Spherical rigidity via Contact dynamics

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Abstract

We prove that a closed connected 2n+1-dimensional smooth manifold M is homeomorphic with the 2n+1-dimensional sphere if and only if it admits a K-contact form with exactly n+1 closed characteristics.

Résumé

On montre qu'une variété M lisse, fermée, connexe de dimension $2n+1$ est homéomorphe à la sphère de même dimension si et seulement si M admet une forme de K-contact avec exactement $n+1$ caractéristiques fermées.

1 Preliminaries

A classical topological application of Morse Theory states that an n-dimensional closed manifold admitting a Morse function with only 2 critical points is homeomorphic with the sphere $\mathbf{S}^n$ ([HIR], p. 54). Since Morse Theory has been extended into Morse-Bott Theory with the notion of nondegenerate critical manifolds ([BOT]), it is quite natural to investigate the rigidity of the sphere using this generalized Morse Theory. This is attempted in this note, but only in the contact geometric setting.

The work presented in this paper is a continuation of that started in [RU1] where a simply connected condition has been removed from the main result.

We start with the basic contact geometric vocabulary that will be used throughout this paper. A contact form on a $2n+1$-dimensional manifold M is a 1-form α such that the identity

$$\alpha \wedge (d\alpha)^n \neq 0$$

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holds everywhere on M. Given such a 1-form, there is always a unique vector field ξ satisfying $\alpha(\xi) = 1$ and $i_\xi d\alpha = 0$. The vector field ξ is called the characteristic vector field of the contact form α and the corresponding 1-dimensional foliation is called a contact flow.

A contact form is said to be almost regular if its characteristic vector field is almost regular, that is: each point in M belongs to a flow box pierced by the flow only a finite number of times. If M is compact, all leaves are circles and since the flow lines are geodesics in an appropriate metric (see below), it follows from a theorem of Wadsley ([WAD]) that the leaves are orbits of a circle action on M. If this action is free, then the contact form is said to be regular.

Every contact manifold (M, α) admits a nonunique riemannian metric g and a (1,1) tensor field J such that the following identities hold ([BLA]):

$$J\xi = 0, \quad \alpha(\xi) = 1, \quad J^2 = -I + \alpha \otimes \xi, \quad \alpha(X) = g(\xi, X)$$
$$g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y), \quad g(X, JY) = d\alpha(X, Y).$$

The tensors g, α, ξ and J will be referred to as structure tensors and g is called a contact metric adapted to α. When the characteristic vector field ξ is Killing with respect to a contact metric g, then the manifold is said to be K-contact. All almost regular contact manifolds are K-contact, but a K-contact form need not be almost regular. However, any K-contact, compact manifold admits an almost regular contact form ([RU2]).

Proposition 1. *Every $2n+1$-dimensional sphere carries a K-contact form with exactly $n+1$ closed characteristics.*

Proof. These K-contact forms come from perturbations of the canonical sasakian form on $\mathbf{S}^{2n+1}$ (see for example [BLA] for the definition of sasakian). Indeed, consider the sphere $\mathbf{S}^{2n+1}$ as the set:

$$\{(z_1, z_2, ..., z_{n+1}) \in \mathbf{C}^{n+1}, |z_1|^2 + |z_2|^2 + ... + z_{n+1}|^2 = 1\}$$

with the periodic $\mathbf{R}$ action $\phi_t \colon (z_1, z_2, ..., z_{n+1}) \mapsto (e^{it}z_1, e^{it}z_2, ..., e^{it}z_{n+1})$. The vector field $\xi = \frac{d}{dt}_{|t=0}\phi_t$ is the characteristic vector field of the standard sasakian (hence K-contact) form α on the unit sphere.

Consider now the isometric perturbation ϕ_t^λ:

$$\phi_t^\lambda \colon (z_1, z_2, ..., z_{n+1}) \mapsto (e^{i\lambda_1 t}z_1, e^{i\lambda_2 t}z_2, ..., e^{it}z_{n+1}),$$

where the $\lambda_j's$ are real constants with $\lambda_{n+1} = 1$. If the λ_j's are rationally independent, then the flow generated by $\xi^\lambda = \frac{d}{dt}_{|t=0}\phi_t^\lambda$ contains exactly $n+1$ compact leaves which are circles. Let us denote by $\mu = \xi^\lambda - \xi$. The vector field μ has the following properties:

$$L_\mu g_0 = 0, \quad [\mu, \xi] = 0, \quad 1 + \alpha(\mu) > 0.$$

It is well known that with such a μ, the vector field $\xi^\lambda = \xi + \mu$ is characteristic for a sasakian form ([KOY], Theorem 7.2). ∎

We will prove that the existence of a K-contact form whose contact flow has the minimum possible number of compact leaves characterizes the sphere up to homeomorphism (diffeomorphism in dimensions ≤ 5). A similar result was proven in [RU1] where the manifolds were redundantly assumed to be simply connected.

2 Morse theory on K-contact manifolds

Let α, ξ, g be K-contact structure tensors on a closed manifold M of dimension $2n+1$. There exist a periodic Killing vector field Z commuting with ξ such that closed characteristics of α are exactly critical circles of the function $\alpha(Z)$ ([BAR]). In [RU1] (see also [RU3]), Morse theory was carried out on closed K-contact manifolds and it was established that *the critical manifolds of the function $\alpha(Z)$ are all nondegenerate and each has even index.* Such functions are known to have a unique local minimum (global minimum if M is compact) and all of their level surfaces are connected ([GST]).

Using a combination of Morse Theory with Carrière's classification of riemannian flows, we establish the following rigidity result for the 3-sphere.

Proposition 2. *A closed 3-manifold is diffeomorphic with the 3-sphere if and only if it carries a K-contact form with exactly 2 closed characteristics.*

Proof. By Proposition 1, the 3-sphere carries a K-contact form with exactly 2 closed characteristics. Conversely, let us denote the 3-manifold by M. Each of the two closed characteristics has a solid torus tubular neighborhood, so M is the union of 2 solid tori glued together by their boundaries. This means that M admits a genus one Heegaard decomposition. The only closed 3-manifolds admitting such a decomposition are lens spaces (including $\mathbf{S}^3$) and $\mathbf{S}^2 \times \mathbf{S}^1$. Being sasakian, M cannot be diffeomorphic with $\mathbf{S}^2 \times \mathbf{S}^1$ ($H_1(M, \mathbf{R})$ must be trivial or have even rank, ([TAC])), therefore the contact flow on M is differentiably conjugate to an isometric flow on a lens space $L_{p,q}$, the quotient of a standard sphere $\mathbf{S}^3$ under a finite abelian group of isometries Γ ([CAR]). So, we may as well assume that $M = L_{p,q}$ is a lens space with a flow ϕ and a smooth function H with the following properties:

(1) H has exactly 2 critical circles which are nondegenerate with even indices 0 and 2 respectively.

(2) The level surfaces of H are leaf closures of ϕ.

The function H lifts to $\mathbf{S}^3$ into a function $\tilde{H}$ with the same properties (1) and (2), ϕ replaced by its lift $\tilde{\phi}$. We point out that the flow $\tilde{\phi}$ is isometric relative to the standard metric g_0 on the unit sphere.

Crucial Remark. *If γ is a geodesic (in the g_0 metric) realizing the distance between two regular levels of $\tilde{H}$, then the gradient vector field $\nabla\tilde{H}$ satisfies $\nabla\tilde{H} = \|\nabla\tilde{H}\|\gamma'$. In particular, the gradient lines joining the two critical circles are geodesic lines (after reparametrization).*

Let us denote the quotient map by

$$\pi\colon \mathbf{S}^3 \to M$$

The two closed characteristics are disjoint, closed geodesics, call them C_1 and C_2. Each is the projection of a great circle γ_1 and γ_2 respectively from $\mathbf{S}^3$ and since γ_1 and γ_2 are disjoint, one can find a totally geodesic 2-sphere $\mathbf{S}^2$ intersecting them transversally and nontrivially in $\mathbf{S}^3$. Moreover, from the above remark, $\nabla\tilde{H}$ is not perpendicular to $\mathbf{S}^2$ except at points where $\mathbf{S}^2$ meets γ_1 and γ_2. As a consequence, the critical set of the restriction $\tilde{H}_{|\mathbf{S}^2}$ is contained in the critical set of $\tilde{H}$. It follows clearly that each critical point of the restriction of $\tilde{H}$ to $\mathbf{S}^2$ is nondegenerate with index zero or two.

The subset $\mathbf{S} = \cup_{g\in\Gamma} g(\mathbf{S}^2)$ is a connected totally geodesic surface in $\mathbf{S}^3$. Moreover, $\mathbf{S}$ intersects γ_1 and γ_2 transversally (and nontrivially). Therefore $\pi(\mathbf{S})$ is a connected hypersurface in M intersecting C_1 and C_2 transversally and nontrivially. The function H induces then a smooth function on $\pi(\mathbf{S})$ whose critical points are all nondegenerate with even indices. Therefore there must be exactly two critical points on $\pi(\mathbf{S})$ which implies, by the classical result mentionned in Section 1, that the surface $\pi(\mathbf{S})$ is a 2-sphere. However, this is only possible if the group Γ is the trivial group $\{0\}$ and therefore $\pi\colon \mathbf{S}^3 \to M$ is a diffeomorphism. ■

3 Spherical rigidity

Given a K-contact manifold M^{2n+1} with $n+1$ closed characteristics, N_0, N_1,..., N_n which are critical circles of a function $\alpha(Z)$ with index $0, 2, ..., 2n$ respectively, we shall denote by M^{2k+1}, $0 \le k \le n$, the closures of the sets $\{x \in M^{2n+1},\ \lim_{t\to\infty} \Psi_t(x) \in N_k\}$, where Ψ_t is the gradient flow for the function $\alpha(Z)$. For an arbitrary vector field Y on M^{2n+1}, one has

$$d(\alpha(Z))(Y) = di_Z\alpha(Y) = g(JZ, Y).$$

Therefore, Ψ_t is generated by the vector field JZ. For the argument to follow, we point out that ξ, Z and JZ are commuting vector fields. The subsets M^{2k+1} constitute a stratification of M^{2n+1} as follows:

$$M^1 \subset M^3 \subset M^5 \subset ... \subset M^{2n-1} \subset M^{2n+1} \qquad 1$$

and each M^{2k+1} is a closed $2k+1$-dimensional submanifold with $k+1$ nondegenerate critical circles for the restriction of $\alpha(Z)$.

Proposition 3. *M^3 is a K-contact submanifold, which is diffeomorphic with the 3-sphere $\mathbf{S}^3$.*

Proof. For $x \in N_0$ or $x \in N_1$; if $v \in T_xM^3$ and $\alpha(v) = 0$, then the fact that $Jv \in T_xM^3$ follows from J-invariance of the splitting $\nu_{N_i} = \nu^-_{N_i} \oplus \nu^+_{N_i}$ for the normal bundle of N_i, $i = 0, 1$ (see Proposition 2 in [RU1]). For $x \in M^3 - (N_0 \cup N_1)$, the tangent space T_xM^3 is spanned by the vectors JZ, ξ and $Z - \alpha(Z)\xi$ and is therefore J-invariant. This implies that M^3 is a K-contact submanifold.

Since M^3 supports a K-contact flow with exactly two closed characteristics, it follows from Prposition 2 that M^3 must be diffeomorphic with $\mathbf{S}^3$. ■

Proposition 4. *A closed K-contact 2n+1-dimensional manifold with exactly $n+1$ closed characteristics is simply connected.*

Proof. Let

$$\pi\colon \widetilde{M} \to M^{2n+1}$$

denote the universal covering of the K-contact manifold M^{2n+1} with exactly n+1 closed characteristics. For $n = 1$, Proposition 4 follows immediately ¿from Proposition 2. So we may assume $n \geq 2$. The universal covering space $\widetilde{M}$ inherits a K-contact form whose closed characteristics are nondegenerate critical circles of a function $\widetilde{\alpha(Z)}$, all of whose critical manifolds have even indices. In particular there is exactly one critical circle of index 0, that is $\pi^{-1}(M^1)$. As a consequence, each connected component of $\pi^{-1}(M^3)$ contains the circle $\pi^{-1}(M^1)$ and hence, $\pi^{-1}(M^3)$ is connected, making it into a covering space of M^3.

Now let $p \in M^3$ and C be a loop in M^{2n+1} based at p. Let γ be a lift of C with initial point q_1 and terminal point q_2 both in $\pi^{-1}(p) \subset \pi^{-1}(M^3)$. Since $\pi^{-1}(M^3)$ is connected, there is a path $\gamma' \in \pi^{-1}(M^3)$ connecting q_1 to q_2. The paths γ' and γ are homotopic in $\widetilde{M}$ and therefore, $\pi(\gamma') \subset M^3$ is a loop based at p, homotopic to C. This proves that the natural homomorphism

$$\pi_1\colon \pi_1(M^3) \to \pi_1(M^{2n+1})$$

is surjective. But since $\pi_1(M^3) \simeq \{0\}$, we conclude that M^{2n+1} is simply connected.■

The subsets M^{2k-1} and N_k are disjoint closed submanifolds of M^{2k+1}, so each has a tubular neighborhood in M^{2k+1}. In fact, one can find open neighborhoods U of M^{2k-1} and V of N_k such that M^{2k-1}, $N_k \simeq \mathbf{S}^1$ and $\mathbf{S}^{2k-1} \times \mathbf{S}^1$ are deformation retracts of U, V and $U \cap V$ respectively. The Mayer-Vietoris sequence for the (integer coefficients) homology yields the following long exact sequence:

$$\dots \to H_*(\mathbf{S}^{2k-1} \times \mathbf{S}^1) \to H_*(M^{2k-1}) \oplus H_*(\mathbf{S}^1) \to H_*(M^{2k+1}) \to H_{*-1}(\mathbf{S}^{2k-1} \times \mathbf{S}^1) \to \dots. \quad (2)$$

Lemma 1. *For $1 \leq k \leq n$, $H_1(M^{2k+1}) \simeq \{0\}$ and for $2 \leq k \leq n$, $H_2(M^{2k-1}) \simeq H_2(M^{2k+1})$.*

Proof. Writing the exact sequence (2) for $* = 2$, one obtains:

$$0 \to H_2(M^{2k-1}) \to H_2(M^{2k+1}) \to \mathbf{Z} \to H_1(M^{2k-1}) \oplus \mathbf{Z} \to 0$$

The lemma follows since by Proposition 4, $H_1(M^{2n+1}) = \{0\}$. ■

Lemma 2. *For $2 \leq k \leq n$, if M^{2k-1} is a homology sphere, then so is M^{2k+1}.*

Proof. Since $H_1(M^{2k+1}) \simeq \{0\}$ by Lemma 1, we need only to show that $H_i(M^{2k+1}) \simeq \{0\}$ fro $2 \leq i \leq 2k$.

Writing the exact sequence for $* = 2l$, $1 < l < k$ and use the hypothesis, one obtains the exact sequence:

$$0 \to H_{2l}(M^{2k+1}) \to 0 \to 0 \to H_{2l-1}(M^{2k+1}) \to 0$$

from which follows that $H_i(M^{2k+1}) \simeq \{0\}$ for $3 \leq i \leq 2k-2$.

Next consider the sequence (2) for $* = 2$ and use Lemma 1 to obtain the exact sequence

$$0 \to H_2(M^{2k+1}) \to \mathbf{Z} \to \mathbf{Z} \to 0$$

leading to $H_2(M^{2k+1}) \simeq \{0\}$.

Using Poincaré Duality in conjunction with the isomorphisms $H_0(M^{2k+1}) \simeq \mathbf{Z}$ and $H_1(M^{2k+1}) \simeq \{0\}$, one obtains the following sequence of isomorphisms:

$$H_{2k}(M^{2k+1}) \simeq H^1(M^{2k+1}) \simeq Hom(H_1(M^{2k+1}), \mathbf{Z}) \simeq \{0\}.$$

Finally using Poincaré Duality again in conjunction with the isomorphisms $H_1(M^{2k+1}) \simeq \{0\}$ and $H_2(M^{2k+1}) \simeq \{0\}$, one obtains the following sequence of isomorphisms:

$$H_{2k-1}(M^{2k+1}) \simeq H^2(M^{2k+1}) \simeq Hom(H_2(M^{2k+1}), \mathbf{Z}) \simeq \{0\}.$$

We have thus shown that $H_i(M^{2k+1}) \simeq \{0\}$ for $2 \leq i \leq 2k$. ■

Theorem 1. *A closed smooth 2n+1-dimensional manifold is homeomorphic with the 2n+1-sphere if and only if it admits a K-contact form with exactly n+1 closed characteristics.*

Proof. We have pointed out in Proposition 1 that odd dimensional spheres admit K-contact forms with exactly the minimal number of closed characteristics (see also [RU2]). So it remains only to prove the if part of our statement.

Let $M^1 \subset M^3 \subset ... \subset M^{2n+1}$ be the stratification of the K-contact manifold M^{2n+1} as in (1). Since M^3 is diffeomorphic with $\mathbf{S}^3$ by Proposition 3, using Lemma 2 and induction on k, we see that M^{2n+1} is a homology sphere. Now, for $n \geq 2$, since M^{2n+1} is simply connected by Proposition 4, our theorem follows from Smale's solution to the generalized Poincaré conjecture ([SMA]). For $n = 1$, the theorem is equivalent to Proposition 2. ■

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