Finite p-groups with few normal subgroups

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Abstract

This paper investigates the finite nonabelian p-groups G with the property that every normal subgroup of G either contains the commutator subgroup G', or is contained in the center of G. We will prove that the nilpotency class of G is 2 or 3 and we will find all such groups with nilpotency class 3.

1 Introduction

Let G be a finite group. Let us denote by $\mathcal{S}(G)$ the set of the subgroups of G, by $\mathcal{N}(G)$ the set of the normal subgroups of G, by $(H]$ the ideal generated by H in the lattice $(\mathcal{S}(G), \subseteq)$, and by $[H)$ the filter generated by H in the same lattice. Then

$$(Z(G)] \cup [G') \subseteq \mathcal{N}(G) \subseteq \mathcal{S}(G). \tag{1}$$

If the group G is abelian, then in (1) we have equalities.

If G is a finite nonabelian group, then the right inclusion becomes an equality iff G is one of the well-known Dedekind groups.

A natural problem is the search of the finite nonabelian groups which realize the equality in the left inequality from (1). These groups have as few normal subgroups as possible. Unfortunately, the family of these groups is too big. (It contains, for example, all the finite simple groups.) Hence, we decided to restrict ourselves to the case of the finite p-groups.

All the groups in discussion will be finite.

All the notation is standard.

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2 FNS-p-groups and AF-p-groups

Definition 2.1. *A nonabelian p-group G is called FNS-group ("Few Normal Subgroups") if*

$$H \trianglelefteq G \Leftrightarrow H \geq G' \text{ or } H \leq Z(G) \quad \forall H \in \mathcal{S}(G). \tag{2}$$

First we will identify some particular families of FNS-groups.

Proposition 2.1. *Let p be a prime and let G be a nonabelian group of order p^3 or p^4. Then G is a FNS-group.*

Proof Let H be a normal subgroup of G.
If $|H| \leq p$, then $H \leq Z(G)$.
If $|H| \geq p^2$, then $|G/H| \leq p^2$, so G/H is abelian and $H \geq G'$. ∎

A much more interesting family of FNS-groups is the family of the AF-p-groups.

Definition 2.2. *A nonabelian p-group G is called AF-group ("Abelian Factors") if every proper quotient of G is abelian.*

Proposition 2.2. *Every AF-p-group is a FNS-p-group.*

Proof Let G be an AF-p-group and let H be a proper normal subgroup of G. Then G/H is abelian, hence $H \geq G'$. ∎

The structure of the AF-p-groups is established by the following result.

Theorem 2.1. *Let p be a prime and let G be a nonabelian group with $|G| = p^n,\ |Z(G)| = p^m,\ 1 \leq m \leq n-2$.*
Then G is a AF-group iff $n-m$ is even and

$$a)\ G \cong \underbrace{Q_8 \curlyvee Q_8 \curlyvee \ldots \curlyvee Q_8}_{\frac{n-m}{2}} \text{ or } G \cong \underbrace{Q_8 \curlyvee Q_8 \curlyvee \ldots \curlyvee Q_8}_{\frac{n-m}{2}-1} \curlyvee D_8,\ \text{if } p^m = 2;$$

$$b)\ G \cong \underbrace{K_{p^{m+2}} \curlyvee K_{p^{m+2}} \curlyvee \ldots \curlyvee K_{p^{m+2}}}_{\frac{n-m}{2}} \text{ or } G \cong \underbrace{L_{p^{m+2}} \curlyvee L_{p^{m+2}} \curlyvee \ldots \curlyvee L_{p^{m+2}}}_{\frac{n-m}{2}},\ \text{if } p^m \neq 2.$$

(Here $K_{p^{m+2}} = \langle a, b | a^{p^{m+1}} = b^p = 1, [a,b] = a^{p^m}\rangle$ and
$L_{p^{m+2}} = \langle a, b, c | a^{p^m} = b^p = c^p = [a,b] = [a,c] = 1, [b,c] = a^{p^{m-1}}\rangle$.
By $G \curlyvee H$ we have denoted the central product of the finite groups G and H, if these groups have cyclic isomorphic centers.)

Proof This is Theorem 3.1 in [7]. ∎

The preceeding examples hint that the family of the FNS-p-groups is large enough and that the members of this family have few properties in common. One of these common properties tells that the nilpotency class of a FNS-p-group is lower than 4.

Proposition 2.3. *Let G be a nonabelian FNS-p-group. Then the nilpotency class of G is 2 or 3.*

Proof Assume that the class of G is greater than 2. Then $Z(G) < Z_2(G) \triangleleft G$. It results $G' \leq Z_2(G)$, hence $G/Z_2(G)$ is abelian and G has class 3. ∎

In the rest of this article we will classify all the FNS-p-groups of class 3 and we start this task with some necessary conditions for a p-group of class 3 to be a FNS-group.

Proposition 2.4. *Let G be a FNS-p-group of class 3. Then $Z(G) < G'$.*

Proof Because G has class 3 we may choose an element x from $G \setminus Z(G)$ such that $x \cdot (G' \cap Z(G))$ has order p in the nontrivial group $G'/(G' \cap Z(G))$. Since $[G', G] \leq G' \cap Z(G)$, we derive that $x \cdot (G' \cap Z(G)) \in Z(G/(G' \cap Z(G)))$.

Let us assume that the conclusion of Proposition 2.4 is false. One may then choose an element y from $Z(G) \setminus G'$ such that $y \cdot (G' \cap Z(G))$ has order p in the group $Z(G)/(G' \cap Z(G))$. It results $y \cdot (G' \cap Z(G)) \in Z(G/(G' \cap Z(G)))$.

Now $(xy) \cdot (G' \cap Z(G))$ is a central element of order p in $G/(G' \cap Z(G))$, hence

$$\langle (xy) \cdot (G' \cap Z(G)) \rangle \trianglelefteq G/(G' \cap Z(G)).$$

We will denote by H the subgroup of G generated by the element xy and the subgroup $G' \cap Z(G)$. Obviously we have $H \trianglelefteq G$ and $|H : (G' \cap Z(G))| = p$. But G is a FNS-group and H is not contained in $Z(G)$, hence $G' \leq H$. One gets

$$1 < G'/(G' \cap Z(G)) \leq H/(G' \cap Z(G)),$$

the last group having order p. It results $G' = H$ and $xy \in G'$, a contradiction. ■

The following result establishes a link between the FNS-groups of nilpotency class 3 and the AF-groups.

Proposition 2.5. *Let G be a FNS-group of nilpotency class 3. Then $G/Z(G)$ is an AF-group.*

Proof Let $\bar{G} = G/Z(G)$ and let $\bar{H}$ be a proper normal subgroup of $\bar{G}$. There exists a proper normal subgroup H of G such that $Z(G) < H$ and $H/Z(G) = \bar{H}$.

One must have $G' \leq H$, so G/H is abelian. But

$$\bar{G}/\bar{H} = (G/Z(G))/(H/Z(G)) \cong G/H.$$

In conclusion, $\bar{G}$ is an AF-group. ■

Corollary 2.1. *If G is a FNS-p-group of class 3, then $|G' : Z(G)| = p$ and $Z_2(G)/Z(G)$ is a cyclic group.*

Proof Let $\bar{G} = G/Z(G)$. Then $\bar{G}$ is an AF-p-group and Lemma 2.1 from [7] tells that such a group has cyclic center, and its commutator subgroup has order p.

But $\bar{G}' = G'/Z(G)$ and $Z(\bar{G}) = Z_2(G)/Z(G)$. ■

Based on Proposition 2.5, we need to study which AF-p-group may be written in the form $G/Z(G)$.

Definition 2.3. *A group H is called* <u>*capable*</u> *if there exists a group G such that $H \cong G/Z(G)$.*

One possesses a very useful necessary condition for a group being capable.

Lemma 2.1. *Let H be a capable group and let $\mathcal{G}$ be a generating system for H. Then*

$$\bigcap \{\langle g \rangle | g \in \mathcal{G}\} = 1.$$

Proof It follows from [2], Lemma 3.1. ■

We may now determine all the capable AF-p-groups.

Theorem 2.2. *The only capable AF-p-groups are:*
a) D_8 *if* $p = 2$;
b) L_{p^3} *if* $p \neq 2$.

Proof Let H be an AF-p-group, that is one of the groups from Theorem 2.1.

Let us assume that H is the central product of r copies ($r \geq 1$) of the group Q_8. Then

$$H = \langle a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r | a_1^4 = 1, a_i^2 = b_i^2 = a_1^2,$$
$$[a_i, a_j] = [b_i, b_j] = 1, [a_i, b_j] = a_1^{2\delta_{ij}} \rangle,$$

where $i, j \in \{1, 2, \ldots, r\}$. Obviously we have

$$\langle a_1 \rangle \cap \langle a_2 \rangle \cap \ldots \cap \langle a_r \rangle \cap \langle b_1 \rangle \cap \langle b_2 \rangle \cap \ldots \cap \langle b_r \rangle = \langle a_1^2 \rangle \neq 1,$$

and consequently H is not capable for any value $r \geq 1$.

Let H be the central product of r copies ($r \geq 1$) of the group Q_8 with one copy of the group D_8. Then

$$H = \langle a_1, \ldots, a_r, b_1, \ldots, b_r, c, d | a_1^4 = d^2 = 1, a_i^2 = b_i^2 = c^2 = a_1^2,$$
$$[a_i, a_j] = [b_i, b_j] = [a_i, c] = [a_i, d] = [b_i, c] = [b_i, d] = 1, [a_i, b_j] = a_1^{2\delta_{ij}}, [c, d] = a_1^2 \rangle,$$

where $i, j \in \{1, 2, \ldots, r\}$. One may choose for H the generating system

$$\mathcal{G} = \{a_1, \ldots, a_r, b_1, \ldots, b_r, c, a_1 d\}$$

and one gets

$$\bigcap \{\langle g \rangle | g \in \mathcal{G}\} = \langle a_1^2 \rangle \neq 1,$$

Hence H is not capable for any $r \geq 1$.

Observing that for every group G of order 16 and of class 3 we have $G/Z(G) \cong D_8$, it results that D_8 is the only capable AF-p-group with the center of order 2.

In the rest of the proof p will be an arbitrary prime.

Let H be the central product of r copies ($r \geq 1$) of the group K_{p^n} ($n \geq 3$). Then

$$H = \langle a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r | a_1^{p^{n-1}} = b_i^p = 1, a_i^p = a_1^p,$$
$$[a_i, a_j] = [b_i, b_j] = 1, [a_i, b_j] = a_1^{\delta_{ij} p^{n-2}} \rangle,$$

where $i, j \in \{1, 2, \ldots, r\}$. Choosing for H the generating system

$$\mathcal{G} = \{a_1, a_2, \ldots, a_r, a_1 b_1, a_2 b_2, \ldots, a_r b_r\}$$

we get

$$\bigcap \{\langle g \rangle | g \in \mathcal{G}\} = \langle a_1^p \rangle \neq 1,$$

hence H is not capable for any $r \geq 1$.

Finally, let H be the central product of r copies ($r \geq 1$) of the group L_{p^n} ($n \geq 3$). Then

$$H = \langle a, b_1, b_2, \ldots, b_r, c_1, c_2, \ldots, c_r | a^{p^{n-2}} = b_i^p = c_i^p = 1,$$

$$[a, b_i] = [a, c_i] = [b_i, b_j] = [c_i, c_j] = 1, [b_i, c_j] = a^{\delta_{ij} p^{n-3}}\rangle,$$

where $i, j \in \{1, 2, \ldots, r\}$. One chooses for H the generating system

$$\mathcal{G} = \{a, ab_1, ab_2, \ldots, ab_r, ac_1, ac_2, \ldots, ac_r\}$$

and one gets

$$\bigcap\{\langle g\rangle | g \in \mathcal{G}\} = \langle a^p\rangle.$$

If $n \geq 4$, then $a^p \neq 1$, hence H is not capable.

If $n = 3$, then $p \neq 2$. Let us assume that there is a group G such that $G/Z(G) \cong H$. Then G is generated by the set

$$\{\alpha, \beta_1, \beta_2, \ldots, \beta_r, \gamma_1, \gamma_2, \ldots, \gamma_r\} \cup Z(G),$$

where $\alpha, \beta_i, \gamma_j$ are preimages in G of the generators a, b_i, c_j from H. For every $i, j \in \{1, 2, \ldots, r\}$ the following relations take place:

$$\alpha^p, \beta_i^p, \gamma_i^p \in Z(G);\ \alpha, \beta_i, \gamma_i \notin Z(G);\ [\alpha, \beta_i], [\alpha, \gamma_i], [\beta_i, \beta_j], [\gamma_i, \gamma_j] \in Z(G);$$

$$[\beta_i, \gamma_i] \in \alpha \cdot Z(G);\ [\beta_i, \gamma_j] \in Z(G) \text{ if } i \neq j.$$

If $r > 1$, for $i \in \{2, 3, \ldots, r\}$ we apply to the elements $\beta_1, \gamma_1^{-1}, \beta_i$ the Witt identity ([3], Proposition 1.4, p. 254) and we get

$$[\beta_1, \gamma_1, \beta_i]^{\gamma_1^{-1}} \cdot [\gamma_1^{-1}, \beta_i^{-1}, \beta_1]^{\beta_i} \cdot [\beta_i, \beta_1^{-1}, \gamma_1^{-1}]^{\beta_1} = 1. \tag{3}$$

The images in H of the elements $[\gamma_1^{-1}, \beta_i^{-1}]$ and $[\beta_i, \beta_1^{-1}]$ are $[c_1^{-1}, b_i^{-1}]$ and respectively $[b_i, b_1^{-1}]$, which are equal to the unity in the group H, hence

$$[\gamma_1^{-1}, \beta_i^{-1}],\ [\beta_i, \beta_1^{-1}] \in Z(G).$$

The relation (3) becomes successively

$$[\beta_1, \gamma_1, \beta_i]^{\gamma_1^{-1}} = 1,$$

$$[\alpha, \beta_i]^{\gamma_1^{-1}} = 1,$$

$$[\alpha, \beta_i] = 1.$$

In a similar way we get $[\alpha, \beta_1] = 1$ and also $[\alpha, \gamma_i] = 1$ for every $i \in \{1, 2, \ldots, r\}$. All these give the contradiction $\alpha \in Z(G)$.

Hence, H is not capable if $r > 1$.

The group L_{p^3} is capable because for every group G of order p^4 and class 3 we have $G/Z(G) \cong L_{p^3}$. ∎

3 The classification of the FNS-p-groups of class 3

In this section we will determine all the FNS-p-groups of nilpotency class 3, using the results from the previous section.

More precisely, we need to solve the "equation" $G/Z(G) \cong D_8$ in the family of the FNS-2-groups of class 3 and the "equation" $G/Z(G) \cong L_{p^3}$ in the family of the FNS-p-groups of class 3 for $p \neq 2$.

Theorem 3.1. *Let G be a FNS-2-group of class 3. Then G is isomorphic to one of the following groups:*
a) $D_{16} = \langle a, b | a^8 = b^2 = 1, [a, b] = a^6 \rangle$,
b) $SD_{16} = \langle a, b | a^8 = b^2 = 1, [a, b] = a^2 \rangle$,
c) $Q_{16} = \langle a^8 = 1, b^2 = a^4, [a, b] = a^6 \rangle$.

Proof We have already proved that

$$G/Z(G) = \langle a, b | a^4 = b^2 = 1, [a, b] = a^2 \rangle \cong D_8.$$

Denoting by α and β the preimages in G of the elements a and b, one obtains the relations:

$$G = \langle \alpha, \beta, Z(G) \rangle;\ \alpha^4, \beta^2 \in Z(G);\ \alpha^2, \beta \notin Z(G);\ [\alpha, \beta] \in \alpha^2 \cdot Z(G).$$

The last relation tells that α and $[\alpha, \beta]$ commute. Then $[\alpha, \beta]^n = [\alpha^n, \beta]$ for every $n \in \mathbf{N}$. In particular, $[\alpha, \beta]^2 = [\alpha^2, \beta] \neq 1$ (otherwise $\alpha^2 \in Z(G)$) and $[\alpha, \beta]^4 = [\alpha^4, \beta] = 1$. Hence the element $[\alpha, \beta]$ has order 4 in G. Let H be the subgroup generated in G by this element. Obviously $H \leq G'$. We have

$$\alpha^{-1}[\alpha, \beta]\alpha = [\alpha, \beta] \cdot [\alpha, \beta, \alpha] = [\alpha, \beta] \cdot [\alpha^2, \alpha] = [\alpha, \beta],$$

$$\beta^{-1}[\alpha, \beta]\beta = [\alpha, \beta] \cdot [\alpha, \beta, \beta] = [\alpha, \beta] \cdot [\alpha^2, \beta] = [\alpha, \beta]^3,$$

hence H is a normal subgroup of G. Moreover, G/H is abelian because its generators commute. Finally, $G' = H \cong \mathbf{Z}_4$. From Corollary 2.1 it results that $|Z(G)| = 2$ and so $|G| = 16$. G must be a group of order 16 and of class 3, but the only groups with these properties are the groups D_{16}, SD_{16} and Q_{16} ([1], Theorem 4.5, p. 194).

On the other hand, these three groups are FNS-groups (Proposition 2.1) and so the proof is complete. ■

In the case of the "equation" $G/Z(G) \cong L_{p^3}$ the orders of the solutions are also bounded.

Proposition 3.1. *Let G be a FNS-p-group ($p \neq 2$) of class 3. Then the order of G does not exceed p^5 and the commutator subgroup of G is elementary abelian.*

Proof We know that

$$G/Z(G) = \langle a, b, c | a^p = b^p = c^p = [a, c] = [b, c] = 1, [a, b] = c \rangle \cong L_{p^3}.$$

Let us denote by α, β, γ the preimages in G of the elements a, b, c. Then

$$G = \langle \alpha, \beta, \gamma, Z(G) \rangle;\ \alpha^p, \beta^p, \gamma^p \in Z(G);\ \alpha, \beta, \gamma \notin Z(G);$$

$$[\alpha,\gamma],[\beta,\gamma]\in Z(G);\ [\alpha,\beta]\in\gamma\cdot Z(G).$$

We get $[\alpha,\gamma]^p=[\alpha^p,\gamma]=1$ and $[\beta,\gamma]^p=[\beta^p,\gamma]=1$.
By induction one may prove that

$$[\alpha,\beta]^n=[\alpha,\beta^n]\cdot[\beta,\gamma]^{\frac{n(n-1)}{2}}\text{ for every }n\in\mathbf{N},$$

hence $[\alpha,\beta]^p=1$.

It follows that the subgroup

$$H=\langle[\alpha,\beta];[\alpha,\gamma];[\beta,\gamma]\rangle$$

of G is elementary abelian and its order does not exceed p^3. Moreover,

$$[\alpha,\gamma];\ [\beta,\gamma]\in Z(G),$$

$$\alpha^{-1}[\alpha,\beta]\alpha=[\alpha,\beta]\cdot[\alpha,\beta,\alpha]=[\alpha,\beta]\cdot[\alpha,\gamma]^{-1}\in H,$$

$$\beta^{-1}[\alpha,\beta]\beta=[\alpha,\beta]\cdot[\alpha,\beta,\beta]=[\alpha,\beta]\cdot[\beta,\gamma]^{-1}\in H,$$

$$\gamma^{-1}[\alpha,\beta]\gamma=[\alpha,\beta]\cdot[\alpha,\beta,\gamma]=[\alpha,\beta]\in H,$$

Hence H is a normal subgroup of G. G/H is obviously an abelian group and therefore we may conclude that $H=G'$.

One deduces $|Z(G)|\le p^2$ and $|G|\le p^5$. ■

In fact, a FNS-p-group of class 3 has at least the order p^4, hence its order is p^4 or p^5.

Proposition 3.2. *For every prime $p\neq 2$ all the four groups of order p^4 and class 3 are FNS-groups.*

Proof It follows from Proposition 2.1. ■

The case of the FNS-groups of order p^5 and class 3 is more complicated. We know already that the commutator subgroup of such a group is elementary abelian of order p^3 and that the center of the group has order p^2 and is contained in the commutator subgroup. These conditions are not sufficient and we will improve them in the following.

Theorem 3.2. *Let p be an odd prime and let G be a group of order p^5 and of class 3, with the properties*

$$Z(G)<G',\ G'\cong\mathbf{Z}_p\times\mathbf{Z}_p\times\mathbf{Z}_p,\ Z(G)\cong\mathbf{Z}_p\times\mathbf{Z}_p.$$

The following statements are equivalent:
a) G is a FNS-group.
b) $[G,G']=Z(G)$.
c) $C_G(G')=G'$.

Proof First we observe that from the hypothesis we get the relations

$$1 < [G, G'] \leq Z(G),\ G' \leq C_G(G') < G,\ G' = Z_2(G).$$

Let us assume that the a) assertion is true and let $x \in G' \setminus Z(G)$. If H is the subgroup generated in G by x and by $[G, G']$, then $H \trianglelefteq G$, but $H \not\leq Z(G)$. Moreover, $|H : [G, G']| = p$. G being FNS-group, one must have $G' \leq H$. Then $[G, G'] < G' \leq H$ and $|H : [G, G']| = p$. These relations give $H = G'$ and $|[G, G']| = p^2 = |Z(G)|$, hence $[G, G'] = Z(G)$.

If the b) assertion is true, then let H be a normal subgroup of G and let

$$N = (H \cap G') \cdot Z(G).$$

Obviously $Z(G) \leq N \leq G'$. Only the cases $N = Z(G)$ or $N = G'$ are possible.
If $N = Z(G)$, then $H \cap G' \leq Z(G)$, hence $[H, G] \leq Z(G)$, $H \leq Z_2(G) = G'$ and so $H \leq Z(G)$.
If $N = G'$, then

$$Z(G) = [G, G'] = [G, N] = [G, (H \cap G') \cdot Z(G)] = [G, H \cap G'] \leq H \cap G',$$

hence $G' = H \cap G'$, and so $G' \leq H$.
In conclusion, G is a FNS-group.

We proved that the first two statements are equivalent and now we will prove the same thing for the last two statements.

Let x be a fixed element from $G' \setminus Z(G)$. The function

$$\varphi : G \to G \quad \varphi(y) = [x, y]\ \forall y \in G$$

is an endomorphism of G whose kernell is $C_G(x) = C_G(G')$ and whose image is $[G, G']$. Hence

$$|C_G(G')| \cdot |[G, G']| = p^5$$

and

$$[G, G'] = Z(G) \Leftrightarrow |[G, G']| = p^2 \Leftrightarrow |C_G(G')| = p^3 \Leftrightarrow C_G(G') = G'.$$

■

Finally we will give another characterization for the FNS-groups of order p^5 and of class 3, based on a method of M. F. Newman.

Definition 3.1. *Let G be a p-group. The series of subgroups*

$$G = P_0(G) \geq P_1(G) \geq P_2(G) \geq \ldots$$

is called the p-central descending series of G if

$$P_{i+1}(G) = [G, P_i(G)] \cdot (P_i(G))^p \quad \textit{for every } i \in \mathbf{N}.$$

Proposition 3.3. *Let G be a p-group and let*

$$G = P_0(G) \geq P_1(G) \geq P_2(G) \geq \ldots$$

be the p-central descending series of G. Then
a) For every $i \in \mathbf{N}$: $P_i(G)$ is a fully invariant subgroup of G.
b) $P_1(G) = \Phi(G)$.
c) $\exists i \in \mathbf{N}$ such that $P_i(G) = 1$.
d) For every $i \in \mathbf{N}$: $P_i(G)/P_{i+1}(G)$ is a elementary abelian p-group (possibly trivial).

Proof All the sentences follow from [5]. ■

Consequently, every p-group G possesses a p-central descending series of finite length

$$G = P_0(G) > P_1(G) > \ldots > P_c(G) > P_{c+1}(G) = 1,$$

the length of these series being greater or equal to the length of the central descending series of G, that is to the nilpotency class of G.

The construction of M. F. Newman, presented in [6], is the following: For every prime p is defined an oriented graph $\mathcal{K}_p$, its vertices being all the finite p-groups (modulo isomorphisms) and (G, H) being an edge iff $G \cong H/P_c(H)$, where $P_c(H)$ is the last nontrivial member of the p-central descending series of H. In this case H is called a direct descendant of G.

The link between the graphs $\mathcal{K}_p$ and the FNS-groups is established by the following result.

Theorem 3.3. *Let p be an odd prime and let G be a group of order p^5. The following sentences are equivalent:*
a) G is a FNS-group of class 3.
b) G is a direct descendant of the group L_{p^3} in the graph $\mathcal{K}_p$.

Proof Let us assume that G is a FNS-group of class 3. From

$$G' \leq \Phi(G) < G, \quad |G : G'| = p^2, \quad |G : \Phi(G)| \geq p^2$$

it results that

$$\Phi(G) = G' \cong \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p.$$

We will compute now, using Theorem 3.2, the p-central descendent series of G.

$$P_1(G) = \Phi(G) = G';$$

$$P_2(G) = [G, P_1(G)] \cdot (P_1(G))^p = [G, G'] \cdot (G')^p = Z(G);$$

$$P_3(G) = [G, P_2(G)] \cdot (P_2(G))^p = [G, Z(G)] \cdot (Z(G))^p = 1.$$

Hence the p-central descendent series of G is

$$G > G' > Z(G) > 1,$$

$c = 2$ and $G/P_c(G) = G/Z(G) \cong L_{p^3}$.

Let us assume now that G is a direct descendant of the group L_{p^3} in $\mathcal{K}_p$. Then G has a p-central descendent series

$$G = P_0(G) > P_1(G) > P_2(G) > \ldots > P_c(G) > P_{c+1}(G) = 1$$

with $G/P_c(G) \cong L_{p^3}$. It follows immediately that

$$|P_c(G)| = p^2, \quad G^p \leq P_c(G), \quad G' \not\leq P_c(G), \quad c \geq 2.$$

From $P_{c+1}(G) = 1$ one derives that $[G, P_c(G)] \cdot (P_c(G))^p = 1$, hence $P_c(G)$ is an elementary abelian subgroup of order p^2, contained in $Z(G)$.
If $P_c(G) < Z(G)$, we get

$$|Z(G)| = p^3, \quad |G'| = p, \quad G/Z(G) \cong \mathbf{Z}_p \times \mathbf{Z}_p,$$

hence $G' \leq \Phi(G) \leq Z(G)$. But $\Phi(G) = P_1(G) > P_c(G)$. It follows

$$Z(G) = \Phi(G) = P_1(G),$$

$$P_c(G) \leq P_2(G) = [G, P_1(G)] \cdot (P_1(G))^p = [G, Z(G)] \cdot (Z(G))^p = (Z(G))^p$$

and therefore $|Z(G) : (Z(G))^p| \leq p$ and so $Z(G)$ is cyclic. Lemma 2.1 from [7] tells that every p-group with cyclic center and with commutator subgroup of order p is an AF-group. Hence G is an AF-group, a contradiction.
Consequently, we showed that

$$Z(G) = P_c(G) \cong \mathbf{Z}_p \times \mathbf{Z}_p, \quad G/Z(G) \cong L_{p^3}$$

and the nilpotency class of G is higher than 2.
Now $Z(G) = P_c(G) = [G, P_{c-1}(G)] \cdot (P_{c-1}(G))^p$, which implies $[G, P_{c-1}(G)] \leq Z(G)$, hence $P_{c-1}(G) \leq Z_2(G)$. Therefore we get

$$Z(G) = P_c(G) < P_{c-1}(G) \leq Z_2(G),$$

which leads, together with the relations $|Z(G)| = p^2$, $|Z_2(G)| \leq p^3$, to

$$P_{c-1}(G) = Z_2(G), \quad |Z_2(G)| = p^3, \quad G/Z_2(G) \cong \mathbf{Z}_p \times \mathbf{Z}_p, \quad \mathrm{cl}(G) = 3.$$

But $G/Z_2(G) \cong \mathbf{Z}_p \times \mathbf{Z}_p$ implies that $\Phi(G) \leq Z_2(G)$. On the other hand,

$$Z_2(G) = P_{c-1}(G) \leq P_1(G) = \Phi(G),$$

hence $Z_2(G) = \Phi(G)$ and $c = 2$.
We have also

$$\Phi(G) = G' \cdot G^p \leq G' \cdot Z(G) \leq Z_2(G) = \Phi(G),$$

hence $G' \cdot Z(G) = \Phi(G)$. It follows

$$Z(G) = P_2(G) = [G, P_1(G)] \cdot (P_1(G))^p = [G, G' \cdot Z(G)] \cdot (G' \cdot Z(G))^p =$$
$$= [G, G'] \cdot (G')^p \leq G'.$$

In this way we get $G' = \Phi(G) > Z(G)$.
Finally, from [3], Proposition 2.13, p. 266, it results

$$\exp(G'/[G, G']) \mid \exp([G, G']) = p,$$

hence $(G')^p \leq [G, G']$. Proposition 2 from [4] confirms that G' is elementary abelian.

Based on Theorem 3.2 we may conclude now that G is a FNS-group of class 3. ■

Corollary 3.1. *For every odd prime p there exist $p+7$ FNS-groups of order p^5 and of class 3.*

Proof In [6] it is stated that the group L_{p^3} has $p+7$ direct descendants of order p^5 in the graph $\mathcal{K}_p$. The statement follows now from the previous result. ■

Resuming all the results obtained in this section, one may present a complete list of the FNS-p-groups of class 3.

Theorem 3.4. *Let p be a prime. Then the only FNS-p-groups of class 3 are:*
a) D_{16}, SD_{16} and Q_{16}, if $p = 2$;
b) The 4 groups of order p^4 and of class 3 and also the $p+7$ direct descendants of order p^5 of the group L_{p^3} in the graph $\mathcal{K}_p$, if $p \neq 2$. ■

4 Comments

Unfortunately, the case of the FNS-p-groups of class 2 seems to be more difficult to study. If G is such a group, then $G/Z(G)$ is abelian. The family of capable abelian p-groups is pretty large, and each "equation" $G/Z(G) \cong H$, where H is a capable abelian p-group, has many solutions in the family of nonabelian p-groups. In general, to select the FNS-groups from these solutions is a hard task. However, there is a particular case which offers a lot of FNS-p-groups of class 2.

Proposition 4.1. *Let G be a p-group whose commutator subgroup has order p. Then G is a FNS-group.*

Proof Let H be a normal subgroup in G. Then

$$\{1\} \leq [G,H] \leq G' \ .$$

We have two possible cases:
-if $[G,H] = \{1\}$, then $H \leq Z(G)$;
-if $[G,H] = G'$, then $G' \leq H$.
Consequently, G is a FNS-group. ■

Corollary 4.1. *Let G be a p-group such that $G/Z(G) \cong \mathbf{Z}_p \times \mathbf{Z}_p$. Then G is a FNS-group.*

Proof The hypothesis $G/Z(G) \cong \mathbf{Z}_p \times \mathbf{Z}_p$ implies $|G'| = p$. ■

It remains open the problem of determining all the FNS-p-groups of class 2, whose commutator subgroup has an order bigger than p.

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