The (outer) automorphism group of a group extension

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Abstract

If $K \rightarrowtail G \twoheadrightarrow Q$ is a group extension, then any automorphism of G which sends K into itself, induces automorphisms respectively on K and on Q. This subgroup of automorphisms of G is denoted by $\mathrm{Aut}(G,K)$ and is called the automorphism group of the extension $K \rightarrowtail G \twoheadrightarrow Q$. After establishing an interesting group action of $\mathrm{Aut}(K) \times \mathrm{Aut}(Q)$ on the set $\mathcal{H}^2(Q,K)$ of all 2-cohomology classes of Q with coefficients in K, a full description of $\mathrm{Aut}(G,K)$ and $\mathrm{Out}(G,K) = \mathrm{Aut}(G,K)/_{\mathrm{Inn}(G)}$ is obtained in terms of various commutative diagrams. This picture is as general as possible, hence covering and further complementing similar ideas developed earlier by C. Wells ([5]), P. Conner & F. Raymond ([1]), D.J.S. Robinson ([3], [4]) and the author ([2]).

1 Notations and preliminaries

If G is a group and $x \in G$, then $\mu(x)$ is the inner automorphism determined by x (sending $y \in G$ to xyx^{-1}), $\mu(G)$ is known as the inner automorphism group $\mathrm{Inn}(G)$ and $\mathrm{Out}(G) = \mathrm{Aut}(G)/_{\mathrm{Inn}(G)}$ is called the outer automorphism group of G. Write $p : \mathrm{Aut}(G) \twoheadrightarrow \mathrm{Out}(G)$ for the natural projection. For a subset X in G, $C_G X$ denotes the centralizer and $N_G X$ is the normalizer of X in G. Let $Z(G)$ be the center of G.

In the sequel of this paper, aspects of group cohomology (with non-abelian coefficients) will be intensively used. Therefore, we review some basic facts of this theory and meanwhile fix additional notations and terminology.

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Each group extension $K \rightarrowtail G \overset{j}{\twoheadrightarrow} Q$ induces, by choosing a normalized section $s : Q \to G$ ($j \circ s = 1$ and $s(1) = 1$) and via conjugation in G, a map (not necessarily a homomorphism!) $\varphi : Q \to \mathrm{Aut}(K)$ sending $x \in Q$ to the K-automorphism $\varphi(x) : k \in K \mapsto s(x)\, k\, s(x)^{-1}$. This induces a homomorphism $\psi : Q \to \mathrm{Out}(K)$ which is called an abstract kernel. We say that the extension realizes ψ or that it is compatible with ψ.

Two extensions G and G', both with kernel K and quotient Q, are equivalent if and only there is a homomorphism $f : G \to G'$ such that

$$\begin{array}{ccccc} K & \rightarrowtail & G & \twoheadrightarrow & Q \\ \| & & \downarrow f & & \| \\ K & \rightarrowtail & G' & \twoheadrightarrow & Q \end{array}$$

commutes. Write $\mathrm{Ext}_\psi(Q, K)$ for the set of equivalence classes of group extensions realizing an abstract kernel $\psi : Q \to \mathrm{Out}(K)$.

A 2-cocycle of Q with coefficients in K is a pair (φ, c), where $\varphi : Q \to \mathrm{Aut}(K)$ and $c : Q \times Q \to K$ are maps satisfying

$$\begin{cases} \varphi(x)\, \varphi(y) = \mu(c(x, y))\, \varphi(xy) \\ \varphi(x)(c(y, z)) \cdot c(x, yz) = c(x, y) \cdot c(xy, z) \end{cases}$$

for all x, y, z in Q. We always assume that φ and c are normalized, i.e. $\varphi(1) = 1$ and, for all $x \in Q$, $c(x, 1) = c(1, x) = 1$. Each 2-cocycle (φ, c) determines an abstract kernel $\psi = p \circ \varphi : Q \to \mathrm{Out}(K)$. For a fixed abstract kernel $\psi : Q \to \mathrm{Out}(K)$, we write $Z^2_\psi(Q, K)$ for the set of all 2-cocycles of Q with coefficients in K and inducing ψ. Two 2-cocycles (φ, c), (φ', c') are cohomologous (write $(\varphi, c) \sim (\varphi', c')$) if and only if there exists a normalized map ("a cochain") $\lambda : Q \to K$ such that, for all $x, y \in Q$:

$$\begin{cases} \varphi'(x) = \mu(\lambda(x))\, \varphi(x) \\ c'(x, y) = \lambda(x) \cdot \varphi(x)(\lambda(y)) \cdot c(x, y) \cdot \lambda(xy)^{-1} \end{cases}$$

Let $H^2_\psi(Q, K)$ denote the set of all cohomology classes of Q with coefficients in K and inducing the abstract kernel ψ.

A 2-cocycle $(\varphi, c) \in Z^2_\psi(Q, K)$ gives rise to an extension $G = K \times_{(\varphi, c)} Q$ of K by Q realizing ψ, with group operation

$$\forall k, l \in K \;\; \forall x, y \in Q \quad (k, x) \cdot_{(\varphi, c)} (l, y) = (k \cdot \varphi(x)(l) \cdot c(x, y), xy).$$

Conversely, by choosing a normalized section $s : Q \to G$ of an extension $K \rightarrowtail G \overset{j}{\twoheadrightarrow} Q$ realizing an abstract kernel ψ, we obtain a 2-cocycle $(\varphi, c) \in Z^2_\psi(Q, K)$ given by $\varphi : Q \to \mathrm{Aut}(K)$ sending $x \in Q \mapsto \mu(s(x))|_K$ and, for $x, y \in Q$, $c(x, y) = s(x)\, s(y)\, s(xy)^{-1}$. This establishes a one-to-one correspondence between $\mathrm{Ext}_\psi(Q, K)$ and $H^2_\psi(Q, K)$. Moreover, if $\mathrm{Ext}_\psi(Q, K)$ is not empty, then $H^2_\psi(Q, Z(K))$ acts on it simply transitively. If we take $\mathcal{H}^2(Q, K) = \bigcup_\psi H^2_\psi(Q, K)$, where ψ runs through all abstract kernels $Q \to \mathrm{Out}(K)$, then $\mathcal{H}^2(Q, K)$ is in one-to-one correspondence with $\mathcal{E}(Q, K) = \bigcup_\psi \mathrm{Ext}_\psi(Q, K)$, the set of all equivalence classes of group extensions with kernel K and quotient Q.

2 Crucial group actions

In this section, we introduce group actions which play a crucial role in the sequel.

Proposition 2.1. *Assume an abstract kernel* $Q \to \mathrm{Out}(K)$*. There is a group action of* $\mathrm{Aut}(K)$ *on* $\mathcal{H}^2(Q,K)$ *defined as follows:*

$$\mathrm{Aut}(K) \times \mathcal{H}^2(Q,K) \to \mathcal{H}^2(Q,K) : (\nu, <\varphi, c>) \mapsto <\mu(\nu)\circ\varphi, \nu\circ c>$$

Proof. Assume $(\varphi, c) \in Z^2_\psi(Q,K)$ and $\nu \in \mathrm{Aut}(K)$. We claim that $(\mu(\nu)\circ\varphi, \nu\circ c) \in Z^2_{\mu(p(\nu))\circ\psi}(Q,K)$. Take x, y, z in Q. First there is

$$\begin{aligned}\mu[\nu(c(x,y))] = \nu\circ\mu(c(x,y))\circ\nu^{-1} &= \nu[\varphi(x)\varphi(y)\varphi(xy)^{-1}]\nu^{-1}\\ &= (\nu\varphi(x)\nu^{-1})(\nu\varphi(y)\nu^{-1})(\nu\varphi(xy)\nu^{-1})^{-1}\end{aligned}$$

and secondly we verify,

$$\begin{aligned}\nu(c(x,y))\cdot\nu(c(xy,z)) = \nu[c(x,y)\cdot c(xy,z)] &= \nu[\varphi(x)(c(y,z))\cdot c(x,yz)]\\ &= (\nu\varphi(x)\nu^{-1})(\nu(c(y,z)))\cdot\nu(c(x,yz)).\end{aligned}$$

If now $(\varphi, c)\sim(\varphi', c')$, then $(\mu(\nu)\circ\varphi, \nu\circ c)\sim(\mu(\nu)\circ\varphi', \nu\circ c')$. Indeed, let $\lambda : Q \to K$ be the cochain such that $\varphi' = \mu(\lambda)\varphi$. Then $\mu(\nu)\circ\varphi' = \mu(\nu\circ\lambda)(\mu(\nu)\circ\varphi)$ and, for each $x, y \in Q$:

$$\begin{aligned}\nu(c'(x,y)) &= \nu[\lambda(x)\cdot\varphi(x)(\lambda(y))\cdot c(x,y)\cdot\lambda(xy)^{-1}]\\ &= (\nu\circ\lambda)(x)\cdot(\nu\varphi(x)\nu^{-1})((\nu\circ\lambda)(y))\cdot\nu(c(x,y))\cdot((\nu\circ\lambda)(xy))^{-1}.\end{aligned}$$

We now easily conclude that we have a group action. ∎

Proposition 2.2. *Assume an abstract kernel* $Q \to \mathrm{Out}(K)$*. There is a group action of* $\mathrm{Aut}(Q)$ *on* $\mathcal{H}^2(Q,K)$ *defined as follows:*

$$\mathrm{Aut}(Q) \times \mathcal{H}^2(Q,K) \to \mathcal{H}^2(Q,K) : (\Phi, <\varphi, c>) \mapsto <\varphi\circ\Phi^{-1}, c\circ(\Phi^{-1}\times\Phi^{-1})>$$

Proof. Take $(\varphi, c) \in Z^2_\psi(Q,K)$ and $\Phi \in \mathrm{Aut}(Q)$. Obviously $(\varphi\circ\Phi^{-1}, c\circ(\Phi^{-1}\times\Phi^{-1})) \in Z^2_{\psi\circ\Phi^{-1}}(Q,K)$. Moreover, if $(\varphi, c)\sim(\varphi', c')$ and $\lambda : Q \to K$ is the corresponding cochain, then $(\varphi\circ\Phi^{-1}, c\circ(\Phi^{-1}\times\Phi^{-1}))\sim(\varphi'\circ\Phi^{-1}, c'\circ(\Phi^{-1}\times\Phi^{-1}))$ via the cochain $\lambda\circ\Phi^{-1} : Q \to K$. Thus we have a group action. ∎

Combining the above results, one easily obtains that

Proposition 2.3. *Assume an abstract kernel* $Q \to \mathrm{Out}(K)$*. There is a group action of* $\mathrm{Aut}(K)\times\mathrm{Aut}(Q)$ *on* $\mathcal{H}^2(Q,K)$ *defined as follows:*

$$(\mathrm{Aut}(K)\times\mathrm{Aut}(Q)) \times \mathcal{H}^2(Q,K) \to \mathcal{H}^2(Q,K) : ((\nu,\Phi), <\varphi, c>) \mapsto {}^{(\nu,\Phi)}<\varphi, c>$$

where ${}^{(\nu,\Phi)}<\varphi, c> = <\mu(\nu)\circ\varphi\circ\Phi^{-1}, \nu\circ c\circ(\Phi^{-1}\times\Phi^{-1})>$.

3 A fundamental automorphism diagram

Fix a group extension $K \rightarrowtail G \twoheadrightarrow Q$. An automorphism σ in $\mathrm{Aut}(G, K)$ restricts to an automorphism of K and consequently induces an automorphism of Q. Write $A : \mathrm{Aut}(G, K) \to \mathrm{Aut}(K)$ for the restriction to K and $B : \mathrm{Aut}(G, K) \to \mathrm{Aut}(Q)$ for the corresponding homomorphism. So, each σ gives rise to a commutative diagram:

$$\begin{array}{ccccc} K & \rightarrowtail & G & \twoheadrightarrow & Q \\ \downarrow A(\sigma) & & \downarrow \sigma & & \downarrow B(\sigma) \\ K & \rightarrowtail & G & \twoheadrightarrow & Q \end{array}$$

Let $\Theta : \mathrm{Aut}(G, K) \to \mathrm{Aut}(K) \times \mathrm{Aut}(Q)$ denote the homomorphism sending $\sigma \mapsto (A(\sigma), B(\sigma))$. A pair $(\nu, \Phi) \in \mathrm{Aut}(K) \times \mathrm{Aut}(Q)$ is called inducible ([5]) if there is an automorphism $\sigma \in \mathrm{Aut}(G, K)$ inducing ν on K and Φ on Q. The set of inducible pairs is hence precisely $\mathrm{Im}(\Theta)$.

An important observation, using the group action introduced in Proposition 2.3, is

Proposition 3.1. *Assume $K \rightarrowtail G \twoheadrightarrow Q$ is an extension determing a cohomology class $a = <\varphi, c> \in \mathcal{H}^2(Q, K)$. Then, for all $\sigma \in \mathrm{Aut}(G, K)$, ${}^{(A(\sigma),B(\sigma))} <\varphi, c> = <\varphi, c>$.*

Proof. Take $G = K \times_{(\varphi,c)} Q$ and consider $\sigma \in \mathrm{Aut}(G, K)$. Construct the map $\xi_\sigma : Q \to K$ defined by $\sigma(1, x) = (\xi_\sigma(x), B(\sigma)(x))$ $(x \in Q)$. For every $(k, x) \in G$,

$$(k, x) = (k, 1) \cdot_{(\varphi,c)} (1, x) = (1, x) \cdot_{(\varphi,c)} (\varphi(x)^{-1}(k), 1).$$

Therefore we find that

$$\sigma(k, 1) \cdot_{(\varphi,c)} \sigma(1, x) = (A(\sigma)(k), 1) \cdot_{(\varphi,c)} (\xi_\sigma(x), B(\sigma)(x)) = (A(\sigma)(k) \cdot \xi_\sigma(x), B(\sigma)(x))$$

while

$$\begin{aligned} \sigma(1, x) \cdot_{(\varphi,c)} \sigma(\varphi(x)^{-1}(k), 1) &= (\xi_\sigma(x), B(\sigma)(x)) \cdot_{(\varphi,c)} (A(\sigma)(\varphi(x)^{-1}(k)), 1) \\ &= (\xi_\sigma(x) \cdot \varphi(B(\sigma)(x))[A(\sigma)(\varphi(x)^{-1}(k))], B(\sigma)(x)). \end{aligned}$$

So ξ_σ must satisfy

$$\forall x \in Q \quad \mu(\xi_\sigma(x)^{-1}) \circ A(\sigma) = \varphi(B(\sigma)(x)) \circ A(\sigma) \circ \varphi(x)^{-1} \tag{1}$$

and because $B(\sigma) \in \mathrm{Aut}(Q)$, this is equivalent to

$$\mu(A(\sigma)) \circ \varphi \circ B(\sigma)^{-1} = \mu(\xi_\sigma \circ B(\sigma)^{-1}) \circ \varphi.$$

Now, take $x, y \in Q$ and obtain that

$$\sigma[(1, x) \cdot_{(\varphi,c)} (1, y)] = \sigma[(c(x, y), 1) \cdot_{(\varphi,c)} (1, xy)] = (A(\sigma)(c(x, y)) \cdot \xi_\sigma(xy), B(\sigma)(xy))$$

which must be equal to

$$\begin{aligned} \sigma(1, x) \cdot_{(\varphi,c)} \sigma(1, y) &= (\xi_\sigma(x), B(\sigma)(x)) \cdot_{(\varphi,c)} (\xi_\sigma(y), B(\sigma)(y)) \\ &= (\xi_\sigma(x) \cdot \varphi(B(\sigma)(x))(\xi_\sigma(y)) \cdot c(B(\sigma)(x), B(\sigma)(y)), B(\sigma)(xy)) \end{aligned}$$

So we have that

$$A(\sigma)(c(x,y)) = \xi_\sigma(x)\cdot\varphi(B(\sigma)(x))(\xi_\sigma(y))\cdot c(B(\sigma)(x),B(\sigma)(y))\cdot\xi_\sigma(xy)^{-1} \qquad (2)$$

and this translates into

$$\begin{aligned}&(A(\sigma)\circ c\circ(B(\sigma)^{-1}\times B(\sigma)^{-1}))(x,y)\\ &=(\xi_\sigma\circ B(\sigma)^{-1})(x)\cdot\varphi(x)[(\xi_\sigma\circ B(\sigma)^{-1})(y)]\cdot c(x,y)\cdot((\xi_\sigma\circ B(\sigma)^{-1})(xy))^{-1}.\end{aligned}$$

We conclude that ${}^{(A(\sigma),B(\sigma))}<\varphi,c>=<\varphi,c>$. ■

As to the converse, one has

Proposition 3.2. *Let $K\rightarrowtail G\twoheadrightarrow Q$ be a group extension determining a cohomology class $a=<\varphi,c>\in\mathcal{H}^2(Q,K)$. Assume $\nu\in\mathrm{Aut}(K)$ and $\Phi\in\mathrm{Aut}(Q)$ such that ${}^{(\nu,\Phi)}<\varphi,c>=<\varphi,c>$. Then there exists $\sigma\in\mathrm{Aut}(G,K)$ such that $\Theta(\sigma)=(\nu,\Phi)$.*

Proof. Write $a=<\varphi,c>$ and consider G as $K\times_{(\varphi,c)}Q$. Let $\nu\in\mathrm{Aut}(K)$ and $\Phi\in\mathrm{Aut}(Q)$. If ${}^{(\nu,\Phi)}<\varphi,c>\;=\;<\varphi,c>$, then there is a cochain $\lambda:Q\to K$ such that

$$\eta: K\times_{(\mu(\nu)\circ\varphi\circ\Phi^{-1},\nu\circ c\circ(\Phi^{-1}\circ\Phi^{-1}))}Q\to K\times_{(\varphi,c)}Q:(k,x)\mapsto(k\cdot\lambda(x),x)$$

is a group isomorphism. Moreover, one can introduce another isomorphism ς as follows:

$$\varsigma: K\times_{(\varphi,c)}Q\to K\times_{(\mu(\nu)\circ\varphi\circ\Phi^{-1},\nu\circ c\circ(\Phi^{-1}\circ\Phi^{-1}))}Q:(k,x)\mapsto(\nu(k),\Phi(x)).$$

Indeed, let $k,k'\in K$ and $x,x'\in Q$, then

$$\begin{aligned}&\varsigma(k,x)\cdot_{(\mu(\nu)\circ\varphi\circ\Phi^{-1},\nu\circ c\circ(\Phi^{-1}\circ\Phi^{-1}))}\varsigma(k',x')\\ &=(\nu(k),\Phi(x))\cdot_{(\mu(\nu)\circ\varphi\circ\Phi^{-1},\nu\circ c\circ(\Phi^{-1}\circ\Phi^{-1}))}(\nu(k'),\Phi(x'))\\ &=(\nu(k)\cdot\nu(\varphi(x)(k'))\cdot\nu(c(x,y)),\Phi(xx'))=\varsigma[(k,x)\cdot_{(\varphi,c)}(k',x')].\end{aligned}$$

Consequently, $\sigma=\eta\circ\varsigma\in\mathrm{Aut}(G,K)$ and $A(\sigma)=\nu$, $B(\sigma)=\Phi$. ■

This leads to an explicit description of the inducible pairs of a group extension.

Theorem 3.3. *If $K\rightarrowtail G\twoheadrightarrow Q$ is an extension compatible with $\psi:Q\to\mathrm{Out}(K)$ and $a\in H^2_\psi(Q,K)$ is the corresponding cohomology class, then*

$$\mathrm{Im}(\Theta)=\mathrm{Stab}_{\mathrm{Aut}(K)\times\mathrm{Aut}(Q)}a.$$

Now recall the following definition (introduced in [5]):

Definition 3.4. *Let $\psi:Q\to\mathrm{Out}(K)$ be an abstract kernel. A pair (ν,Φ) in $\mathrm{Aut}(K)\times\mathrm{Aut}(Q)$ is called compatible with respect to ψ if and only if $\psi\circ\Phi=\mu(p(\nu))\circ\psi$. The set of all compatible pairs with respect to ψ is denoted by $\mathrm{Comp}(\psi)$.*

Note that

Proposition 3.5. *Assume $K\rightarrowtail G\twoheadrightarrow Q$ is an extension compatible with $\psi:Q\to\mathrm{Out}(K)$. Then $\mathrm{Im}(\Theta)\subseteq\mathrm{Comp}(\psi)\subseteq p^{-1}(N_{\mathrm{Out}(K)}\psi(Q))\times\mathrm{Aut}(Q)$.*

Proof. If $(\nu,\Phi)\in\mathrm{Aut}(K)\times\mathrm{Aut}(Q)$ stabilizes the cohomology class $a\in H^2_\psi(Q,K)$ corresponding to the given extension, then $\mu(p(\nu))\circ\psi=\psi\circ\Phi$. This also implies that $p(\nu)$ belongs to the normalizer of $\psi(Q)$ in $\mathrm{Out}(K)$. ■

For an abstract kernel $\psi : Q \to \mathrm{Out}(K)$, the restriction map $\mathrm{Aut}(K) \to \mathrm{Aut}(Z(K))$ induces a map $\mathrm{Out}(K) \to \mathrm{Aut}(Z(K))$, which defines a Q-module structure on $Z(K)$ (also write ψ for this map). It is well known ([5]) that

Theorem 3.6. *Let $K \rightarrowtail G \twoheadrightarrow Q$ be a group extension realizing an abstract kernel $\psi : Q \to \mathrm{Out}(K)$. Then*

$$\mathrm{Ker}(\Theta) = \mathrm{Ker}(A) \cap \mathrm{Ker}(B) \cong Z^1_\psi(Q, Z(K)).$$

Proof. To fix notations, we sketch the proof of this result. Let $a = <\varphi, c>$, $G = K \times_{(\varphi,c)} Q$ and $\sigma \in \mathrm{Ker}(\Theta)$. Then the map $\xi_\sigma : Q \to K$, defined by $\sigma(1, x) = (\xi_\sigma(x), x)$, takes its values in $Z(K)$ (because of (1)) and, translating (2), it satisfies $\xi_\sigma(xy) = \xi_\sigma(x) \cdot \varphi(x)(\xi_\sigma(y))$ (for all $x, y \in Q$), which means that $\xi_\sigma \in Z^1_\psi(Q, Z(K))$. It is now an easy exercise to show that $\xi : \mathrm{Ker}(\Theta) \to Z^1_\psi(Q, Z(K))$ sending $\sigma \mapsto \xi_\sigma$ is an isomorphism. ∎

Definition 3.7. *For each extension $K \rightarrowtail G \overset{j}{\twoheadrightarrow} Q$, realizing an abstract kernel $\psi : Q \to \mathrm{Out}(K)$, we define $\overline{B}^1_\psi(Q, Z(K)) = \{\xi_{\mu(g)} \parallel g \in C_G K, j(g) \in Z(Q)\}$.*

At first sight, this notion might look rather peculiar but in the following proposition, we exhibit its importance in this context.

Proposition 3.8. *Assume $K \rightarrowtail G \twoheadrightarrow Q$ is a group extension compatible with $\psi : Q \to \mathrm{Out}(K)$. Then*

1. *$B^1_\psi(Q, Z(K))$ is a subgroup of $\overline{B}^1_\psi(Q, Z(K))$.*

2. *$\overline{B}^1_\psi(Q, Z(K))$ is a subgroup of $Z^1_\psi(Q, Z(K))$.*

3. *$\mathrm{Ker}(\Theta) \cap \mathrm{Inn}(G) \cong \overline{B}^1_\psi(Q, Z(K)) \cong (j^{-1}(Z(Q)) \cap C_G K)/_{Z(G)}$.*

Proof. Let $G = K \times_{(\varphi,c)} Q$. Take $\gamma \in B^1_\psi(Q, Z(K))$ sending $x \in Q \mapsto k_0 \cdot \varphi(x)(k_0^{-1})$, for some $k_0 \in Z(K)$. Then $\lambda = \xi_{\mu(k_0,1)} \in \overline{B}^1_\psi(Q, Z(K))$. Now take $k_0 \in K$ and $x_0 \in Q$. Then $\mu(k_0, x_0) \in \mathrm{Ker}(\Theta)$ if and only if $(k_0, x_0) \in C_G(K)$ and $x_0 \in Z(Q)$. Therefore, by definition and because of the proof of Proposition 3.6, $\overline{B}^1_\psi(Q, Z(K))$ is exactly the subgroup of $Z^1_\psi(Q, Z(K)) \cong \mathrm{Ker}(\Theta)$ corresponding to $\mathrm{Ker}(\Theta) \cap \mathrm{Inn}(G)$. ∎

Now define

Definition 3.9. *Let $K \rightarrowtail G \twoheadrightarrow Q$ be a group extension compatible with $\psi : Q \to \mathrm{Out}(K)$. Write $\overline{H}^1_\psi(Q, Z(K)) = Z^1_\psi(Q, Z(K))/_{\overline{B}^1_\psi(Q, Z(K))}$.*

Obviously, $\overline{B}^1_\psi(Q, Z(K))/_{B^1_\psi(Q, Z(K))} \rightarrowtail H^1_\psi(Q, Z(K)) \twoheadrightarrow \overline{H}^1_\psi(Q, Z(K))$ is exact.

We are now ready to summarize the above results in the following

Theorem 3.10. *If $\psi : Q \to \mathrm{Out}(K)$ is an abstract kernel and $K \rightarrowtail G \overset{j}{\twoheadrightarrow} Q$ is an extension compatible with ψ determining a cohomology class $a \in H^2_\psi(Q, K)$, then there is a commutative diagram of groups and group homomorphisms such that both the rows and the columns are exact sequences:*

$$
\begin{array}{ccccc}
\overline{B}^1_\psi(Q, Z(K)) & \rightarrowtail & \mathrm{Inn}(G) & \overset{\Theta}{\twoheadrightarrow} & G/(j^{-1}(Z(Q)) \cap C_G K) \\
\downarrow & & \downarrow & & \downarrow \\
Z^1_\psi(Q, Z(K)) & \rightarrowtail & \mathrm{Aut}(G, K) & \overset{\Theta}{\twoheadrightarrow} & \mathrm{Stab}_{\mathrm{Aut}(K)\times\mathrm{Aut}(Q)}a \\
\downarrow & & \downarrow & & \downarrow \\
\overline{H}^1_\psi(Q, Z(K)) & \rightarrowtail & \mathrm{Out}(G, K) & \twoheadrightarrow & Q_0
\end{array}
\tag{3}
$$

Proof. Most of the proof of this theorem is already finished by combining Theorems 3.3, 3.6 and Proposition 3.8. To understand the upper-right corner of the diagram, it is sufficient to realize that $\mathrm{Inn}(G) \cong G/Z(G)$. ■

Remark 3.11. The middle row in (3) is the main component in the diagram and is conceptually the exact sequence originally due to C. Wells ([5]) although he did not describe the image of Θ concretely (as being a stabilizer). Additionally, here we also obtain information on the outer automorphisms of the group extension.

4 Alternative automorphism diagrams

Using the terminology and notations introduced above, what can be done to reach alternative diagrams which provide extra information on the (outer) automorphism group of a group extension $K \rightarrowtail G \twoheadrightarrow Q$, is concentrating on the homomorphisms $A : \mathrm{Aut}(G, K) \to \mathrm{Aut}(K)$ and $B : \mathrm{Aut}(G, K) \to \mathrm{Aut}(Q)$ respectively, rather than on $\Theta : \mathrm{Aut}(G, K) \to \mathrm{Aut}(K) \times \mathrm{Aut}(Q)$ as we did above.

For example, it is natural to define the following subgroup of $\mathrm{Aut}(K)$:

Definition 4.1. *Assume K and Q are groups such that $\mathcal{H}^2(Q, K)$ is non-empty. For each element $a =< \varphi, c >\in \mathcal{H}^2(Q, K)$, we define the subgroup $\mathcal{M}_{K,a}$ of* $\mathrm{Aut}(K)$ *as*

$$\mathcal{M}_{K,a} = \{\nu \in \mathrm{Aut}(K) \parallel \exists \Phi \in \mathrm{Aut}(Q) \textit{ such that } {}^{(\nu,\Phi)}< \varphi, c >=< \varphi, c >\}.$$

Of course,

Proposition 4.2. *Assume $K \rightarrowtail G \twoheadrightarrow Q$ an extension determing a cohomology class $a \in \mathcal{H}^2(Q, K)$. Then,*

$$\mathrm{Im}(A) = \mathcal{M}_{K,a} \subseteq p^{-1}(N_{\mathrm{Out}(K)}\psi(Q)).$$

Proof. Because of Proposition 3.1, we immediately have that for each $\sigma \in \mathrm{Aut}(G, K)$, $A(\sigma) \in \mathcal{M}_{K,a}$. Conversely, if $\nu \in \mathcal{M}_{K,a}$ with $a =< \varphi, c >$, then there exists $\Phi \in \mathrm{Aut}(Q)$ such that ${}^{(\nu,\Phi)}< \varphi, c > =< \varphi, c >$ and because of Proposition 3.2, there is an automorphism σ of G such that $A(\sigma) = \nu$. ■

Analogously, we introduce the following subgroup of $\mathrm{Aut}(Q)$:

Definition 4.3. *Let K and Q be groups such that $\mathcal{H}^2(Q,K)$ is non-empty. For each element $a = <\varphi, c> \in \mathcal{H}^2(Q,K)$, we define the subgroup $\mathcal{M}_{Q,a}$ of $\mathrm{Aut}(Q)$ as*

$$\mathcal{M}_{Q,a} = \{\Phi \in \mathrm{Aut}(Q) \parallel \exists \nu \in \mathrm{Aut}(K) \text{ such that } {}^{(\nu,\Phi)}<\varphi, c> = <\varphi, c>\}.$$

Then

Proposition 4.4. *If $K \rightarrowtail G \twoheadrightarrow Q$ is a group extension determing a cohomology class $a \in \mathcal{H}^2(Q,K)$, then $\mathrm{Im}(B) = \mathcal{M}_{Q,a}$.*

Proof. Proposition 3.1 implies that for each $\sigma \in \mathrm{Aut}(G,K)$, $B(\sigma) \in \mathcal{M}_{Q,a}$. Conversely, for any $\Phi \in \mathcal{M}_{Q,a}$, with $a = <\varphi, c>$, there exists $\nu \in \mathrm{Aut}(K)$ such that ${}^{(\nu,\Phi)}<\varphi, c> = <\varphi, c>$ and it follows from Proposition 3.2 that there exists $\sigma \in \mathrm{Aut}(G,K)$ such that $B(\sigma) = \Phi$. ■

Referring to the action of $\mathrm{Aut}(K)$ on $\mathcal{H}^2(Q,K)$ (as introduced in Proposition 2.1), it is now also easy to conclude that

Proposition 4.5. *Let $K \rightarrowtail G \twoheadrightarrow Q$ be an extension. Write $\psi : Q \to \mathrm{Aut}(K)$ for the induced abstract kernel and $a \in H^2_\psi(Q,K)$ for the corresponding cohomology class. Then*

$$A(\mathrm{Ker}(B)) = \mathrm{Stab}_{\mathrm{Aut}(K)} a \subseteq p^{-1}(C_{\mathrm{Out}(K)} \psi(Q)).$$

Proof. Assume $\sigma \in \mathrm{Aut}(G,K)$ such that $B(\sigma) = 1_Q$, the identity on Q. Because of Proposition 3.1, $(A(\sigma), 1_Q)$ stabilizes a. It immediately follows that $A(\sigma) \in \mathrm{Stab}_{\mathrm{Aut}(K)} a$. Conversely, let $\nu \in \mathrm{Stab}_{\mathrm{Aut}(K)} a$. Then $(\nu, 1_Q)$ stabilizes a and the desired result now follows from Proposition 3.2. Finally, note that if $\nu \in \mathrm{Aut}(K)$ belongs to the stabilizer of $a \in H^2_\psi(Q,K)$, then $\mu(p(\nu)) \circ \psi = \psi$. So $\mathrm{Stab}_{\mathrm{Aut}(K)} a$ is contained in the inverse image in $\mathrm{Aut}(K)$ of the centralizer of $\psi(Q)$ in $\mathrm{Out}(K)$. ■

Using the action of $\mathrm{Aut}(Q)$ on $\mathcal{H}^2(Q,K)$ (as introduced in Proposition 2.2), we have

Proposition 4.6. *Assume $K \rightarrowtail G \twoheadrightarrow Q$ is a group extension compatible with an abstract kernel $\psi : Q \to \mathrm{Out}(K)$ and with cohomology class $a \in H^2_\psi(Q,K)$. Then*

$$B(\mathrm{Ker}(A)) = \mathrm{Stab}_{\mathrm{Aut}(Q)} a \subseteq \{\Phi \in \mathrm{Aut}(Q) \parallel \psi \circ \Phi = \psi\}.$$

Proof. If for $\sigma \in \mathrm{Aut}(G,K)$, $A(\sigma) = 1_K$, the identity on K, then $(1_K, B(\sigma))$ belongs to the stabilizer of a (Proposition 3.1). Thus $B(\sigma) \in \mathrm{Stab}_{\mathrm{Aut}(Q)} a$. Conversely, if $\Phi \in \mathrm{Stab}_{\mathrm{Aut}(Q)} a$, then $(1_K, \Phi)$ stabilizes $a \in H^2_\psi(Q,K)$ (observe that $\psi \circ \Phi = \psi$) and applying Proposition 3.2 finishes the proof. ■

Combining these properties, we conclude with

Theorem 4.7. *Let $\psi : Q \to \mathrm{Out}(K)$ be an injective abstract kernel and $K \rightarrowtail G \overset{j}{\twoheadrightarrow} Q$ is an extension realizing ψ and determining a cohomology class $a \in H^2_\psi(Q,K)$. Then there are two commutative diagrams built with exact rows and columns:*

$$\begin{tikzcd}
\Gamma_A^0 = \mathrm{Ker}(A)\cap\mathrm{Inn}(G) \arrow[r, tail] \arrow[d, tail] & \mathrm{Inn}(G) \arrow[r, two heads, "A"] \arrow[d, tail] & G/_{C_GK} \arrow[d, tail] \\
\Gamma_A^1=\mathrm{Ker}(A) \arrow[r, tail] \arrow[d, two heads] & \mathrm{Aut}(G,K) \arrow[r, two heads, "A"] \arrow[d, two heads] & \mathcal{M}_{K,a} \arrow[d, two heads] \\
\Gamma_A^2 = \mathrm{Ker}(A)/_{(\mathrm{Ker}(A)\cap\mathrm{Inn}(G))} \arrow[r, tail] & \mathrm{Out}(G,K) \arrow[r, two heads] & Q_1
\end{tikzcd} \tag{4}$$

and

$$\begin{tikzcd}
\overline{B}^1_\psi(Q,Z(K)) \arrow[r, tail] \arrow[d, tail] & \Gamma_A^0 \arrow[r, two heads, "B"] \arrow[d, tail] & \mathrm{Ker}(\psi)/_{(\mathrm{Ker}(\psi)\cap Z(Q))} \arrow[d, tail] \\
Z^1_\psi(Q,Z(K)) \arrow[r, tail] \arrow[d, two heads] & \Gamma_A^1 \arrow[r, two heads, "B"] \arrow[d, two heads] & \mathrm{Stab}_{\mathrm{Aut}(Q)}a \arrow[d, two heads] \\
\overline{H}^1_\psi(Q,Z(K)) \arrow[r, tail] & \Gamma_A^2 \arrow[r, two heads] & Q_3
\end{tikzcd} \tag{5}$$

where

$$Q_1 = \mathcal{M}_{K,a}/_{A(\mathrm{Inn}(G))} \cong \overline{\mathcal{M}_{K,a}}/_{\psi(Q)}$$

with $\overline{\mathcal{M}_{K,a}} = \mathcal{M}_{K,a}/_{\mathrm{Inn}(K)} \subseteq \mathrm{Out}(K)$ *and* $Q_1 \rightarrowtail N_{\mathrm{Out}(K)}\psi(Q)/_{\psi(Q)}$.

Proof. This easily follows from Propositions 4.2 and 4.6. The reader only has to note that $B(\mathrm{Ker}(A)\cap\mathrm{Inn}(G))$ is isomorphic to

$$\left(C_GK/_{Z(G)}\right)/_{\left((j^{-1}(Z(Q))\cap C_GK)/_{Z(G)}\right)} \cong \mathrm{Ker}(\psi)/_{(\mathrm{Ker}(\psi)\cap Z(Q))}$$

because $\overline{B}^1_\psi(Q,Z(K)) \cong (j^{-1}(Z(Q))\cap C_GK)/_{Z(G)}$ (Proposition 3.8), $\Gamma_A^0 \cong C_GK/_{Z(G)}$ and $\mathrm{Ker}(\psi) \cong C_GK/_{Z(K)}$. $\mathrm{Inn}(K)$ clearly is normal in $\mathcal{M}_{K,a}$. Since $A(\mathrm{Inn}(G)) \cong G/_{C_GK}$ and $\mathrm{Inn}(K) \cong K/_{Z(K)}$, it follows that $A(\mathrm{Inn}(G))/_{\mathrm{Inn}(K)} \cong Q/_{\mathrm{Ker}(\psi)} \cong \psi(Q)$. Therefore, $Q_1 \cong \overline{\mathcal{M}_{K,a}}/_{\psi(Q)}$ and the announced embedding follows from Proposition 4.2. ∎

Somehow "dual" to the above is the following:

Theorem 4.8. *Assume $K \rightarrowtail G \overset{j}{\twoheadrightarrow} Q$ is a group extension compatible with $\psi : Q \to \mathrm{Out}(K)$ and corresponding to a cohomology class $a \in H^2_\psi(Q,K)$. Then the following are commutative diagrams built with exact rows and columns:*

$$\begin{tikzcd}
\Gamma_B^0 = \mathrm{Ker}(B)\cap\mathrm{Inn}(G) \arrow[r, tail] \arrow[d, tail] & \mathrm{Inn}(G) \arrow[r, two heads, "B"] \arrow[d, tail] & \mathrm{Inn}(Q) \arrow[d, tail] \\
\Gamma_B^1=\mathrm{Ker}(B) \arrow[r, tail] \arrow[d, two heads] & \mathrm{Aut}(G,K) \arrow[r, two heads, "B"] \arrow[d, two heads] & \mathcal{M}_{Q,a} \arrow[d, two heads] \\
\Gamma_B^2 = \mathrm{Ker}(B)/_{(\mathrm{Ker}(B)\cap\mathrm{Inn}(G))} \arrow[r, tail] & \mathrm{Out}(G,K) \arrow[r, two heads] & Q_2
\end{tikzcd} \tag{6}$$

and

$$\begin{array}{ccccc} \overline{B}^1_\psi(Q,Z(K)) & \rightarrowtail & \Gamma^0_B & \overset{A}{\twoheadrightarrow} & p^{-1}(\psi(Z(Q))) \\ \downarrow & & \downarrow & & \downarrow \\ Z^1_\psi(Q,Z(K)) & \rightarrowtail & \Gamma^1_B & \overset{A}{\twoheadrightarrow} & \mathrm{Stab}_{\mathrm{Aut}(K)}a \\ \downarrow & & \downarrow & & \downarrow \\ \overline{H}^1_\psi(Q,Z(K)) & \rightarrowtail & \Gamma^2_B & \twoheadrightarrow & Q_4 \end{array} \tag{7}$$

where

$$Q_4 = \mathrm{Stab}_{\mathrm{Aut}(K)}a/_{A(\mathrm{Ker}(B)\cap\mathrm{Inn}(G))} \cong \overline{\mathrm{Stab}}_{\mathrm{Aut}(K)}a/_{\psi(Z(Q))}$$

with

$$\overline{\mathrm{Stab}}_{\mathrm{Aut}(K)}a = \mathrm{Stab}_{\mathrm{Aut}(K)}a/_{\mathrm{Inn}(K)} \subseteq \mathrm{Out}(K)$$

and

$$Q_4 \rightarrowtail C_{\mathrm{Out}(K)}\psi(Q)/_{\psi(Z(Q))}.$$

Proof. This is an immediate consequence of Propositions 4.4 and 4.5. Only observe that if $a = <\varphi, c>$ and $(k,x) \in G = K \times_{(\varphi,c)} Q$, then $A(\mu(k,x)) = \mu(k) \circ \varphi(x)$ and $B(\mu(k,x)) = \mu(x)$. This implies that $\mathrm{Ker}(B) \cap \mathrm{Inn}(G) \cong j^{-1}(Z(Q))/_{Z(G)}$ and $A(\mathrm{Ker}(B)\cap\mathrm{Inn}(G))$ is the preimage in $\mathrm{Aut}(K)$ of $\psi(Z(Q))$. Obviously, $\mathrm{Inn}(K)$ is contained in $\mathrm{Stab}_{\mathrm{Aut}(K)}a$ and hence Q_4 is isomorphic to $\overline{\mathrm{Stab}}_{\mathrm{Aut}(K)}a/_{\psi(Z(Q))}$. And because of Proposition 4.5, Q_4 hence embeds into $C_{\mathrm{Out}(K)}\psi(Q)/_{\psi(Z(Q))}$. ∎

The relation between diagram (3) and the alternative automorphism diagrams established above, is given by:

Proposition 4.9. *If $K \rightarrowtail G \overset{j}{\twoheadrightarrow} Q$ is a group extension inducing an abstract kernel $\psi : Q \to \mathrm{Out}(K)$ and a cohomology class $a \in H^2_\psi(Q,K)$, then the following are commutative diagrams built with exact rows and columns:*

$$\begin{array}{ccccc} \mathrm{Ker}(\psi)/_{(\mathrm{Ker}(\psi)\cap Z(Q))} & \rightarrowtail & G/_{(j^{-1}(Z(Q))\cap C_GK)} & \twoheadrightarrow & G/_{C_GK} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Stab}_{\mathrm{Aut}(Q)}a & \rightarrowtail & \mathrm{Stab}_{\mathrm{Aut}(K)\times\mathrm{Aut}(Q)}a & \twoheadrightarrow & \mathcal{M}_{K,a} \\ \downarrow & & \downarrow & & \downarrow \\ Q_3 & \rightarrowtail & Q_0 & \twoheadrightarrow & Q_1 \end{array}$$

and

$$\begin{array}{ccccc} p^{-1}(\psi(Z(Q))) & \rightarrowtail & G/_{(j^{-1}(Z(Q))\cap C_GK)} & \twoheadrightarrow & \mathrm{Inn}(Q) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Stab}_{\mathrm{Aut}(K)}a & \rightarrowtail & \mathrm{Stab}_{\mathrm{Aut}(K)\times\mathrm{Aut}(Q)}a & \twoheadrightarrow & \mathcal{M}_{Q,a} \\ \downarrow & & \downarrow & & \downarrow \\ Q_4 & \rightarrowtail & Q_0 & \twoheadrightarrow & Q_2 \end{array}$$

Proof. This follows immediately when applying the canonical epimorphisms $\mathrm{Aut}(K) \times \mathrm{Aut}(Q) \twoheadrightarrow \mathrm{Aut}(K)$ and $\mathrm{Aut}(K) \times \mathrm{Aut}(Q) \twoheadrightarrow \mathrm{Aut}(Q)$ on $\mathrm{Im}(\Theta)$. ∎

We conclude with a "cube automorphism diagram", which arises by glueing together diagrams (4), (5), (6) and (7).

Theorem 4.10. *Let $K \rightarrowtail G \twoheadrightarrow Q$ be an extension realizing $\psi : Q \to \mathrm{Out}(K)$ and with $a \in H^2_\psi(Q, K)$ as cohomology class. Then the following "cube" is a commutative diagram built with exact rows and columns:*

$$\begin{tikzcd}
 & & & \mathrm{Ker}(\psi)/(\mathrm{Ker}(\psi)\cap Z(Q)) \arrow[rrr] \arrow[ddd] & & & \mathrm{Inn}(Q) \arrow[rr] \arrow[ddd] & & \Delta_1 \arrow[ddd] \\
 & \Gamma^0_A \arrow[urr, "B"] \arrow[rrr] \arrow[ddd] & & & \mathrm{Inn}(G) \arrow[urr, "B"] \arrow[rrr, "A"] \arrow[ddd] & & & G/C_G K \arrow[ur] \arrow[ddd] & \\
\overline{B}^1_\psi(Q, Z(K)) \arrow[ur] \arrow[rr] \arrow[ddd] & & \Gamma^0_B \arrow[urr] \arrow[rrr, "A"] \arrow[ddd] & & & p^{-1}(\psi(Z(Q))) \arrow[urr] \arrow[ddd] & & & \\
 & & & \mathrm{Stab}_{\mathrm{Aut}(Q)} a \arrow[rrr] \arrow[ddd] & & & \mathcal{M}_{Q,a} \arrow[rr] \arrow[ddd] & & \Delta_2 \arrow[ddd] \\
 & \Gamma^1_A \arrow[urr, "B"] \arrow[rrr] \arrow[ddd] & & & \mathrm{Aut}(G, K) \arrow[urr, "B"] \arrow[rrr, "A"] \arrow[ddd] & & & \mathcal{M}_{K,a} \arrow[ur] \arrow[ddd] & \\
Z^1_\psi(Q, Z(K)) \arrow[ur] \arrow[rr] \arrow[ddd] & & \Gamma^1_B \arrow[urr] \arrow[rrr, "A"] \arrow[ddd] & & & \mathrm{Stab}_{\mathrm{Aut}(K)} a \arrow[urr] \arrow[ddd] & & & \\
 & & & Q_3 \arrow[rrr] & & & Q_2 \arrow[rr] & & \Delta_3 \\
 & \Gamma^2_A \arrow[urr] \arrow[rrr] & & & \mathrm{Out}(G, K) \arrow[urr] \arrow[rrr] & & & Q_1 \arrow[ur] & \\
\overline{H}^1_\psi(Q, Z(K)) \arrow[ur] \arrow[rr] & & \Gamma^2_B \arrow[urr] \arrow[rrr] & & & Q_4 \arrow[urr] & & &
\end{tikzcd}$$

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