On dual Euler-Simpson formulae

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Abstract

The dual Euler-Simpson formulae are given. A number of inequalities, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or functions in L_p-spaces, are proved. The results are applied to obtain the error estimates for some quadrature rules..

1 Introduction

One of the elementary quadrature rules of closed type is the Simpson's rule based on the Simpson's formula [4, p. 45]

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880}f^{(4)}(\xi), \tag{1.1}$$

where $a \leq \xi \leq b$. A simple quadrature rule of open type, which is closely related to the Simpson's rule, is based on the following three-point formula [4, p. 71]

$$\begin{aligned}\int_a^b f(t)\mathrm{d}t &= \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] \\ &\quad + \frac{7(b-a)^5}{23040}f^{(4)}(\eta),\end{aligned} \tag{1.2}$$

where $a \leq \eta \leq b$. The formulae (1.1) and (1.2) are valid for any function f which has a continuous fourth derivative $f^{(4)}$ on $[a,b]$. P. S. Bullen [3] proved that, under certain convexity assumptions on f, the three-point quadrature rule based on the

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formula (1.2) is more accurate than the Simpson's quadrature rule. As pointed in [3], the formula (1.2) naturally appears in a pair with the formula (1.1) and we call it the dual Simpson formula. This dual Simpson formula is the key notion in this paper.

In the recent paper [5] the following two identities, named the extended Euler formulae, have been proved:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + T_n(x) + R_n^1(x) \tag{1.3}$$

and

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + T_{n-1}(x) + R_n^2(x), \tag{1.4}$$

where $T_0(x) = 0$ and

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right], \tag{1.5}$$

for $m \geq 1$, while

$$R_n^1(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]} B_n^*\left(\frac{x-t}{b-a}\right)\mathrm{d}f^{(n-1)}(t)$$

and

$$R_n^2(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]}\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right]\mathrm{d}f^{(n-1)}(t).$$

Here, as in the rest of the paper, we write $\int_{[a,b]} g(t)d\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi : [a,b] \to \mathbf{R}$ of bounded variation, and $\int_a^b g(t)dt$ for the Riemann integral. The identities (1.3) and (1.4) extend the well known formula for the expansion of an arbitrary function in Bernoulli polynomials [6, p. 17]. They hold for every function $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n \geq 1$, and for every $x \in [a,b]$. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k \geq 0$, are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t) = B_k(t), \text{ for } 0 \leq t < 1,$$

$$B_k^*(t+1) = B_k^*(t), \text{ for } t \in \mathbf{R}.$$

The Bernoulli polynomials $B_k(t)$, $k \geq 0$ are uniquely determined by the following identities

$$B_k'(t) = kB_{k-1}(t),\ k \geq 1;\quad B_0(t) = 1 \tag{1.6}$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1},\ k \geq 0. \tag{1.7}$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have

$$B_0(t) = 1,\ B_1(t) = t - \frac{1}{2},\ B_2(t) = t^2 - t + \frac{1}{6},\ B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \tag{1.8}$$

so that $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. From (1.7) it follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that $B_k^*(t)$ are continuous functions for $k \geq 2$. Moreover, using (1.6) we get

$$B_k^{*\prime} = kB_{k-1}^*(t),\ k \geq 1 \tag{1.9}$$

and this holds for every $t \in \mathbf{R}$ when $k \geq 3$, and for every $t \in \mathbf{R} \setminus \mathbf{Z}$ when $k = 1, 2$.

The aim of this paper is to establish a generalizations of the dual Simpson formula (1.2) and then, to give various error estimates for the quadrature rules which are based on such generalizations of (1.2).

In Section 2 we use the extended Euler formulae (1.3) and (1.4) to obtain two new integral identities. We call these new identities the dual Euler-Simpson formulae, since they generalize the dual Simpson formula (1.2).

In Section 3 we prove a number of inequalities related to the dual Euler-Simpson formulae, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or functions from the L_p-spaces.

Finally, in Section 4, we consider the repeated dual Euler-Simpson quadrature rule and the repeated modified dual Euler-Simpson quadrature rule which are based on the dual Euler-Simpson formulae. We give the error estimates for these quadrature rules when they are applied to the functions of various classes. Also, we show that, under certain assumptions on the involved functions, these quadrature rules can be more accurate than the analogous repeated Euler-Simpson quadrature rule considered in [6] .

2 Dual Euler-Simpson formulae

For $k \geq 1$ define the functions $G_k(t)$ and $F_k(t)$ as

$$G_k(t) := 2B_k^*\left(\frac{1}{4} - t\right) - B_k^*\left(\frac{1}{2} - t\right) + 2B_k^*\left(\frac{3}{4} - t\right),\ t \in \mathbf{R}$$

and

$$F_k(t) := G_k(t) - \tilde{B}_k,\ t \in \mathbf{R},$$

where

$$\tilde{B}_k := G_k(0) = 2B_k\left(\frac{1}{4}\right) - B_k\left(\frac{1}{2}\right) + 2B_k\left(\frac{3}{4}\right),\ k \geq 1.$$

Obviously, $G_k(t)$ and $F_k(t)$ are periodic functions of period 1 and continuous for $k \geq 2$. Thus, it is enough to know the behavior of these functions on the interval $[0,1]$. We shall investigate this behavior in the next section.

Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geq 1$. We introduce the following notation

$$D(a,b) := \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right].$$

Further, we define $\tilde{T}_0(a,b) := 0$ and, for $1 \leq m \leq n$,

$$\tilde{T}_m(a,b) := \frac{b-a}{3}\left[2T_m\left(\frac{3a+b}{4}\right) - T_m\left(\frac{a+b}{2}\right) + 2T_m\left(\frac{a+3b}{4}\right)\right],$$

where $T_m(x)$ is given by (1.5). It is easy to see that

$$\tilde{T}_m(a,b) = \frac{1}{3}\sum_{k=1}^{m} \frac{(b-a)^k}{k!}\tilde{B}_k\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]. \tag{2.1}$$

In the next theorem we establish two formulae which play the key role in this paper. We call them the dual Euler-Simpson formulae.

Theorem 1. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n \geq 1$. Then*

$$\int_a^b f(t)\mathrm{d}t = D(a,b) - \tilde{T}_n(a,b) + \tilde{R}_n^1(a,b), \tag{2.2}$$

where

$$\tilde{R}_n^1(a,b) = \frac{(b-a)^n}{3(n!)}\int_{[a,b]} G_n\left(\frac{t-a}{b-a}\right)\mathrm{d}f^{(n-1)}(t).$$

Also,

$$\int_a^b f(t)\mathrm{d}t = D(a,b) - \tilde{T}_{n-1}(a,b) + \tilde{R}_n^2(a,b), \tag{2.3}$$

where

$$\tilde{R}_n^2(a,b) = \frac{(b-a)^n}{3(n!)}\int_{[a,b]} F_n\left(\frac{t-a}{b-a}\right)\mathrm{d}f^{(n-1)}(t).$$

Proof. Put

$$x = \frac{3a+b}{4},\ \frac{a+b}{2},\ \frac{a+3b}{4}$$

in formula (1.3) to get three new formulae. Then multiply these new formulae by

$$\frac{2(b-a)}{3},\quad -\frac{b-a}{3},\ \frac{2(b-a)}{3},$$

respectively, and add. The result is the formula (2.2). The formula (2.3) is obtained from (1.4) by the same procedure. ∎

Remark 1. *Suppose that $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on $[a,b]$, for some $n \geq 1$. In this case (2.2) holds with*

$$\tilde{R}_n^1(a,b) = \frac{(b-a)^n}{3(n!)}\int_a^b G_n\left(\frac{t-a}{b-a}\right)f^{(n)}(t)\mathrm{d}t,$$

while (2.3) holds with

$$\tilde{R}_n^2(a,b) = \frac{(b-a)^n}{3(n!)}\int_a^b F_n\left(\frac{t-a}{b-a}\right)f^{(n)}(t)\mathrm{d}t.$$

By direct calculation and using (1.8), we get

$$\tilde{B}_1 = \tilde{B}_2 = \tilde{B}_3 = 0.$$

This implies, by (2.1),

$$\tilde{T}_1(a,b) = \tilde{T}_2(a,b) = \tilde{T}_3(a,b) = 0.$$

Also,

$$F_1(t) = G_1(t) = \begin{cases} -3t, & 0 \le t \le 1/4 \\ -3t+2, & 1/4 < t \le 1/2 \\ -3t+1, & 1/2 < t \le 3/4 \\ -3t+3, & 3/4 < t \le 1 \end{cases}, \tag{2.4}$$

$$F_2(t) = G_2(t) = \begin{cases} 3t^2, & 0 \le t \le 1/4 \\ 3t^2-4t+1, & 1/4 \le t \le 1/2 \\ 3t^2-2t, & 1/2 \le t \le 3/4 \\ 3t^2-6t+3, & 3/4 \le t \le 1 \end{cases} \tag{2.5}$$

and

$$F_3(t) = G_3(t) = \begin{cases} -3t^3, & 0 \le t \le 1/4 \\ -3t^3+6t^2-3t+\frac{3}{8}, & 1/4 \le t \le 1/2 \\ -3t^3+3t^2-\frac{3}{8}, & 1/2 \le t \le 3/4 \\ -3t^3+9t^2-9t+3, & 3/4 \le t \le 1 \end{cases}. \tag{2.6}$$

Applying (2.2) with $n = 1,\ 2,\ 3$ we get the identities

$$\begin{aligned} \int_a^b f(t)\mathrm{d}t - D(a,b) &= \frac{b-a}{3}\int_{[a,b]} G_1\left(\frac{t-a}{b-a}\right)\mathrm{d}f(t) \\ &= \frac{(b-a)^2}{6}\int_{[a,b]} G_2\left(\frac{t-a}{b-a}\right)\mathrm{d}f'(t) \\ &= \frac{(b-a)^3}{18}\int_{[a,b]} G_3\left(\frac{t-a}{b-a}\right)\mathrm{d}f''(t). \end{aligned}$$

The same identities are obtained from (2.3) with $n = 1,\ 2,\ 3$, since $\tilde{T}_0(a,b) = \tilde{T}_1(a,b) = \tilde{T}_2(a,b) = 0$ and $F_k(t) = G_k(t)$ for $k = 1,\ 2,\ 3$, while (2.3) with $n = 4$ yields the identity

$$\int_a^b f(t)\mathrm{d}t - D(a,b) = \frac{(b-a)^4}{72}\int_{[a,b]} F_4\left(\frac{t-a}{b-a}\right)\mathrm{d}f'''(t).$$

3 Some inequalities related to dual Euler-Simpson formulae

In this section we use the dual Euler-Simpson formulae established in Theorem 1 to prove a number of inequalities for various classes of functions. First, we need some properties of the functions $G_k(t)$ and $F_k(t)$ defined in the previous section. As we noted earlier, it is enough to know the behavior of these functions on the interval $[0,1]$

The Bernoulli polynomials have a property of symmetry with respect to $\frac{1}{2}$, that is [1, 23.1.8]

$$B_k(1-t) = (-1)^k B_k(t),\ 0 \le t \le 1,\ k \ge 1. \tag{3.1}$$

Setting $t = \frac{1}{2}$ and $t = \frac{1}{4}$ in (3.1) we get

$$B_k\left(\frac{1}{2}\right) = (-1)^k B_k\left(\frac{1}{2}\right) \text{ and } B_k\left(\frac{3}{4}\right) = (-1)^k B_k\left(\frac{1}{4}\right),$$

respectively. This implies

$$B_{2k-1}\left(\frac{1}{2}\right) = 0,\ B_{2k-1}\left(\frac{3}{4}\right) + B_{2k-1}\left(\frac{1}{4}\right) = 0,\ B_{2k}\left(\frac{3}{4}\right) = B_{2k}\left(\frac{1}{4}\right),\ k \ge 1,$$

so that we have

$$\begin{aligned}\tilde{B}_{2k-1} &= 2B_{2k-1}\left(\frac{1}{4}\right) - B_{2k-1}\left(\frac{1}{2}\right) + 2B_{2k-1}\left(\frac{3}{4}\right) \\ &= 0,\ k \geq 1, \end{aligned} \tag{3.2}$$

and

$$\tilde{B}_{2k} = 2B_{2k}\left(\frac{1}{4}\right) - B_{2k}\left(\frac{1}{2}\right) + 2B_{2k}\left(\frac{3}{4}\right) = 4B_{2k}\left(\frac{1}{4}\right) - B_{2k}\left(\frac{1}{2}\right),\ k \geq 1.$$

Also, we have [1, 23.1.21, 23.1.22]

$$B_{2k}\left(\frac{1}{2}\right) = -\left(1 - 2^{1-2k}\right) B_{2k},\ B_{2k}\left(\frac{1}{4}\right) = -2^{-2k}\left(1 - 2^{1-2k}\right) B_{2k},\ k \geq 1,$$

which gives the formula

$$\tilde{B}_{2k} = \left(8 \cdot 2^{-4k} - 6 \cdot 2^{-2k} + 1\right) B_{2k},\ k \geq 1. \tag{3.3}$$

Now, by (3.2) we have

$$F_{2k-1}(t) = G_{2k-1}(t),\ k \geq 1. \tag{3.4}$$

Also,

$$F_{2k}(t) = G_{2k}(t) - \tilde{B}_{2k},\ k \geq 1, \tag{3.5}$$

where $\tilde{B}_{2k}$ is given by (3.3). Further, the points 0 and 1 are the zeros of $F_n(t)$, that is

$$F_n(0) = F_n(1) = 0,\ n \geq 1.$$

As we shall see below, 0 and 1 are the only zeros of $F_n(t)$ for $n = 2k$, $k \geq 2$, while for $n = 2k - 1$, $k \geq 2$ we have

$$F_{2k-1}\left(\frac{1}{2}\right) = G_{2k-1}\left(\frac{1}{2}\right) = 2B_{2k-1}\left(\frac{3}{4}\right) - B_{2k-1} + 2B_{2k-1}\left(\frac{1}{4}\right) = 0.$$

We shall see that 0, $\frac{1}{2}$ and 1 are the only zeros of $F_{2k-1}(t) = G_{2k-1}(t)$, for $k \geq 2$. Also, note that for $n = 2k$, $k \geq 1$ we have

$$G_{2k}(0) = G_{2k}(1) = \tilde{B}_{2k} = \left(8 \cdot 2^{-4k} - 6 \cdot 2^{-2k} + 1\right) B_{2k}$$

and

$$G_{2k}\left(\frac{1}{2}\right) = 4B_{2k}\left(\frac{1}{4}\right) - B_{2k} = \left(8 \cdot 2^{-4k} - 4 \cdot 2^{-2k} - 1\right) B_{2k},$$

while

$$F_{2k}\left(\frac{1}{2}\right) = G_{2k}\left(\frac{1}{2}\right) - \tilde{B}_{2k} = -2\left(1 - 2^{-2k}\right) B_{2k}. \tag{3.6}$$

Lemma 1. *For $k \geq 2$ we have*

$$G_k(1-t) = (-1)^k G_k(t),\ 0 \leq t \leq 1,$$

and

$$F_k(1-t) = (-1)^k F_k(t),\ 0 \leq t \leq 1.$$

Proof. As we noted in Introduction, the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$. Therefore, for $k \geq 2$ and $0 \leq t \leq 1$ we have

$$B_k^*\left(\frac{1}{4}-t\right)=\begin{cases} B_k\left(\frac{1}{4}-t\right), & 0 \leq t \leq \frac{1}{4} \\ B_k\left(\frac{5}{4}-t\right), & \frac{1}{4} \leq t \leq 1, \end{cases}$$

and, using (3.1),

$$\begin{aligned} B_k^*\left(\frac{3}{4}+t\right) &= \begin{cases} B_k\left(\frac{3}{4}+t\right), & 0 \leq t \leq \frac{1}{4} \\ B_k\left(-\frac{1}{4}+t\right), & \frac{1}{4} \leq t \leq 1 \end{cases} \\ &= \begin{cases} (-1)^k B_k\left(\frac{1}{4}-t\right), & 0 \leq t \leq \frac{1}{4} \\ (-1)^k B_k\left(\frac{5}{4}-t\right), & \frac{1}{4} \leq t \leq 1. \end{cases} \end{aligned}$$

Comparing the above equalities, we see that

$$B_k^*\left(\frac{3}{4}+t\right)=(-1)^k B_k^*\left(\frac{1}{4}-t\right), \ 0 \leq t \leq 1.$$

By the similar observation we get

$$B_k^*\left(\frac{1}{2}+t\right)=(-1)^k B_k^*\left(\frac{1}{2}-t\right), \ 0 \leq t \leq 1,$$

and

$$B_k^*\left(\frac{1}{4}+t\right)=(-1)^k B_k^*\left(\frac{3}{4}-t\right), \ 0 \leq t \leq 1.$$

Using these identities, we get

$$\begin{aligned} G_k(1-t) &= 2B_k^*\left(-\frac{3}{4}+t\right)-B_k^*\left(-\frac{1}{2}+t\right)+2B_k^*\left(-\frac{1}{4}+t\right) \\ &= 2B_k^*\left(\frac{1}{4}+t\right)-B_k^*\left(\frac{1}{2}+t\right)+2B_k^*\left(\frac{3}{4}+t\right) \\ &= (-1)^k\left[2B_k^*\left(\frac{3}{4}-t\right)-B_k^*\left(\frac{1}{2}-t\right)+2B_k^*\left(\frac{1}{4}-t\right)\right] \\ &= (-1)^k G_k(t), \end{aligned}$$

which proves the first identity. Further, we have $\tilde{B}_k=(-1)^k\tilde{B}_k$, since (3.2) holds, so that

$$F_k(1-t)=G_k(1-t)-\tilde{B}_k=(-1)^k\left[G_k(t)-\tilde{B}_k\right]=(-1)^k F_k(t),$$

which proves the second identity. ∎

Note that the identities established in Lemma 1 are valid for $k=1$, too, except at the points $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ of discontinuity of $F_1(t)=G_1(t)$.

Lemma 2. *For $k \geq 2$ the function $G_{2k-1}(t)$ has no zeros in the interval $\left(0, \frac{1}{2}\right)$. The sign of this function is determined by*

$$(-1)^{k-1}G_{2k-1}(t)>0, \ 0<t<\frac{1}{2}.$$

Proof. For $k=2$, $G_3(t)$ is given by (2.6) and it is easy to see that

$$-G_3(t) > 0,\ 0 < t < \frac{1}{2}.$$

Thus, our assertion is true for $k=2$. Now, assume that $k \geq 3$. Then $2k-1 \geq 5$ and $G_{2k-1}(t)$ is continuous and twice differentiable function. Using (1.9) we get

$$G'_{2k-1}(t) = -(2k-1)G_{2k-2}(t)$$

and

$$G''_{2k-1}(t) = (2k-1)(2k-2)G_{2k-3}(t).$$

We know that 0 and $\frac{1}{2}$ are the zeros of $G_{2k-1}(t)$. Let us suppose that some α, $0 < \alpha < \frac{1}{2}$, is also a zero of $G_{2k-1}(t)$. Then inside each of the intervals $(0,\alpha)$ and $\left(\alpha, \frac{1}{2}\right)$ the derivative $G'_{2k-1}(t)$ must have at least one zero, say β_1, $0 < \beta_1 < \alpha$ and β_2, $\alpha < \beta_2 < \frac{1}{2}$. Therefore, the second derivative $G''_{2k-1}(t)$ must have at least one zero inside the interval (β_1, β_2). Thus, from the assumption that $G_{2k-1}(t)$ has a zero inside the interval $\left(0, \frac{1}{2}\right)$, it follows that $(2k-1)(2k-2)G_{2k-3}(t)$ also has a zero inside this interval. From this it follows that $G_3(t)$ would have a zero inside the interval $\left(0, \frac{1}{2}\right)$, which is not true. Thus, $G_{2k-1}(t)$ can not have a zero inside the interval $\left(0, \frac{1}{2}\right)$. To determine the sign of $G_{2k-1}(t)$, note that

$$G_{2k-1}\left(\frac{1}{4}\right) = 2B_{2k-1}(0) - B_{2k-1}\left(\frac{1}{4}\right) + 2B_{2k-1}\left(\frac{1}{2}\right) = -B_{2k-1}\left(\frac{1}{4}\right).$$

We have [1, 23.1.14]

$$(-1)^k B_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2},$$

which implies

$$(-1)^{k-1} G_{2k-1}\left(\frac{1}{4}\right) = (-1)^k B_{2k-1}\left(\frac{1}{4}\right) > 0.$$

Consequently, we have

$$(-1)^{k-1} G_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2}.$$

∎

Corollary 1. *For $k \geq 2$ the functions $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly increasing on the interval $\left(0, \frac{1}{2}\right)$, and strictly decreasing on the interval $\left(\frac{1}{2}, 1\right)$. Consequently,* 0 *and* 1 *are the only zeros of $F_{2k}(t)$ in the interval $[0,1]$ and*

$$\max_{t \in [0,1]} |F_{2k}(t)| = 2\left(1 - 2^{-2k}\right)|B_{2k}|,\ k \geq 2.$$

Also, we have

$$\max_{t \in [0,1]} |G_{2k}(t)| = \left(1 + 4 \cdot 2^{-2k} - 8 \cdot 2^{-4k}\right)|B_{2k}|,\ k \geq 2.$$

Proof. Using (1.9) we get

$$\left[(-1)^k F_{2k}(t)\right]' = \left[(-1)^k G_{2k}(t)\right]' = 2k(-1)^{k-1} G_{2k-1}(t)$$

and $(-1)^{k-1}G_{2k-1}(t) > 0$ for $0 < t < \frac{1}{2}$, by the Lemma 2. Thus, $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly increasing on the interval $\left(0, \frac{1}{2}\right)$. Also, by the Lemma 1, we have $F_{2k}(1-t) = F_{2k}(t)$, $0 \le t \le 1$ and $G_{2k}(1-t) = G_{2k}(t)$, $0 \le t \le 1$, which implies that $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly decreasing on the interval $\left(\frac{1}{2}, 1\right)$. Further, $F_{2k}(0) = F_{2k}(1) = 0$, which implies that $|F_{2k}(t)|$ achieves its maximum at $t = \frac{1}{2}$, that is

$$\max_{t\in[0,1]} |F_{2k}(t)| = \left|F_{2k}\left(\frac{1}{2}\right)\right| = 2\left(1 - 2^{-2k}\right)|B_{2k}|.$$

Also,

$$\begin{aligned}
&\max_{t\in[0,1]} |G_{2k}(t)| \\
= \;& \max\left\{|G_{2k}(0)|, \left|G_{2k}\left(\frac{1}{2}\right)\right|\right\} \\
= \;& \max\left\{\left(8\cdot 2^{-4k} - 6\cdot 2^{-2k} + 1\right)|B_{2k}|, \left(1 + 4\cdot 2^{-2k} - 8\cdot 2^{-4k}\right)|B_{2k}|\right\} \\
= \;& \left(1 + 4\cdot 2^{-2k} - 8\cdot 2^{-4k}\right)|B_{2k}|,
\end{aligned}$$

which completes the proof. ∎

Corollary 2. *Assume $k \ge 2$. Then we have*

$$\int_0^1 |F_{2k-1}(t)|\,\mathrm{d}t = \int_0^1 |G_{2k-1}(t)|\,\mathrm{d}t = \frac{2}{k}\left(1 - 2^{-2k}\right)|B_{2k}|.$$

Also, we have

$$\int_0^1 |F_{2k}(t)|\,\mathrm{d}t = \left|\tilde{B}_{2k}\right| = \left(8\cdot 2^{-4k} - 6\cdot 2^{-2k} + 1\right)|B_{2k}|$$

and

$$\int_0^1 |G_{2k}(t)|\,\mathrm{d}t \le 2\left|\tilde{B}_{2k}\right| = 2\left(8\cdot 2^{-4k} - 6\cdot 2^{-2k} + 1\right)|B_{2k}|.$$

Proof. Using (1.9) it is easy to see that

$$G'_m(t) = -mG_{m-1}(t),\ m \ge 3. \tag{3.7}$$

By (3.4) we have $\int_0^1 |F_{2k-1}(t)|dt = \int_0^1 |G_{2k-1}(t)|dt$. Now, using Lemma 1, Lemma 2 and (3.7) we get

$$\begin{aligned}
\int_0^1 |G_{2k-1}(t)|\,\mathrm{d}t = 2\left|\int_0^{\frac{1}{2}} G_{2k-1}(t)\mathrm{d}t\right| &= 2\left|-\frac{1}{2k}\, G_{2k}(t)|_0^{\frac{1}{2}}\right| \\
&= \frac{1}{k}\left|G_{2k}\left(\frac{1}{2}\right) - G_{2k}(0)\right| = \frac{2}{k}\left(1 - 2^{-2k}\right)|B_{2k}|,
\end{aligned}$$

which proves the first assertion. By the Corollary 1, $F_{2k}(t)$ does not change the sign on the interval $(0,1)$. Therefore, using (3.5) and (3.7), we get

$$\begin{aligned}\int_0^1 |F_{2k}(t)|\,\mathrm{d}t &= \left|\int_0^1 F_{2k}(t)\mathrm{d}t\right| = \left|\int_0^1 \left[G_{2k}(t) - \tilde{B}_{2k}\right]\mathrm{d}t\right| \\ &= \left|-\frac{1}{2k+1}\,G_{2k+1}(t)|_0^1 - \tilde{B}_{2k}\right| = \left|\tilde{B}_{2k}\right|\end{aligned}$$

and $\tilde{B}_{2k}$ is given by (3.3). This proves the second assertion. Finally, we use (3.5) again and the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,\mathrm{d}t = \int_0^1 \left|F_{2k}(t) + \tilde{B}_{2k}\right|\mathrm{d}t \le \int_0^1 |F_{2k}(t)|\,\mathrm{d}t + \left|\tilde{B}_{2k}\right| = 2\left|\tilde{B}_{2k}\right|,$$

which proves the third assertion. ■

Theorem 2. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \ge 1$. Then*

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + \tilde{T}_{n-1}(a,b)\right| \le \frac{(b-a)^{n+1}}{3(n!)}\int_0^1 |F_n(t)|\,\mathrm{d}t \cdot L. \tag{3.8}$$

Also, we have

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + \tilde{T}_{n}(a,b)\right| \le \frac{(b-a)^{n+1}}{3(n!)}\int_0^1 |G_n(t)|\,\mathrm{d}t \cdot L. \tag{3.9}$$

Proof. For any integrable function $\Phi : [a,b] \to \mathbf{R}$ we have

$$\left|\int_{[a,b]} \Phi(t)\mathrm{d}f^{(n-1)}(t)\right| \le \int_a^b |\Phi(t)|\,\mathrm{d}t \cdot L, \tag{3.10}$$

since $f^{(n-1)}$ is L-Lipschitzian function. Applying (3.10) with $\Phi(t) = F_n\left(\frac{t-a}{b-a}\right)$, we get

$$\begin{aligned}&\left|\frac{(b-a)^n}{3(n!)}\int_{[a,b]} F_n\left(\frac{t-a}{b-a}\right)\mathrm{d}f^{(n-1)}(t)\right| \\ \le\ & \frac{(b-a)^n}{3(n!)}\int_a^b \left|F_n\left(\frac{t-a}{b-a}\right)\right|\mathrm{d}t \cdot L \\ =\ & \frac{(b-a)^{n+1}}{3(n!)}\int_0^1 |F_n(t)|\,\mathrm{d}t \cdot L.\end{aligned}$$

Applying the above inequality, we get the inequality (3.8) from the identity (2.3). Similarly, we can apply the inequality (3.10) with $\Phi(t) = G_n\left(\frac{t-a}{b-a}\right)$, and then use the identity (2.2), to obtain the inequality (3.9). ■

As we have already noted in Section 2, we have

$$\tilde{T}_0(a,b) = \tilde{T}_1(a,b) = \tilde{T}_2(a,b) = \tilde{T}_3(a,b) = 0. \tag{3.11}$$

Moreover, since $\tilde{B}_2 = 0$ and $\tilde{B}_{2k-1} = 0$, $k \geq 1$, we have

$$\tilde{T}_m(a,b) = \frac{1}{3}\sum_{k=2}^{\left[\frac{m}{2}\right]} \frac{(b-a)^{2k}}{(2k)!}\tilde{B}_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right],\ m \geq 4, \tag{3.12}$$

where $\left[\frac{m}{2}\right]$ is the greatest integer less than or equal to $\frac{m}{2}$.

Corollary 3. *Let $f : [a,b] \to \mathbf{R}$ be given function.*
If f is L-Lipschitzian on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b)\right| \leq \frac{5}{24}(b-a)^2 \cdot L.$$

If f' is L-Lipschitzian on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b)\right| \leq \frac{5}{324}(b-a)^3 \cdot L.$$

If f'' is L-Lipschitzian on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b)\right| \leq \frac{1}{576}(b-a)^4 \cdot L.$$

If f''' is L-Lipschitzian on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b)\right| \leq \frac{7}{23040}(b-a)^5 \cdot L.$$

Proof. Using (2.4) and (2.5) we get

$$\int_0^1 |F_1(t)|\,\mathrm{d}t = \frac{5}{8} \text{ and } \int_0^1 |F_2(t)|\,\mathrm{d}t = \frac{5}{54},$$

respectively. Therefore, using (3.11) and applying (3.8) with $n = 1$ and $n = 2$, we get the first and the second inequality, respectively. Using the Corollary 2, we get

$$\int_0^1 |F_3(t)|\,\mathrm{d}t = \frac{1}{32} \text{ and } \int_0^1 |F_4(t)|\,\mathrm{d}t = \frac{7}{320}.$$

Now, the third inequality follows from (3.8) with $n = 3$ and (3.11), while the fourth one follows from (3.8) with $n = 4$ and (3.11). ∎

Corollary 4. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geq 3$. Set $D_1(a,b) := 0$ and for any integer r such that $2 \leq r \leq \frac{n}{2}$ define*

$$D_r(a,b) := \frac{1}{3}\sum_{i=2}^{r} \frac{(b-a)^{2i}}{(2i)!}\tilde{B}_{2i}\left[f^{(2i-1)}(b) - f^{(2i-1)}(a)\right]. \tag{3.13}$$

If $n = 2k-1,\ k \geq 2,$ *then*

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + D_{k-1}(a,b)\right| \leq \frac{4(b-a)^{2k}}{3\left[(2k)!\right]}\left(1-2^{-2k}\right)|B_{2k}|\cdot L.$$

If $n = 2k,\ k \geq 2,$ *then*

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + D_{k-1}(a,b)\right| \leq \frac{(b-a)^{2k+1}}{3\left[(2k)!\right]}\left(8\cdot 2^{-4k} - 6\cdot 2^{-2k}+1\right)|B_{2k}|\cdot L$$

and

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + D_{k}(a,b)\right| \leq \frac{2(b-a)^{2k+1}}{3\left[(2k)!\right]}\left(8\cdot 2^{-4k} - 6\cdot 2^{-2k}+1\right)|B_{2k}|\cdot L.$$

Proof. For $n = 2k-1$, by (3.12) we have that $\tilde{T}_{n-1}(a,b) = D_{k-1}(a,b)$. Thus, the first inequality follows from Corollary 2 and (3.8). For $n = 2k$, by (3.12) we have that $\tilde{T}_{n-1}(a,b) = D_{k-1}(a,b)$ and $\tilde{T}_n(a,b) = D_k(a,b)$. Now, the second inequality follows from Corollary 2 and (3.8), while the third one follows from Corollary 2 and (3.9). ∎

Remark 2. *Suppose that* $f : [a,b] \to \mathbf{R}$ *is such that* $f^{(n)}$ *exists and is bounded on* $[a,b]$, *for some* $n \geq 1$ *In this case we have for all* $t, s \in [a,b]$

$$\left|f^{(n-1)}(t) - f^{(n-1)}(s)\right| \leq \|f^{(n)}\|_\infty \cdot |t-s|,$$

which means that $f^{(n-1)}$ *is an* $\|f^{(n)}\|_\infty$*-Lipschitzian function on* $[a,b]$. *Therefore, the inequalities established in Theorem 2 hold with* $L = \|f^{(n)}\|_\infty$. *Consequently, under appropriate assumptions on* f, *the inequalities from Corollary 3 hold with* $L = \|f'\|_\infty,\ \|f''\|_\infty,\ \|f'''\|_\infty,\ \|f''''\|_\infty$, *respectively. However, a similar observation can be made for the results of the Corollary 4.*

Theorem 3. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[a,b]$ *for some* $n \geq 1$. *Then*

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + \tilde{T}_{n-1}(a,b)\right| \leq \frac{(b-a)^n}{3(n!)}\max_{t\in[0,1]}|F_n(t)|\cdot V_a^b(f^{(n-1)}) \tag{3.14}$$

and

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + \tilde{T}_{n}(a,b)\right| \leq \frac{(b-a)^n}{3(n!)}\max_{t\in[0,1]}|G_n(t)|\cdot V_a^b(f^{(n-1)}), \tag{3.15}$$

where $V_a^b(f^{(n-1)})$ *is the total variation of* $f^{(n-1)}$ *on* $[a,b]$.

Proof. If $\Phi : [a,b] \to \mathbf{R}$ is bounded on $[a,b]$ and the Riemann-Stieltjes integral $\int_{[a,b]}\Phi(t)df^{(n-1)}(t)$ exists, then

$$\left|\int_{[a,b]}\Phi(t)\mathrm{d}f^{(n-1)}(t)\right| \leq \max_{t\in[a,b]}|\Phi(t)|\cdot V_a^b(f^{(n-1)}). \tag{3.16}$$

We apply the estimate (3.16) to $\Phi(t) = F_n\left(\frac{t-a}{b-a}\right)$ to obtain

$$\begin{aligned} & \left|\frac{(b-a)^n}{3(n!)} \int_{[a,b]} F_n\left(\frac{t-a}{b-a}\right) \mathrm{d}f^{(n-1)}(t)\right| \\ \leq \; & \frac{(b-a)^n}{3(n!)} \max_{t\in[a,b]} \left|F_n\left(\frac{t-a}{b-a}\right)\right| \cdot V_a^b(f^{(n-1)}) \\ = \; & \frac{(b-a)^n}{3(n!)} \max_{t\in[0,1]} |F_n(t)| \cdot V_a^b(f^{(n-1)}). \end{aligned}$$

Now, we use the above inequality and the identity (2.3) to obtain (3.14) In the same manner, we apply the estimate (3.16) to $\Phi(t) = G_n\left(\frac{t-a}{b-a}\right)$, and then use the identity (2.2), to obtain the inequality (3.15). ∎

Corollary 5. *Let $f : [a,b] \to \mathbf{R}$ be given function.*
If f is a continuous function of bounded variation on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b)\right| \leq \frac{5}{12}(b-a) \cdot V_a^b(f).$$

If f' is a continuous function of bounded variation on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b)\right| \leq \frac{1}{24}(b-a)^2 \cdot V_a^b(f').$$

If f'' is a continuous function of bounded variation on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b)\right| \leq \frac{5}{1296}(b-a)^3 \cdot V_a^b(f'').$$

If f''' is a continuous function of bounded variation on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b)\right| \leq \frac{1}{1152}(b-a)^4 \cdot V_a^b(f''').$$

Proof. From the explicit expressions (2.4), (2.5) and (2.6), we get

$$\begin{aligned} \max_{t\in[0,1]} |F_1(t)| &= -F_1\left(\frac{3}{4}\right) = \frac{5}{4}, \\ \max_{t\in[0,1]} |F_2(t)| &= -F_2\left(\frac{1}{2}\right) = \frac{1}{4} \end{aligned}$$

and

$$\max_{t\in[0,1]} |F_3(t)| = -F_3\left(\frac{1}{3}\right) = \frac{5}{72},$$

respectively. Therefore, using (3.11) and applying (3.14) with $n = 1$, $n = 2$ and $n = 3$, we get the first, the second and the third inequality, respectively. Further, using the Corollary 1, we get

$$\max_{t\in[0,1]} |F_4(t)| = \frac{1}{16}.$$

Now, the fourth inequality follows from (3.14) with $n = 4$ and (3.11). ∎

Corollary 6. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n \geq 3$. Define $D_r(a,b)$, $r \geq 0$ as in the Corollary 4.*
If $n = 2k-1$, $k \geq 2$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + D_{k-1}(a,b)\right| \leq \frac{(b-a)^{2k-1}}{3\left[(2k-1)!\right]} \max_{t\in[0,1]} |F_{2k-1}(t)| \cdot V_a^b(f^{(2k-2)}).$$

If $n = 2k$, $k \geq 2$, then

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + D_{k-1}(a,b)\right| \leq \frac{2(b-a)^{2k}}{3\left[(2k)!\right]} \left(1 - 2^{-2k}\right) |B_{2k}| \cdot V_a^b(f^{(2k-1)})$$

and

$$\begin{aligned} &\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + D_k(a,b)\right| \\ \leq\ &\frac{(b-a)^{2k}}{3\left[(2k)!\right]} \left(1 + 4\cdot 2^{-2k} - 8\cdot 2^{-4k}\right) |B_{2k}| \cdot V_a^b(f^{(2k-1)}). \end{aligned}$$

Proof. The argument is similar to that used in the proof of Corollary 4. We apply Theorem 3 and use the formulae established in Corollary 1. ∎

Remark 3. *Suppose that $f : [a,b] \to \mathbf{R}$ is such that $f^{(n)} \in L_1[a,b]$ for some $n \geq 1$ In this case $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ and we have*

$$V_a^b(f^{(n-1)}) = \int_a^b \left|f^{(n)}(t)\right| \mathrm{d}t = \|f^{(n)}\|_1,$$

Therefore, the inequalities established in Theorem 3 hold with $\|f^{(n)}\|_1$ in place of $V_a^b(f^{(n-1)})$. However, a similar observation can be made for the results of the Corollaries 5 and 6.

Theorem 4. *Assume (p,q) is a pair of conjugate exponents, that is*

$$1 < p, q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1 \quad or \quad p = \infty,\ q = 1.$$

Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a,b]$ for some $n \geq 1$. Then we have

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + \tilde{T}_{n-1}(a,b)\right| \leq K(n,p)(b-a)^{n+\frac{1}{q}} \cdot \|f^{(n)}\|_p, \tag{3.17}$$

where

$$K(n,p) = \frac{1}{3(n!)} \left[\int_0^1 |F_n(t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}}.$$

Also, we have

$$\left|\int_a^b f(t)\mathrm{d}t - D(a,b) + \tilde{T}_n(a,b)\right| \leq K^*(n,p)(b-a)^{n+\frac{1}{q}} \cdot \|f^{(n)}\|_p, \tag{3.18}$$

where

$$K^*(n,p) = \frac{1}{3(n!)} \left[\int_0^1 |G_n(t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}}.$$

Proof. Applying the Hölder inequality we have

$$\begin{aligned}
&\left|\frac{(b-a)^n}{3(n!)}\int_a^b F_n\left(\frac{t-a}{b-a}\right)f^{(n)}(t)\mathrm{d}t\right| \\
\le\ & \frac{(b-a)^n}{3(n!)}\left[\int_a^b \left|F_n\left(\frac{t-a}{b-a}\right)\right|^q \mathrm{d}t\right]^{\frac{1}{q}}\cdot\left\|f^{(n)}\right\|_p \\
=\ & \frac{(b-a)^{n+\frac{1}{q}}}{3(n!)}\left[\int_0^1 |F_n(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}}\cdot\left\|f^{(n)}\right\|_p \\
=\ & K(n,p)(b-a)^{n+\frac{1}{q}}\cdot\|f^{(n)}\|_p
\end{aligned}$$

Using the above inequality, by the Remark 1, from (2.3) we get the estimate(3.14) In the same manner, from (2.2) we get the estimate (3.18). ∎

Remark 4. *For $p=\infty$ we have*

$$K(n,\infty)=\frac{1}{3(n!)}\int_0^1 |F_n(t)|\,\mathrm{d}t$$

and

$$K^*(n,\infty)=\frac{1}{3(n!)}\int_0^1 |G_n(t)|\,\mathrm{d}t.$$

The results established in Theorem 4 for $p=\infty$ coincide with the results of Theorem 2 with $L=\|f^{(n)}\|_\infty$. Moreover, by Remark 2 and Corollary 3, we have

$$\left|\int_a^b f(t)\mathrm{d}t-D(a,b)\right|\le K(n,\infty)\,(b-a)^{n+1}\cdot\|f^{(n)}\|_\infty,\quad n=1,2,3,4,$$

where

$$K(1,\infty)=\frac{5}{24},\ K(2,\infty)=\frac{5}{324},\ K(3,\infty)=\frac{1}{576},\ K(4,\infty)=\frac{7}{23040}.$$

Further, by Remark 2 and Corollary 4, for $k\ge 2$ we have

$$\left|\int_a^b f(t)\mathrm{d}t-D(a,b)+D_{k-1}(a,b)\right|\le K(2k-1,\infty)(b-a)^{2k}\cdot\|f^{(2k-1)}\|_\infty,$$

$$\left|\int_a^b f(t)\mathrm{d}t-D(a,b)+D_{k-1}(a,b)\right|\le K(2k,\infty)(b-a)^{2k+1}\cdot\|f^{(2k)}\|_\infty$$

and

$$\left|\int_a^b f(t)\mathrm{d}t-D(a,b)+D_k(a,b)\right|\le K^*(2k,\infty)(b-a)^{2k+1}\cdot\|f^{(2k)}\|_\infty,$$

where

$$K(2k-1,\infty)=\frac{4\left(1-2^{-2k}\right)}{3\left[(2k)!\right]}\left|B_{2k}\right|,$$

$$K(2k,\infty)=\frac{8\cdot 2^{-4k}-6\cdot 2^{-2k}+1}{3\left[(2k)!\right]}\left|B_{2k}\right|$$

and

$$K^*(2k,\infty)\le\frac{2\left(8\cdot 2^{-4k}-6\cdot 2^{-2k}+1\right)}{3\left[(2k)!\right]}\left|B_{2k}\right|.$$

Remark 5. *Let us define for* $p = 1$

$$K(n,1) := \frac{1}{3(n!)} \max_{t\in[0,1]} |F_n(t)|$$

and

$$K^*(n,1) := \frac{1}{3(n!)} \max_{t\in[0,1]} |G_n(t)| .$$

Then, using Remark 3 and Theorem 3, we can extend the results established in Theorem 4 to the pair $p = 1$, $q = \infty$. *This means that if we set* $\frac{1}{q} = 0$, *then (3.17) and (3.18) hold for* $p = 1$. *Also, by Remark 3 and Corollary 5, we have*

$$\left| \int_a^b f(t)\mathrm{d}t - D(a,b) \right| \leq K(n,1)\,(b-a)^n \cdot \|f^{(n)}\|_1, \quad n = 1,2,3,4,$$

where

$$K(1,1) = \frac{5}{12},\ K(2,1) = \frac{1}{24},\ K(3,1) = \frac{5}{1296},\ K(4,1) = \frac{1}{1152}.$$

Further, by Remark 3 and Corollary 6, for $k \geq 2$ *we have*

$$\left| \int_a^b f(t)\mathrm{d}t - D(a,b) + D_{k-1}(a,b) \right| \leq K(2k-1,1)(b-a)^{2k-1} \cdot \|f^{(2k-1)}\|_1,$$

$$\left| \int_a^b f(t)\mathrm{d}t - D(a,b) + D_{k-1}(a,b) \right| \leq K(2k,1)(b-a)^{2k} \cdot \|f^{(2k)}\|_1$$

and

$$\left| \int_a^b f(t)\mathrm{d}t - D(a,b) + D_k(a,b) \right| \leq K^*(2k,1)(b-a)^{2k} \cdot \|f^{(2k)}\|_1,$$

where

$$K(2k-1,1) = \frac{1}{3\left[(2k-1)!\right]} \max_{t\in[0,1]} |F_{2k-1}(t)| ,$$

$$K(2k,1) = \frac{2\left(1-2^{-2k}\right)}{3\left[(2k)!\right]} |B_{2k}|$$

and

$$K^*(2k,1) = \frac{2\left(1+4\cdot 2^{-2k} - 8\cdot 2^{-4k}\right)}{3\left[(2k)!\right]} |B_{2k}| .$$

4 Error estimates for dual Euler-Simpson quadrature formulae

Let us divide the interval $[a,b]$ into ν subintervals of equal length $h=\frac{1}{\nu}(b-a)$. Assume that $f:[a,b]\to\mathbf{R}$ is such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n\geq 1$. We consider the repeated dual Euler-Simpson formula

$$\int_a^b f(t)\mathrm{d}t = D_\nu(f)-\sigma_{n-1}(f)+\rho_n(f) \tag{4.1}$$

and the repeated modified dual Euler-Simpson formula

$$\int_a^b f(t)\mathrm{d}t = D_\nu(f)-\sigma_n(f)+\tilde{\rho}_n(f), \tag{4.2}$$

where

$$\begin{aligned} D_\nu(f) &= \sum_{i=1}^{\nu} D\left(a+(i-1)\,h, a+ih\right) \\ &= \frac{h}{3}\sum_{i=1}^{\nu}\left[2f(a+(i-3/4)h)-f(a+(i-1/2)h)+2f(a+(i-1/4)h)\right] \end{aligned}$$

and

$$\sigma_m(f)=\sum_{i=1}^{\nu}\tilde{T}_m\left(a+(i-1)h,a+ih\right),\ m\geq 0.$$

Because of (3.11) we have

$$\sigma_0(f)=\sigma_1(f)=\sigma_2(f)=\sigma_3(f)=0, \tag{4.3}$$

while for $m\geq 4$, using (3.12) we get

$$\begin{aligned} \sigma_m(f) &= \sum_{i=1}^{\nu}\frac{1}{3}\sum_{j=2}^{\left[\frac{m}{2}\right]}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(a+ih)-f^{(2j-1)}(a+(i-1)h)\right] \\ &= \frac{1}{3}\sum_{j=2}^{\left[\frac{m}{2}\right]}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\sum_{i=1}^{\nu}\left[f^{(2j-1)}(a+ih)-f^{(2j-1)}(a+(i-1)h)\right] \\ &= \frac{1}{3}\sum_{j=2}^{\left[\frac{m}{2}\right]}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right]. \end{aligned} \tag{4.4}$$

The remainders $\rho_n(f)$ and $\tilde{\rho}_n(f)$ can be written as

$$\rho_n(f)=\sum_{i=1}^{\nu}\rho_n(f;i),\quad \tilde{\rho}_n(f)=\sum_{i=1}^{\nu}\tilde{\rho}_n(f;i), \tag{4.5}$$

where, for $i=1,\cdots,\nu$,

$$\rho_n(f;i)=\int_{a+(i-1)h}^{a+ih} f(t)\mathrm{d}t-D\left(a+(i-1)\,h,a+ih\right)+\tilde{T}_{n-1}\left(a+(i-1)h,a+ih\right)$$

and

$$\tilde{\rho}_n(f;i)=\int_{a+(i-1)h}^{a+ih} f(t)\mathrm{d}t-D\left(a+(i-1)\,h,a+ih\right)+\tilde{T}_{n}\left(a+(i-1)h,a+ih\right).$$

We shall apply the results from the preceding section to obtain some estimates for the remainders $\rho_n(f)$ and $\tilde{\rho}_n(f)$. Before doing this, note that for $n = 2k-1$, $k \geq 3$, we have

$$\sigma_{2k-2}(f) = \sigma_{2k-1}(f) = \frac{1}{3}\sum_{j=2}^{k-1} \frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right].$$

Thus

$$\rho_{2k-1}(f) = \tilde{\rho}_{2k-1}(f),$$

so that (4.1) and (4.2) coincide in this case. This shows that (4.2) can be interesting only when $n = 2k$, $k \geq 2$. In this case we have

$$\begin{aligned} \tilde{\rho}_{2k}(f) &= \rho_{2k}(f) + \sigma_{2k}(f) - \sigma_{2k-1}(f) \\ &= \rho_{2k}(f) + \frac{h^{2k}}{3\,[(2k)!]}\tilde{B}_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]. \end{aligned}$$

In fact we have

$$\tilde{\rho}_{2k-2}(f) = \rho_{2k}(f), \;\; k \geq 3.$$

Therefore, for $k \geq 3$ we can approximate $\int_a^b f(t)\mathrm{d}t$ by

$$D_\nu(f) - \frac{1}{3}\sum_{j=2}^{k-1} \frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

using either (4.1) with $n = 2k-1$ or (4.2) with $n = 2k-2$. To obtain the error estimate for this approximation, if we apply (4.1), then we must assume that $f^{(2k-2)}$ is a continuous function of bounded variation on $[a,b]$. To do this via the formula (4.2), it is enough to assume that $f^{(2k-3)}$ is a continuous function of bounded variation on $[a,b]$

Theorem 5. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geq 1$.*
For $n = 1$, 2, 3, 4 we have, respectively,

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f)\right| \leq \frac{5}{24}\nu h^2 \cdot L,$$

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f)\right| \leq \frac{5}{324}\nu h^3 \cdot L,$$

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f)\right| \leq \frac{1}{576}\nu h^4 \cdot L,$$

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f)\right| \leq \frac{7}{23040}\nu h^5 \cdot L.$$

If $n = 2k-1$, $k \geq 3$, then

$$\begin{aligned} &\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f) + \frac{1}{3}\sum_{j=2}^{k-1} \frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right| \\ \leq\; & \frac{4\nu h^{2k}}{3\,[(2k)!]}\left(1 - 2^{-2k}\right)|B_{2k}| \cdot L. \end{aligned}$$

If $n = 2k,\ k \geq 3,$ *then*

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f) + \frac{1}{3}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right|$$
$$\leq \frac{\nu h^{2k+1}}{3\left[(2k)!\right]}\left(8\cdot 2^{-4k} - 6\cdot 2^{-2k} + 1\right)|B_{2k}|\cdot L.$$

If $n = 2k,\ k \geq 2,$ *then*

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f) + \frac{1}{3}\sum_{j=2}^{k}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right|$$
$$\leq \frac{2\nu h^{2k+1}}{3\left[(2k)!\right]}\left(8\cdot 2^{-4k} - 6\cdot 2^{-2k} + 1\right)|B_{2k}|\cdot L.$$

Proof. Applying (3.8) and (3.9) we get for $i = 1, \cdots, \nu$, respectively,

$$|\rho_n(f;i)| \leq \frac{h^{n+1}}{3(n!)}\int_0^1 |F_n(t)|\,\mathrm{d}t\cdot L$$

and

$$|\tilde{\rho}_n(f;i)| \leq \frac{h^{n+1}}{3(n!)}\int_0^1 |G_n(t)|\,\mathrm{d}t\cdot L.$$

Using the above estimates and the triangle inequality, we get from (4.5)

$$|\rho_n(f)| \leq \sum_{i=1}^{\nu}|\rho_n(f;i)| \leq \frac{\nu h^{n+1}}{3(n!)}\int_0^1 |F_n(t)|\,\mathrm{d}t\cdot L$$

and

$$|\tilde{\rho}_n(f)| \leq \sum_{i=1}^{\nu}|\tilde{\rho}_n(f;i)| \leq \frac{\nu h^{n+1}}{3(n!)}\int_0^1 |G_n(t)|\,\mathrm{d}t\cdot L.$$

Now, we use (4.3) and (4.4) and the rest of the argument is quite the same as for the Corollaries 3 and 4. ■

Remark 6. *Instead of the assumption that* $f^{(n-1)}$ *is an L-Lipschitzian function on* $[a,b]$,*we can use the stronger assumption that* $f^{(n)}$ *exists and is bounded on* $[a,b]$, *for some* $n \geq 1$ *In this case Theorem 5 applies with* L *replaced by* $\left\|f^{(n)}\right\|_\infty$ *(see Remark 2).*

Theorem 6. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[a,b]$ *for some* $n \geq 1$.
For $n = 1,\ 2,\ 3,\ 4$ *we have, respectively,*

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f)\right| \leq \frac{5}{12}h\cdot V_a^b(f),$$

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f)\right| \leq \frac{1}{24}h^2\cdot V_a^b(f'),$$

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f)\right| \leq \frac{5}{1296}h^3 \cdot V_a^b(f''),$$

$$\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f)\right| \leq \frac{1}{1152}h^4 \cdot V_a^b(f''').$$

If $n = 2k-1,\ k \geq 3,$ *then*

$$\begin{aligned}&\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f) + \frac{1}{3}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right| \\ \leq\ & \frac{h^{2k-1}}{3\left[(2k-1)!\right]}\max_{t\in[0,1]}|F_{2k-1}(t)| \cdot V_a^b(f^{(2k-2)}).\end{aligned}$$

If $n = 2k,\ k \geq 3,$ *then*

$$\begin{aligned}&\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f) + \frac{1}{3}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right| \\ \leq\ & \frac{2h^{2k}}{3\left[(2k)!\right]}\left(1 - 2^{-2k}\right)|B_{2k}| \cdot V_a^b(f^{(2k-1)}).\end{aligned}$$

If $n = 2k,\ k \geq 2,$ *then*

$$\begin{aligned}&\left|\int_a^b f(t)\mathrm{d}t - D_\nu(f) + \frac{1}{3}\sum_{j=2}^{k}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]\right| \\ \leq\ & \frac{h^{2k}}{3\left[(2k)!\right]}\left(1 + 4\cdot 2^{-2k} - 8\cdot 2^{-4k}\right)|B_{2k}| \cdot V_a^b(f^{(2k-1)}).\end{aligned}$$

Proof. Applying (3.14) and (3.15) we get for $i = 1, \cdots, \nu$, respectively,

$$|\rho_n(f;i)| \leq \frac{h^n}{3(n!)}\max_{t\in[0,1]}|F_n(t)| \cdot V_{a+(i-1)h}^{a+ih}(f^{(n-1)})$$

and

$$|\tilde{\rho}_n(f;i)| \leq \frac{h^n}{3(n!)}\max_{t\in[0,1]}|G_n(t)| \cdot V_{a+(i-1)h}^{a+ih}(f^{(n-1)}).$$

Using the above estimates and the triangle inequality, we get from (4.5)

$$\begin{aligned}|\rho_n(f)| &\leq \sum_{i=1}^{\nu}|\rho_n(f;i)| \\ &\leq \frac{h^n}{3(n!)}\max_{t\in[0,1]}|F_n(t)| \cdot \sum_{i=1}^{\nu}V_{a+(i-1)h}^{a+ih}(f^{(n-1)}) \\ &= \frac{h^n}{3(n!)}\max_{t\in[0,1]}|F_n(t)| \cdot V_a^b(f^{(n-1)})\end{aligned}$$

and similarly

$$|\tilde{\rho}_n(f)| \leq \frac{h^n}{3(n!)}\max_{t\in[0,1]}|G_n(t)| \cdot V_a^b(f^{(n-1)})..$$

Now, we use (4.3) and (4.4) and argue similarly as in the Corollaries 5 and 6. ∎

Remark 7. *If $f : [a, b] \to \mathbf{R}$ is such that $f^{(n)} \in L_1[a, b]$ for some $n \geq 1$, then $f^{(n-1)}$ is a continuous function of bounded variation on $[a, b]$ and $V_a^b(f^{(n-1)}) = \|f^{(n)}\|_1$. Therefore, Theorem 6 applies with $\|f^{(n)}\|_1$ in place of $V_a^b(f^{(n-1)})$ (see Remark 3).*

Theorem 7. *Assume (p, q) is a pair of conjugate exponents, that is*

$$1 < p, q < \infty, \ \frac{1}{p} + \frac{1}{q} = 1 \quad or \quad p = \infty, \ q = 1.$$

Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a, b]$ for some $n \geq 1$. Then we have

$$|\rho_n(f)| \leq \nu K(n, p) h^{n+\frac{1}{q}} \cdot \|f^{(n)}\|_p$$

and

$$|\tilde{\rho}_n(f)| \leq \nu K^*(n, p) h^{n+\frac{1}{q}} \cdot \|f^{(n)}\|_p,$$

where $K(n, p)$ and $K^*(n, p)$ are defined as in Theorem 4.

Proof. For $i = 1, \cdots, \nu$ consider the function $g_i(t) = f^{(n)}(t)$, $t \in [a + (i-1)h, a + ih]$. Obviously we have

$$\|g_i\|_p \leq \|f^{(n)}\|_p,$$

where the norm $\|g_i\|_p$ is taken over the interval $[a + (i-1)h, a + ih]$, while the norm $\|f^{(n)}\|_p$ is taken over the interval $[a, b]$. Applying (3.17) and (3.18) and using the above inequality, we get for $i = 1, \cdots, \nu$

$$|\rho_n(f; i)| \leq K(n, p) h^{n+\frac{1}{q}} \cdot \|g_i\|_p \leq K(n, p) h^{n+\frac{1}{q}} \cdot \|f^{(n)}\|_p$$

and

$$|\tilde{\rho}_n(f; i)| \leq K^*(n, p) h^{n+\frac{1}{q}} \cdot \|g_i\|_p \leq K^*(n, p) h^{n+\frac{1}{q}} \cdot \|f^{(n)}\|_p.$$

The result follows from (4.5) by the triangle inequality. ■

In the following discussion we assume that $f : [a, b] \to \mathbf{R}$ has a continuous derivative of order n, for some $n \geq 1$. In this case we can use (2.3) and the second formula from Remark 1 to obtain, for $i = 1, \cdots, \nu$,

$$\begin{aligned} \rho_n(f; i) &= \frac{h^n}{3(n!)} \int_{a+(i-1)h}^{a+ih} F_n\left(\frac{t - a - (i-1)h}{h}\right) f^{(n)}(t) \mathrm{d}t \\ &= \frac{h^{n+1}}{3(n!)} \int_0^1 F_n(s) f^{(n)}(a + (i-1)h + hs) \mathrm{d}s. \end{aligned}$$

Therefore, by (4.5) we get

$$\rho_n(f) = \frac{h^{n+1}}{3(n!)} \int_0^1 F_n(s) \Phi_n(s) \mathrm{d}s, \tag{4.6}$$

where

$$\Phi_n(s) = \sum_{i=1}^{\nu} f^{(n)}(a + (i-1)h + hs), \quad 0 \leq s \leq 1. \tag{4.7}$$

Similarly, we get

$$\tilde{\rho}_n(f) = \frac{h^{n+1}}{3(n!)} \int_0^1 G_n(s)\Phi_n(s)\mathrm{d}s.$$

Obviously, $\Phi_n(s)$ is a continuous function on $[0,1]$ and

$$\begin{aligned}\int_0^1 \Phi_n(s)\mathrm{d}s &= h^{-1}\sum_{i=1}^{\nu}\left[f^{(n-1)}(a+ih) - f^{(n-1)}(a+(i-1)h)\right] \\ &= h^{-1}\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]. \end{aligned} \tag{4.8}$$

From the discussion given at the beginning of this section it follows that it is the most interesting to consider the repeated dual Euler-Simpson formula (4.1) for $n = 2k$, $k \geq 2$, which can be rewritten as

$$\int_a^b f(t)\mathrm{d}t = D_\nu(f) - \frac{1}{3}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right] + \rho_{2k}(f). \tag{4.9}$$

We assume the sum on the right hand side in the above equality to be zero when $k = 2$.

Theorem 8. *If $f : [a,b] \to \mathbf{R}$ is such that $f^{(2k)}$ is a continuous function on $[a,b]$, for some $k \geq 2$, then there exists a point $\eta \in [a,b]$ such that*

$$\rho_{2k}(f) = -\nu\frac{h^{2k+1}}{3\left[(2k)!\right]}\tilde{B}_{2k}f^{(2k)}(\eta). \tag{4.10}$$

Proof. Using (4.6) we can rewrite $\rho_{2k}(f)$ as

$$\rho_{2k}(f) = (-1)^k\frac{h^{2k+1}}{3\left[(2k)!\right]}J_k, \tag{4.11}$$

where

$$J_k = \int_0^1 (-1)^k F_{2k}(s)\Phi_{2k}(s)\mathrm{d}s. \tag{4.12}$$

If

$$m = \min_{t\in[a,b]} f^{(2k)}(t), \quad M = \max_{t\in[a,b]} f^{(2k)}(t),$$

then from (4.7) we get

$$\nu m \leq \Phi_{2k}(s) \leq \nu M, \quad 0 \leq s \leq 1.$$

On the other side, from Corollary 1 it follows that

$$(-1)^k F_{2k}(s) \geq 0, \quad 0 \leq s \leq 1,$$

which implies

$$\nu m\int_0^1 (-1)^k F_{2k}(s)\mathrm{d}s \leq J_k \leq \nu M\int_0^1 (-1)^k F_{2k}(s)\mathrm{d}s.$$

We have already calculated in the proof of Corollary 2 that $\int_0^1 F_{2k}(s)\mathrm{d}s = -\tilde{B}_{2k}$, so that we have

$$\nu m(-1)^{k-1}\tilde{B}_{2k} \leq J_k \leq \nu M(-1)^{k-1}\tilde{B}_{2k}.$$

By the continuity of $f^{(2k)}(s)$ on $[a,b]$, it follows that there must exist a point $\eta \in [a,b]$ such that

$$J_k = \nu(-1)^{k-1}\tilde{B}_{2k}f^{(2k)}(\eta).$$

Combining this with (4.11) we get (4.10). ∎

Remark 8. *The repeated dual Euler-Simpson formula (4.9) is a generalization of the dual Simpson formula (1.2). Namely from (4.10) for $k = 2$ and $\nu = 1$ we get*

$$\rho_4(f) = \frac{7(b-a)^5}{23040} f^{(4)}(\eta)$$

and (4.9) reduces to (1.2).

Remark 9. *In [6, p. 222] the following repeated Euler-Simpson formula has been considered:*

$$\begin{aligned} \int_a^b f(t)\mathrm{dt} &= S_\nu(f) \\ &\quad + \frac{1}{3}\sum_{j=2}^{k-1} \frac{h^{2j}}{(2j)!}(1 - 4\cdot 2^{-2j}) B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right] + \rho_{S,2k}(f), \end{aligned}$$

where

$$S_\nu(f) = \frac{h}{6}\sum_{i=1}^{\nu} \left[f(a + (i-1)h) + 4f(a + (i - 1/2)h) + f(a + ih)\right].$$

It has been proved that, under the assumptions of Theorem 8, there exists a point $\xi \in [a, b]$ such that [6, p. 225]

$$\rho_{S,2k}(f) = \nu \frac{h^{2k+1}}{3\left[(2k)!\right]}(1 - 4\cdot 2^{-2k}) B_{2k} f^{(2k)}(\xi).$$

We can compare the remainders $\rho_{S,2k}(f)$ and $\rho_{2k}(f)$. From (3.3) and (4.10) we get

$$\begin{aligned} \rho_{2k}(f) &= -\nu \frac{h^{2k+1}}{3\left[(2k)!\right]}\left(8\cdot 2^{-4k} - 6\cdot 2^{-2k} + 1\right) B_{2k} f^{(2k)}(\eta) \\ &= -\nu \frac{h^{2k+1}}{3\left[(2k)!\right]}\left(1 - 4\cdot 2^{-2k}\right)(1 - 2\cdot 2^{-2k}) B_{2k} f^{(2k)}(\eta). \end{aligned}$$

This gives

$$\frac{\rho_{2k}(f)}{\rho_{S,2k}(f)} = -(1 - 2\cdot 2^{-2k})\frac{f^{(2k)}(\eta)}{f^{(2k)}(\xi)}.$$

Therefore, if $f^{(2k)}$ does not change the sign on $[a, b]$, then $\rho_{2k}(f)$ and $\rho_{S,2k}(f)$ have opposite signs. Moreover, if $f^{(2k)}(t) \geq 0$, $a \leq t \leq b$, then

$$(-1)^{k-1} I_{S,2k}(f;\nu) \leq (-1)^{k-1}\int_a^b f(t)\mathrm{dt} \leq (-1)^{k-1} I_{2k}(f;\nu),$$

where

$$I_{S,2k}(f;\nu) = S_\nu(f) + \frac{1}{3}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}(1 - 4\cdot 2^{-2j}) B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]$$

and

$$I_{2k}(f;\nu) = D_\nu(f) - \frac{1}{3}\sum_{j=2}^{k-1}\frac{h^{2j}}{(2j)!}\tilde{B}_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right].$$

On the contrary, if $f^{(2k)}(t) \leq 0$, $a \leq t \leq b$, *then*

$$(-1)^{k-1} I_{2k}(f;\nu) \leq (-1)^{k-1} \int_a^b f(t)\mathrm{d}t \leq (-1)^{k-1} I_{S,2k}(f;\nu).$$

Also note that for the numerical coefficients

$$K = -\nu \frac{h^{2k+1}}{3\left[(2k)!\right]} \left(8 \cdot 2^{-4k} - 6 \cdot 2^{-2k} + 1\right) B_{2k}$$

and

$$K_S = \nu \frac{h^{2k+1}}{3\left[(2k)!\right]} (1 - 4 \cdot 2^{-2k}) B_{2k}$$

we have

$$\frac{7}{8} \leq -\frac{K}{K_S} = 1 - 2 \cdot 2^{-2k} < 1, \quad k \geq 2.$$

Thus, if $f^{(2k)}$ *changes very slowly, then the approximate equality* $\int_a^b f(t)\mathrm{d}t = I_{2k}(f;\nu)$ *will be more accurate than the approximate equality* $\int_a^b f(t)\mathrm{d}t = I_{S,2k}(f;\nu)$.

Theorem 9. *If* $f : [a,b] \to \mathbf{R}$ *is such that* $f^{(2k)}$ *is a continuous function on* $[a,b]$, *for some* $k \geq 2$, *and does not change the sign on* $[a,b]$, *then there exists a point* $\theta \in [0,1]$ *such that*

$$\rho_{2k}(f) = -\theta \frac{h^{2k}}{3\left[(2k)!\right]} 2(1 - 2^{-2k}) B_{2k} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]. \tag{4.13}$$

Proof. Suppose that $f^{(2k)}(t) \geq 0$, $a \leq t \leq b$. Then from (4.7) we get

$$\Phi_{2k}(s) \geq 0, \quad 0 \leq s \leq 1.$$

From Corollary 1 it follows that

$$0 \leq (-1)^k F_{2k}(s) \leq (-1)^k F_{2k}\left(\frac{1}{2}\right), \quad 0 \leq s \leq 1.$$

Therefore, if J_k is given by (4.12), then

$$0 \leq J_k \leq (-1)^k F_{2k}\left(\frac{1}{2}\right) \int_0^1 \Phi_{2k}(s)\mathrm{d}s.$$

Using (3.6) and (4.8), we get

$$0 \leq J_k \leq (-1)^{k-1} 2\left(1 - 2^{-2k}\right) B_{2k} h^{-1} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right],$$

which means that there must exist a point $\theta \in [0,1]$ such that

$$J_k = \theta (-1)^{k-1} 2\left(1 - 2^{-2k}\right) B_{2k} h^{-1} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right].$$

Combining this with (4.11) we get (4.13). The argument is the same when $f^{(2k)}(t) \leq 0$, $a \leq t \leq b$, since in that case we get

$$(-1)^{k-1} 2\left(1 - 2^{-2k}\right) B_{2k} h^{-1} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \leq J_k \leq 0.$$

■

Remark 10. *If we approximate* $\int_a^b f(t)\mathrm{d}t$ *by* $I_{2k}(f;\nu)$, *then the next approximation will be* $I_{2k+2}(f;\nu)$. *The difference* $\Delta_{2k}(f;\nu) = I_{2k+2}(f;\nu) - I_{2k}(f;\nu)$ *is equal to the last term in* $I_{2k+2}(f;\nu)$, *that is*

$$\begin{aligned}\Delta_{2k}(f;\nu) &= -\frac{h^{2k}}{3\left[(2k)!\right]}\tilde{B}_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \\ &= -\frac{h^{2k}}{3\left[(2k)!\right]}(1-2\cdot 2^{-2k})(1-4\cdot 2^{-2k})B_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right].\end{aligned}$$

We see that, under the assumptions of Theorem 9, $\rho_{2k}(f)$ *and* $\Delta_{2k}(f;\nu)$ *are of the same sign. Moreover, we have*

$$\frac{\rho_{2k}(f)}{\Delta_{2k}(f;\nu)} = \frac{2\theta(1-2^{-2k})}{(1-2\cdot 2^{-2k})(1-4\cdot 2^{-2k})} \le \frac{20}{7}\theta,\ \ k \ge 2.$$

Thus, we have the following estimate for the remainder $\rho_{2k}(f)$*:*

$$|\rho_{2k}(f)| \le \frac{20}{7}\left|\Delta_{2k}(f;\nu)\right|,\ \ k \ge 2.$$

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