|v\|_{2,p}$$

where c_1 and c_0 are the constants of the imbeddings of $W^{2,p}(\Omega)$ in $C^1(\overline{\Omega})$ and $C(\overline{\Omega})$ respectively (see e.g. [1] or [5]).

Hence, for $\|u-u_0\|_{1,\infty} + c_1\|\alpha-\alpha_0\|_p + c_0\|\beta-\beta_0\|_p \le \frac{1}{\sqrt{3}c(u_0)}$ small enough,

$$\frac{1}{t} < \frac{1}{c(u_0)} - \sqrt{3}\|u-u_0\|_{1,\infty} - c_1\|\alpha-\alpha_0\|_p - c_0\|\beta-\beta_0\|_p = \frac{1}{c} \le \frac{1}{c(u)}$$

and the result holds.

Let $u_0 \in W^{2,p}(\Omega)$ be a solution of (2_{t_0}) for some t_0. We define recursively the sequence $\{u_n\}_{n\in\mathrm{I\!N}}$, where u_{n+1} is the solution of the quasilinear problem

$$(4) \begin{cases} Q_{u_{n+1}}u_{n+1} = (t_0+\varepsilon)(h'(u_n)(u_{n+1}-u_n) + h(u_n)) & \text{in } \Omega \\ u_{n+1} = g \quad \text{in } \partial\Omega \end{cases}$$

In order to prove that the sequence is well defined for ε small enough, we'll state the following regularity result, which shows that $u_n \in C^{2,\gamma}(\overline{\Omega})$ for every n:

Lemma 3
Let $u \in W^{2,p}(\Omega)$ be a solution of

$$\begin{cases} Q_u u = F(x,y,u) & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases}$$

where $F \in C^\gamma(\overline{\Omega} \times \mathbb{R})$. Then $u \in C^{2,\gamma}(\overline{\Omega})$.
<u>Proof</u>
As $W^{2,p}(\Omega) \hookrightarrow C^{1,\gamma}(\overline{\Omega})$, the problem

$$\begin{cases} Q_u z = F(x,y,u) & \text{in } \Omega \\ z = g & \text{in } \partial\Omega \end{cases}$$

admits a unique solution $z \in C^{2,\gamma}(\overline{\Omega})$, and by the uniqueness in $W^{2,p}(\Omega)$ we conclude that $z = u$.
Theorem 4
There exists $\varepsilon > 0$ such that $\{u_n\}_{n\in\mathbb{N}}$ is well defined, and converges in $W^{2,p}(\Omega)$ to a solution of $(2_{t_0+\varepsilon})$.
<u>Proof</u>
Let us first note that for fixed $v \in B_R(u_0) \subset W^{2,p}(\Omega)$ and $u \in W^{2,p}(\Omega)$, we have:

$$Q_u u - Q_v v = Q_u(u-v) + (DF_2(\nabla v)v_{xx} + DF_1(\nabla v)v_{yy} - DG(\nabla v)v_{xy})\,\nabla(u-v) + r(\nabla u)$$

where the remainder r satisfies:

$$\|r(\nabla u)\|_p \leq \overline{c}\|\nabla(u-v)\|_\infty^2$$

for some constant $\overline{c}$ independent of u and v. Moreover, if $\xi \in L^\infty(\Omega, \mathbb{R}^2)$ is a mean value between ∇u and ∇v, and $L_{v,\xi,u}$ the linear operator given by

$$L_{v,\xi,u} w = Q_u w + (DF_2(\xi)v_{xx} + DF_1(\xi)v_{yy} - DG(\xi)v_{xy})\,\nabla w - (t_0+\varepsilon)h'(v)w$$

then by lemma 2 there exist constants c, R such that if $v \in C^2(\overline{\Omega})$, $\|v - u_0\|_{2,p} \leq R$ and $\|u - u_0\|_{1,\infty} \leq c_1 R$, then

$$\|w\|_{2,p} \leq c\|L_{v,\xi,u} w\|_p$$

for every $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Choosing R and ε small enough, we'll see that (4) is uniquely solvable. Indeed, uniqueness follows from the assumption $h' \geq 0$ (using for example [5], theorem 10.2), and existence may be proved by fixed point methods in the following way: for u_1, writing $z = u_1 - u_0$ and $L_z = L_{u_0,\nabla u_0, z+u_0}$, problem (4) is equivalent to

$$\begin{cases} L_z z = \varepsilon h(u_0) + r(\nabla(z)) \text{ in } \Omega \\ z = 0 \quad \text{in } \partial\Omega \end{cases}$$

Let $T : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ be the continuous operator defined by $Tz = w$, where $w \in W^{2,p}(\Omega)$ is the unique solution of the linear problem

$$\begin{cases} L_z w = \varepsilon h(u_0) + r(\nabla(z)) \text{ in } \Omega \\ w = 0 \quad \text{in } \partial\Omega \end{cases}$$

Then for $\|z\|_{1,\infty} \leq \overline{R} \leq c_0 R$ and a compact set K containing a neighborhood of $u_0(\overline{\Omega})$ we have:

$$\|Tz\|_{2,p} \leq \|L_z(Tz)\|_p = c\|\varepsilon h(u_0) + r(\nabla z)\|_p \leq c(\varepsilon\|h\|_{\infty,K} + \overline{c}\overline{R}^2)$$

and by the compactness of the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ we conclude that the closure of $T(\{\|z\|_{1,\infty} \leq \overline{R}\})$ is compact. Furthermore,

$$\|Tz\|_{1,\infty} \leq c_0 c(\varepsilon\|h\|_{\infty,K} + \overline{c}\overline{R}^2) \leq \overline{R}$$

if ε and $\overline{R}$ are small enough. By Schauder theorem, we conclude that T has a fixed point z, and then $u_1 = z + u_0$ is a solution of (4).

Let us assume that the sequence is well defined up to u_{n+1}. Then, for $n > 0$

$$Q_{u_{n+1}} u_{n+1} - Q_{u_n} u_n - (t_0+\varepsilon)h'(u_n)(u_{n+1} - u_n) =$$

$$(t_0+\varepsilon)[h(u_n) - h_(u_{n-1}) - h'(u_{n-1})(u_n - u_{n-1})] = (t_0+\varepsilon)\frac{h''(s)}{2}(u_n - u_{n-1})^2$$

for some mean value $s \in L^\infty(\Omega)$.

Moreover, if $u_j \in B_R(u_0) \subset W^{2,p}(\Omega)$ for $j = 1, ..., n+1$ then

$$\|u_{n+1} - u_n\|_{2,p} \leq c\|Q_{u_{n+1}} u_{n+1} - Q_{u_n} u_n - (t_0+\varepsilon)h'(u_n)(u_{n+1} - u_n)\|_p,$$

and we conclude that

$$\|u_{n+1} - u_n\|_{2,p} \leq c\frac{(t_0+\varepsilon)}{2}\|h''\|_{\infty,K}\|u_n - u_{n-1}\|_p\|u_n - u_{n-1}\|_\infty \leq$$

$$\leq \frac{cc_0}{2}(t_0+\varepsilon)\|h''\|_{\infty,K}\|u_n - u_{n-1}\|^2_{2,p}$$

for $n > 0$.

Thus, by induction

$$\|u_{n+1} - u_n\|_{2,p} \leq (\frac{cc_0}{2}(t_0+\varepsilon)\|h''\|_{\infty,K}\|u_1 - u_0\|_{2,p})^{2^n-1}\|u_1 - u_0\|_{2,p}$$

and as

$$\|u_1 - u_0\|_{2,p} \leq c\varepsilon\|h(u_0)\|_p,$$

if ε satisfies

$$c(\varepsilon) = \frac{c^2 c_0}{2}(t_0+\varepsilon)\|h''\|_{\infty,K}\varepsilon\|h(u_0)\|_p < 1$$

then

$$\|u_{n+1} - u_0\|_{2,p} \leq \sum_{0\leq j\leq n}\|u_{j+1} - u_j\|_{2,p} \leq \frac{c\varepsilon\|h(u_0)\|_p}{1 - c(\varepsilon)}.$$

Choosing ε small, $\|u_1 - u_0\|_{2,p} \leq R$, and then we may assume as inductive hypothesis that the sequence is well defined up to u_n and that $u_j \in B_R(u_0)$. As before, if $z = u_{n+1} - u_n$, problem (4) is equivalent to

$$\begin{cases} L_z z = (t_0+\varepsilon)\frac{h''(s)}{2}(u_n - u_{n-1})^2 + r(\nabla(z)) \text{ in } \Omega \\ z = 0 \quad \text{in } \partial\Omega \end{cases}$$

where $L_z := L_{u_n,\nabla u_n,z+u_n}$ and defining an operator $T : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ we obtain for $\|z\|_{1,\infty} \leq \overline{R} \leq c_0 R$:

$$\|Tz\|_{1,\infty} \leq c_0 c \left(c(\varepsilon)^{2^n-1} \varepsilon \|h(u_0)\|_p + \overline{c}\overline{R}^2 \right)$$

Then, it suffices to consider for example $\varepsilon \leq (c_0 R)^2$ such that $c(\varepsilon) << 1$ and

$$c_0 c(\|h(u_0)\|_p + \overline{c})\sqrt{\varepsilon} \leq 1,$$

since in that case taking $\overline{R} = \sqrt{\varepsilon}$ we obtain $\|Tz\|_{1,\infty} \leq \overline{R}$, and the existence of u_{n+1} can be deduced from Schauder theorem.

Furthermore, as $\|u_{n+1} - u_n\|_{2,p} \leq c(\varepsilon)^{2^n-1}\|u_1 - u_0\|_{2,p}$, $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{2,p}(\Omega)$, and the proof is complete.

Remark:

A sequence $\{u_n\}_{n\in\mathbb{N}}$ may be also defined recursively by the *linear* problems

$$\begin{cases} Q_{u_n} u_{n+1} = (t_0 + \varepsilon)(h'(u_n)(u_{n+1} - u_n) + h(u_n)) & \text{in } \Omega \\ u_{n+1} = g \quad \text{in } \partial\Omega \end{cases}$$

In this case, convergence can be guaranteed for ε small enough if $\|u_0\|_{2,p}$ is small.

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