On algebraic curves over a finite field with many rational points

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Abstract

In [12], a new upper bound for the number of $\mathbb{F}_q$-rational points on an absolutely irreducible algebraic plane curve defined over a finite field $\mathbb{F}_q$ of degree $d < \sqrt{q} - 2$ was obtained. The present paper is a continuation of [12] and the main result is a similar upper bound for the case $d = \sqrt{q} - 2$.

1 Introduction

Let X be a projective, geometrically irreducible, non-singular, algebraic curve of genus g defined over a finite field $\mathbb{F}_q$. By the Hasse-Weil theorem, the number N_1 of the $\mathbb{F}_q$-rational points of X has the upper bound

$$N_1 \leq q + 1 + 2g\sqrt{q}. \tag{1.1}$$

Consider X over the algebraic closure $\overline{\mathbb{F}}_q$ equipped with the action of the Frobenius morphism associated to $\mathbb{F}_q$. Let g_d^2 be a simple, not necessarily complete, base-point-free linear series on X cut out by a linear system defined over $\mathbb{F}_q$. The morphism π associated to g_d^2 maps X into a (possible singular) plane curve $\pi(X)$ of degree d defined over $\mathbb{F}_q$. Every absolutely irreducible plane curve C can be obtained in this way, and g_d^2 is cut out on C by the linear system Σ_1 of all lines of the plane. The set $X(\mathbb{F}_q)$ of all $\mathbb{F}_q$-rational points of X turns out to be a subset of the possible

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larger set $\tilde{X}(\mathbb{F}_q)$ consisting of all places of X centred at $\mathbb{F}_q$-rational points on the model $\pi(X)$. For the size N of the latter set,

$$N \leq q + 1 + (d-1)(d-2)\sqrt{q}, \tag{1.2}$$

which coincides with (1.1) when $\pi(X)$ is non-singular, see [13]. The problem of determining N_1 and N have been considered in connection with coding theory, cyclotomy, graph colourings, finite geometry and Waring's problems, among others. In the last decade, several authors have used the Stöhr-Voloch method [15] to obtain improvements on (1.1) and (1.2) under some extra conditions on d or for special families of curves; see for instance [7], [9], [11], and [12].

The upper bound given in [12] (and quoted in the next section) is valid for $3 \leq d \leq \sqrt{q} - 3$, and it improves (1.2) for $\frac{3}{4}(\sqrt{q}+2) \leq d \leq \sqrt{q} - 3$. In this paper we show that the techniques used in [12] can be developed further to deal with the case $d = \sqrt{q} - 2$. It remains to show whether such an upper bound holds true for $d = \sqrt{q} - 1$. In the affirmative case, this will improve the constant term in the estimate on the size of the second largest k-arc in $PG(2,q)$, q odd, given in [12]; see also [10] and [17].

2 The main result

The reader is assumed to be familiar with the terminology used in [15]. Let X be a projective, geometrically irreducible, non-singular, algebraic curve of genus g defined over a finite field $\mathbb{F}_q$, and let $\mathbb{F}_q(X)$ be the field of rational functions on X. We consider X as the algebraic curve $X(\overline{\mathbb{F}}_q)/\overline{\mathbb{F}}_q$ equipped with the action of the Frobenius map relative to $\mathbb{F}_q$. The order of a rational function h at a point $P \in X$ will be denoted by $\nu_P(h)$. Let Σ_1 be a linear system cutting out on X a simple, base-point-free, linear series g^2_d defined over $\mathbb{F}_q$, and let $\pi : X \to PG(2, \mathbb{F}_q)$ be the morphism, say $\pi{=}(x_0, x_1, x_2)$, associated with Σ_1. For each point $P \in X$, we have the place (or branch) $\pi(P){=}((t^{e_P}x_0)P, (t^{e_P}x_1)P, (t^{e_P}x_2)P)$ on the model $\pi(X)$ where $e_P := -min\{\nu_P(x_0), \nu_P(x_1), \nu_P(x_2)\}$ and t is a local parameter of X at P. The centre of the place $\pi(P)$ is the point in $PG(2, \overline{\mathbb{F}}_q)$ whose coordinates are the constant terms of the components of $\pi(P)$. Clearly, the centre U of a place $\pi(P)$ is $\mathbb{F}_q$-rational (that is, U lies on $PG(2, \mathbb{F}_q)$) for any $\mathbb{F}_q$-rational point P of X, but the converse is not always true. If this happens then U is a $\mathbb{F}_q$-rational singular point of the plane curve $\pi(X)$, and more than one places of $\pi(X)$ is centred at U. As in [15], $\pi(X)$ will be considered as a parametrized curve in $PG(2, \overline{\mathbb{F}}_q)$ and points P of X will be viewed as places of $\pi(X)$.

The (Σ_1, P) order sequence at a point $P \in X$ is defined as the triple (j_0, j_1, j_2), where $j_0 = 0, j_1, j_2$ are, in increasing order, the intersection multiplicities at the place $\pi(P)$ of $\pi(X)$ with the lines of the plane. Almost all points of X have the same order sequence which is the Σ_1-order sequence of X and denoted by $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$.

If $\pi(X)$ is not the locus of its inflections, then the Σ_1-order sequence at a generic point is $(0,1,2)$ and X is said to be classical for Σ_1. The Σ_1-order sequence of X is also characterized as the sequence of smallest natural numbers, in increasing order, for which the Wronskian $\det(D_t^{(\varepsilon_i)} x_j) \neq 0$, where $D_t^{(j)}$ is the j-th Hasse derivative with respect to a separating variable t.

As in [12], two types of points P in X with respect to g^2_d are distinguished, namely regular points and inflections depending on whether the Σ_1-order sequence at P satisfies $j_2 = 2j_1$ or not. In [12] a new technique was developed for counting the places of $\pi(X)$ centred at $\mathbb{F}_q$-rational points, in which regular points and inflections do not play a symmetrical role. For this, the set of all places of $\pi(X)$ centred at $\mathbb{F}_q$-rational points is split into two subset S_1 and S_2 consisting of all regular points and inflections of X, respectively. Now put

$$\sum_{P\in S_1} j_1(P) = M_q,$$

$$\sum_{P\in S_2} j_1(P) = M'_q.$$

Then $M_q + M'_q \geq N$, and equality holds if and only if no place of $\pi(X)$ centred at a $\mathbb{F}_q$-rational point is singular; in particular $M_q + M'_q = N_1$ for a non-singular plane model $\pi(X)$ of X. In [12] the following upper bound was obtained for $2M_q + M'_q$.

Theorem 2.1. *Let X be a projective, geometrically irreducible, non-singular, algebraic curve defined over a finite field $\mathbb{F}_q$. Assume that X admits a simple, not necessarily complete, base-point-free linear series g^2_d over $\mathbb{F}_q$. If $\mathbb{F}_q$ has characteristic $p \geq 3$, and q is a square for $p = 3$, and*

$$3 \leq d \begin{cases} \leq \sqrt{q} - 3 & for\ q \neq 3^6, 5^5, \\ \leq 22 & for\ q = 3^6, \\ \leq 48 & for\ q = 5^5, \\ \leq \min\{\frac{(q-5\sqrt{q}+1)}{20}, \frac{(q-5\sqrt{q}+57)}{24}\} & for\ q \leq 23^2, \end{cases}$$

then
(i) $2M_q + M'_q \leq d(q - \sqrt{q} + 1)$;
(ii) $2M_q + M'_q = d(q - \sqrt{q} + 1)$ *if and only if* $d = \frac{1}{2}(\sqrt{q} + 1)$, *in which case the curve is maximal.*

It should be noticed that, in case (ii), $\pi(X)$ turns out to be $\mathbb{F}_q$-isomorphic to the Fermat curve of equation $X^{(\sqrt{q}+1)/2} + Y^{(\sqrt{q}+1)/2} + 1 = 0$; see [2]. For further applications of the Stöhr-Voloch theory to maximal curves, see [3],[5], and [6].

In this paper we investigate the case $d = \sqrt{q} - 2$. Our main result is the following theorem.

Theorem 2.2. *Let X, q, g^2_d be as in Theorem 2.1, and let $q > 23^2$. If $d = \sqrt{q} - 2$, then*

$$2M_q + M'_q \ \leq\ d(q - \tfrac{1}{2}\sqrt{q} - \tfrac{9}{2}) - 3.$$

As in [12], the above theorem may be phrased using classical terminology; see [14] and [16]. If C is an absolutely irreducible, plane curve of degree d defined over $\mathbb{F}_q$, two types of place are distinguished, both centred at $\mathbb{F}_q$-rational points: (a) *the regular places of order* r, that is, places of order and class equal to r; (b) *the irregular places of order* r, that is, places of order r and class different from r. Then M_q and M'_q are the numbers of places of type (a) and type (b) respectively, each counted r times, and Theorem 2.2 is equivalent to the following result.

Theorem 2.3. *Let C be an absolutely irreducible, plane curve of degree $d=\sqrt{q}-2$, defined over $\mathbb{F}_q$. If $q>23^2$ then*

$$2M_q+M_q^{'}\ \le\ d(q-\tfrac{1}{2}\sqrt{q}-\tfrac{9}{2})-3.$$

3 Non-classical and Frobenius non-classical curves with respect to a linear series g^5_{2d}

For the purposes of this paper we also need to consider the 5-dimensional linear series on X, defined as the simple, not necessarily complete, base-point-free linear series g^5_{2d} cut out on $\pi(X)$ by the linear series Σ_2 of all conics. Note that $\Sigma_2=2\Sigma_1$. Hence g^5_{2d} contains g^2_d.

By the Σ_2-order sequence at $P\in X$ we mean the increasing sequence $(j_0,j_1,j_2,j_3,j_4,j_5)$ which gives the possible intersection numbers of $\pi(X)$ with conics at the place $\pi(P)$. All points, except for a finite number, have the same Σ_2-order sequence: $(\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)$ and in general $\varepsilon_i\le j_i$ $(0\le i\le 5)$ holds. If X is classical for Σ_1, the Σ_2-order sequence is $(0,1,2,3,4,\varepsilon_5)$ where ε_5 gives the intersection number at the place $\pi(P)$ associated to a generic point $P\in X$ with the osculating conic at $\pi(P)$. Then X is called classical or non-classical for Σ_2 according as $\varepsilon_5=5$ or $\varepsilon_5>5$. By a result of [8], if $p\ge 5$ and X is non-classical for Σ_2, then $\varepsilon_5=p^\nu$.

From the Σ_2-order sequence $(\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)$ it is possible to extract an increasing subsequence of five elements $\nu_0=0,\nu_1,...,\nu_4$ for which the following determinant does not vanish:

$$\begin{vmatrix} x_0^q & \dots & x_5^q \\ D_t^{(\nu_0)}(x_0) & & D_t^{(\nu_0)}(x_5) \\ \vdots & & \vdots \\ D_t^{(\nu_4)}(x_0) & \dots & D_t^{(\nu_4)}(x_5) \end{vmatrix},$$

where $D_t^{(i)}$ is the i-th Hasse derivative and $x_0,...,x_5$ are the coordinate functions of the morphism associated to Σ_2. The sequence $\nu_0=0,\nu_1,...,\nu_4$, in increasing order, is called the $\mathbb{F}_q$-Frobenius Σ_2-order sequence. There is a further notion of classical curve associated to Frobenius order sequences: X is called Frobenius classical for Σ_2 if the $\mathbb{F}_q$-Frobenius Σ_2-order sequence is $(0,1,2,3,4)$; otherwise the curve is called Frobenius non-classical.

4 Preliminary results

Let X, π, Σ_1 and g^2_d be as in the previous sections. Assume that X is classical for Σ_1, and consider the ramification divisor R of Σ_1 which is given by

$$R = \operatorname{div}\left(\det\begin{pmatrix} x_0 & x_1 & x_2 \\ D_t(x_0) & D_t(x_1) & D_t(x_2) \\ D_t^{(2)}(x_0) & D_t^{(2)}(x_1) & D_t^{(2)}(x_2)\end{pmatrix}\right) + 3\operatorname{div}(dt) + 3E,$$

where $E = \sum e_P P$, and t is a separating variable. To compute $\nu_P(R)$ take a local parameter t at P. We may assume that one of the coordinate functions, say x_0, satisfies $\nu_P(x_0) = 0$. Then $e_P = 0$ and $\nu_P(\operatorname{div}(dt)) = 0$. Put $x = x_1/x_0,\ y = x_2/x_0$. Then

$$\nu_P(R) = \nu_P[D_t(x)D_t^2(y) - D_t^2(x)D_t(y)]; \tag{4.3}$$

by [15] Cor. 1.7,

$$\nu_P(R) = j_1 + j_2 - 3 \quad \textit{when} \quad j_1 j_2(j_2 - j_1) \not\equiv 0 \pmod p. \tag{4.4}$$

Since X is classical for Σ_1, the generalized Plücker formula counting the Weierstrass points gives $\deg R = 3(2g-2) + 3d$; see [15] p.6. Hence

$$\sum_{P\in X} \nu_P(R) = 3(2g-2) + 3d. \tag{4.5}$$

Since the curve X is defined over $\mathbb{F}_q$, we can also consider the divisor S, as in [15] Section 2, defined by:

$$S = \operatorname{div}\left(\det\begin{pmatrix} x_0^q & x_1^q & x_2^q \\ x_0 & x_1 & x_2 \\ D_t(x_0) & D_t(x_1) & D_t(x_2)\end{pmatrix}\right) + \operatorname{div}(dt) + (q+2)E.$$

To compute $\nu_P(S)$ we assume as before that $\nu_P(x_0) = 0$. Then $e_P = 0$, $\nu_P(\operatorname{div}(dt)) = 0$, and hence:

$$\nu_P(S) = \nu_P[(x - x^q)D_t(y) - (y - y^q)D_t(x)] \tag{4.6}$$

where $x = x_1/x_0,\ y = x_2/x_0$. The Stöhr-Voloch theorem applied to Σ_1 states that $\deg S = (2g-2) + (q+2)d$; hence

$$\sum_{P\in X} \nu_P(S) = (2g-2) + (q+2)d. \tag{4.7}$$

Next we give a useful formula for $\nu_P(S)$.

For a point $P \in X$ choose a local parameter t. Without loss of generality, we may again assume that $\nu_P(x_0) = 0$. Then $e_P = 0$, $\nu_P(\operatorname{div}(dt)) = 0$. Put $x = x_1/x_0,\ y = x_2/x_0$. Then a parametrization of the place $\pi(P)$ is given by:

$$\begin{cases} x(t) = & a + m_{11}t^{j_1} + \cdots, \\ y(t) = & b + m_{21}t^{j_1} + b_{j_2}t^{j_2} + \cdots, \end{cases} \tag{4.8}$$

where (a, b) is the centre of the place $\pi(P)$, the tangent l to $\pi(P)$ has equation: $m_{21}(x - a) - m_{11}(y - b) = 0$, and $(0, j_1, j_2)$ is the Σ_1-order sequence at P. To quote the result of the computation of $\nu_p(S)$ given in [12], Section 7, two sets of points of X need to be distinguished, namely:

$$B_1 = \Big\{P \in X\backslash\tilde{X}(\mathbb{F}_q)\ :\ m_{21}(a-a^q) - m_{11}(b-b^q) = 0\Big\};$$

$$B_2 = \Big\{P \in X\backslash\tilde{X}(\mathbb{F}_q)\ :\ m_{21}(a-a^q) - m_{11}(b-b^q) \neq 0\Big\}.$$

Assume that $j_1 j_2 \not\equiv 0 \pmod p$. Then, by [11] Prop. 4.4 (see also [12] Prop. 7.4), and [15] Prop. 2.4.(a)),

$$\nu_P(S) = \begin{cases} j_1 + j_2 - 1 & \textit{for } P \in \tilde{X}(\mathbb{F}_q), \\ j_2 - 1 & \textit{for } P \in B_1, \\ j_1 - 1 & \textit{for } P \in B_2. \end{cases} \tag{4.9}$$

5 Proof of Theorem 2.2

We keep all previous notation. The starting point of the proofs of Theorem 2.2 are five lemmas stated in [12].

Lemma 5.1. ([12] Prop. 1.1) *Assume that X satisfies the following conditions:*
(h1) $2M_q + M'_q \geq d(q - \sqrt{q} + 1)$;
(h2)

$$3 \leq d \begin{cases} \leq \sqrt{q} & \textit{when } q > 23^2, q \neq 3^6, 5^5, \\ \leq 22 & \textit{when } q = 3^6, \\ \leq 48 & \textit{when } q = 5^5, \\ \leq \min\{\frac{(q-5\sqrt{q}+1)}{20}, \frac{(q-5\sqrt{q}+57)}{24}\} & \textit{when } q \leq 23^2; \end{cases}$$

(h3) $q \geq 16$*;*
(h4) $p \geq 3$*, and q is a square when $p = 3$.*
Then
(i) g^2_d *is classical;*
(ii) q *is a square;*
(iii) *the Σ_2-order sequence is* $(0, 1, 2, 3, 4, \sqrt{q})$*;*
(iv) *the $\mathbb{F}_q$-Frobenius Σ_2-order sequence is* $(0, 1, 2, 3, \sqrt{q})$*;*
(v) $d \geq \frac{1}{2}(\sqrt{q} + 1)$.

Lemma 5.2. ([12] Prop. 4.1)
Suppose that both $d < \sqrt{q}$ and (h4) hold. If X has the above properties (i),(ii),(iii), and (iv), then the order sequence $(0, j_1, j_2)$ at $P \in X$ with respect to g^2_d satisfies either $j_2 = 2j_1$, or $j_2 = \frac{1}{2}(\sqrt{q} + j_1)$, or $j_2 = \sqrt{q} - j_1$.

Lemma 5.3. ([12] Prop. 9.2) *Suppose that both $d < \sqrt{q}$ and (h1) hold. If X has property (iv), then the set of points P of X splits into three types according of order sequence (j_0, j_1, j_2) at P with respect to g_d^2:*

$$(0,1,2),(0,2,4); \tag{5.1}$$

$$(0,1,\tfrac{1}{2}(\sqrt{q}+1)); \tag{5.2}$$

$$(0,1,\sqrt{q}-1),(0,2,\sqrt{q}-2). \tag{5.3}$$

The above possibilities were also described in terms of the model $\pi(X)$ in [12] Section 5. As before, let (4.8) be a parametrization of the place $\pi(P)$.

Lemma 5.4. (i) *If (5.1) holds and $a \neq a^q$ or $b \neq b^q$, then* $m_{21}(a-a^q) - m_{11}(b-b^q) \neq 0$.
(ii) *If (5.2) holds and $a \neq a^q$ or $b \neq b^q$, then $m_{21}(a-a^q) - m_{11}(b-b^q) = 0$.*
(iii) *If (5.3) holds then $a \neq a^q$ or $b \neq b^q$ and $m_{21}(a-a^q) - m_{11}(b-b^q) \neq 0$.*

In [12] Section 8, an $\mathbb{F}_q$-birational model $\mathcal{Z}$ of X defined in 5-dimensional space was considered and some of its properties were established. Here we limit ourselves to quoting two results. First, [12] Prop. 8.5 states that if $\pi(X)$ has a singular point then $\deg \mathcal{Z} \geq 2\sqrt{q}$. Another result on $\mathcal{Z}$ is that $3\deg\mathcal{Z} = 2\tau + \rho$, where τ and ρ denote the number of points $P \in X$ of type (5.3) and (5.2) each counted j_1 times; see [12] Prop. 9.3. From these results we deduce the following.

Lemma 5.5. *If $\pi(X)$ is a singular curve, then $\tau + 2\rho \geq 6\sqrt{q}$.*

From now on let $d = \sqrt{q} - 2$ and $q > 23^2$. To prove Theorem 2.2, assume on the contrary that X satisfies the condition:

$$2M_q + M_q' > d(q - \tfrac{1}{2}\sqrt{q} - \tfrac{9}{2}) - 3. \tag{5.4}$$

Since (5.4) implies (h1), the above lemmas are valid for X, and have the following corollary.

Corollary 5.4. *If $P \in \tilde{X}(\mathbb{F}_q)$, then either $\nu_P(S) \in \{2,5\}$ or $\nu_P(S) = \frac{1}{2}(\sqrt{q}+1)$, according as (5.1) or (5.2) holds. If $P \in X \backslash \tilde{X}(\mathbb{F}_q)$, then either $\nu_P(S) = \frac{1}{2}(\sqrt{q}-1)$ or $\nu_P(S) = 1$ according as $P \in B_1$ or $P \in B_2$.*

Our next step is to show that $\pi(X)$ is actually a non-singular plane model of X. We need the following result.

Proposition 5.1. *Let C be an irreducible plane curve of degree $d \le \sqrt{q}-2$ and genus g. If C has at least τ places with Σ_1-order sequence $(0,2,\sqrt{q}-2)$ where Σ_1 is the linear series of all lines, then*

$$2g-2 \le d(d-3)-(\sqrt{q}-3)\tau. \tag{5.5}$$

Proof. Since $d \le \sqrt{q}-2$, each of the τ places is centred at a double point of C. Hence C has at least τ double points. On the other hand, $g=(n-1)(n-2)/2-\sum r_i(r_i-1)/2$ where C has the singular points $P_1,...,P_s$ with multiplicity $r_1,...,r_s$ including the infinitely near points. For the concept of infinitely near point, the reader is referred to [4] Cap. 20, [14] Chapters 14 and 23, and [1]. In particular, the method in [4] p. 447, or in [1] Section 3.2, shows that each double point of C which is the centre of a place with (Σ_1,P) order sequence $(0,2,j_2)$, j_2 odd, has at least $\frac{1}{2}(j_2-3)$ infinitely near double points. Applying this for the case $j_2=\sqrt{q}-2$ gives the proposition.

Proposition 5.2. *The curve $\pi(X)$ has no singular points. In particular, $\tau=0$.*

Proof. Let λ denote the number of all points $P\in X$ with order sequence $(0,2,4)$. From Proposition 5.4,

$$\sum_{P\in X}\nu_P(S)=2M_q+M_q{}'+\tfrac{1}{2}(\sqrt{q}-1)\rho+\tau+\lambda,$$

which together with (4.7) gives the following result:

$$2M_q+M_q^{'} \le 2g-2+(q+2)d-\tfrac{1}{2}(\sqrt{q}-1)\rho-\tau. \tag{5.6}$$

Taking into account (5.5) we obtain:

$$2M_q+M_q^{'} \le d(d-3)+(q+2)d-\tfrac{1}{4}(\sqrt{q}-1)(2\rho+\tau)-\tfrac{3}{4}(\sqrt{q}-\tfrac{7}{3})\tau.$$

By (5.5), this yields:

$$2M_q+M_q^{'} \le d(q+d-1)-\tfrac{3}{2}(q-\sqrt{q}).$$

Since $d=\sqrt{q}-2$, the expression on the right-hand side can also be written as $d(q-\frac{1}{2}\sqrt{q}-\frac{9}{2})-3$, and this shows that (5.4) cannot hold.

Note that Proposition 5.2 together with Corollary 5.4 have the following corollary.

Corollary 5.6. *If $P\in\tilde{X}(\mathbb{F}_q)$, then either $\nu_P(S)=2$ or $\nu_P(S)=\frac{1}{2}(\sqrt{q}+1)$, according as (5.1) or (5.2) holds. If $P\in X\backslash\tilde{X}(\mathbb{F}_q)$, then $P\in B_1$ and $\nu_P(S)=\frac{1}{2}(\sqrt{q}-1)$. In particular, B_2 is empty.*

Next we compute the exact value of ρ. Since $\pi(X)$ is non-singular, the only points $P\in X$ for which $\nu_P(R)$ is positive are the ρ points with order sequence $(0,1,\frac{1}{2}(\sqrt{q}+1))$. Since by (4.4) each of them has weight $\nu_P(R)=\frac{1}{2}(\sqrt{q}-3)$, from (4.5) it follows that

$$\rho=6d(d-2)/(\sqrt{q}-3). \tag{5.7}$$

Since $d=\sqrt{q}-2$ we find that

$$\rho=6(\sqrt{q}-3)-6/(\sqrt{q}-3); \tag{5.8}$$

but this is a contradiction, as ρ must be integer. Thus (5.4) is impossible, and Theorem 2.2 has been proved.

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