On Infinitesimally k-Flat Homogeneous Spaces

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1 Introduction

A k-flat in a Riemannian manifold M is a k-dimensional, totally geodesic, complete, connected, flat submanifold. A homogeneous Riemannian manifold M is said to be k-flat homogeneous if every geodesic in M lies in a k-flat and if the isometry group of M acts transitively on the set of pairs (p, T), where T is a k-flat in M and $p \in T$. A well-known result by Tits and Wang says that a 1-flat homogeneous space, or equivalently a two-point homogeneous space, is symmetric (for an elegant proof see [6]). This was generalized for arbitrary $k \geq 2$ to k-flat homogeneous spaces by Heintze, Palais, Terng and Thorbergsson in [2] for the compact case and by the second author in [3] and [4] for the general case. In this paper we investigate in how far these results are infinitesimal phenomena.

An infinitesimal curvature model (V, g, R) consists of a finite-dimensional real vector space V, a positive definite inner product g on V, and an algebraic curvature tensor R. An infinitesimal k-flat in (V, g, R) is a k-dimensional linear subspace F of V such that $R(X, Y)Z = 0$ for all $X, Y, Z \in F$. Let $\mathcal{A}$ be the group of automorphisms of g and R, i.e. the isometries A of (V, g) satisfying $R(AX, AY)AZ = AR(X, Y)Z$ for all $X, Y, Z \in V$. We say that (V, g, R) is infinitesimally k-flat homogeneous if every one-dimensional linear subspace of V is contained in an infinitesimal k-flat in (V, g, R) and if $\mathcal{A}$ acts transitively on the set of infinitesimal k-flats in (V, g, R). A Riemannian manifold M with metric g and curvature tensor R is said to be infinitesimally k-flat homogeneous if for every $p \in M$ the infinitesimal curvature model (T_pM, g_p, R_p) is infinitesimally k-flat homogeneous.

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Example. Let M be a connected Riemannian symmetric space of rank k. It is well-known that the isotropy subgroup K_p at p of the full isometry group of M acts transitively on the set of k-flats in M containing p. Moreover, if T is such a k-flat, it follows from the Gauss equation that $F = T_pT$ is an infinitesimal k-flat in (T_pM, g_p, R_p). Conversely, given any infinitesimal k-flat F in (T_pM, g_p, R_p), the image of F under the exponential map of M at p is a k-flat in M. Since $K_p \subset \mathcal{A}_p$ it follows that (T_pM, g_p, R_p) is infinitesimally k-flat homogeneous. Hence a Riemannian symmetric space of rank k is infinitesimally k-flat homogeneous.

The Riemannian manifolds which have at every point the same infinitesimal curvature model as some symmetric space, are characterized by the property that $R_p(X,Y)\cdot R_p = 0$ for all $p \in M$ and $X, Y \in T_pM$, where $R_p(X,Y)$ acts as a derivation on R_p. Riemannian manifolds with this property are known as semi-symmetric spaces. Their local classification has been achieved by Szabó [5]. So the infinitesimal analoga of the results described above would be: If M is an infinitesimally k-flat homogeneous space then M is semi-symmetric.

In Section 2 we show that infinitesimally 1-flat homogeneous spaces are related to the Osserman Conjecture about the Jacobi operator of Riemannian manifolds. This implies that infinitesimally 1-flat homogeneous spaces of dimension $n \geq 3$ and $0 \neq n(\text{mod } 4)$ are locally symmetric. For manifolds whose dimension is a multiple of four this remains an open problem.

In Section 3 we show that some cones over Riemannian symmetric spaces of rank one are infinitesimally 2-flat homogeneous, but not always semi-symmetric. This implies that k-flat rigidity of symmetric spaces is not an infinitesimal phenomenon for $k = 2$.

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2 Infinitesimally 1-flat homogeneous spaces

Let M be an infinitesimally 1-flat homogeneous space. We fix a point $p \in M$ and choose a unit tangent vector $X \in T_pM$. The Jacobi operator of M with respect to X is the self-adjoint endomorphism

$$R_X : T_pM \to T_pM \ , \ Y \mapsto R_XY := R(Y,X)X$$

of (T_pM, g_p). Let Y be an eigenvector of R_X with eigenvalue κ. For any $A \in \mathcal{A}_p$ we have

$$R_{AX}AY = R(AY, AX)AX = AR(Y,X)X = AR_XY = \kappa AY \ .$$

Since M is infinitesimally 1-flat homogeneous it follows that the spectrum of the Jacobi operator is independent of the choice of the unit tangent vector X at p. Riemannian manifolds with such a property are known as pointwise Osserman spaces [1].

The only known examples of pointwise Osserman spaces are two-dimensional Riemannian manifolds, four-dimensional self-dual Einstein manifolds and Riemannian manifolds which are locally isometric to two-point homogeneous spaces. A

Riemannian manifold which is locally isometric to a two-point homogeneous space is infinitesimally 1-flat homogeneous. For a two-dimensional Riemannian manifold M we have $\mathcal{A}_p = O(T_pM, g_p)$ for all $p \in M$, which implies that M is infinitesimally 1-flat homogeneous. The results in [1] also imply that any infinitesimally 1-flat homogeneous space M with $\dim M = 2m+1$ or $\dim M = 4m+2$ for some $m \geq 1$ is a real space form (in both cases) or a complex space form (only in the second case). We summarize the previous discussion about infinitesimally 1-flat homogeneous spaces in

Theorem 1. *The following statements hold:*

(a) Every infinitesimally 1-flat homogeneous space is a pointwise Osserman space;

(b) Every two-dimensional Riemannian manifold is infinitesimally 1-flat homogeneous;

(c) An odd-dimensional Riemannian manifold is infinitesimally 1-flat homogeneous if and only if it is a space of constant sectional curvature;

(d) A $(4m+2)$*-dimensional* $(m \geq 1)$ *Riemannian manifold is infinitesimally 1-flat homogeneous if and only if it is a space of constant sectional curvature or a Kähler manifold of constant holomorphic sectional curvature.*

From Theorem 1 we conclude that an infinitesimally 1-flat homogeneous space of dimension $n \geq 3$ and $0 \neq n (\mathrm{mod}\ 4)$ is locally symmetric. If $0 = n (\mathrm{mod}\ 4)$ this remains an open problem.

3 Infinitesimally 2-flat homogeneous spaces

We first describe some properties of the curvature tensor of cones. Let I be some open interval in $\mathbb{R}$ equipped with the canonical Riemannian metric dt^2 and let $a, b \in \mathbb{R}$ such that $a \neq 0$ and $f(t) = at + b > 0$ for all $t \in I$. Let M be a Riemannian manifold with Riemannian metric g. Then the cone $M_I^{a,b}$ is the smooth manifold $I \times M$ equipped with the Riemannian metric $\pi_1^* dt^2 + (f^2 \circ \pi_1)\pi_2^* g$, where $\pi_1 : I \times M \to I$ and $\pi_2 : I \times M \to M$ denote the canonical projections. The following lemma can be obtained by a straightforward calculation.

Lemma 1. *Let* $M_I^{a,b}$ *be a cone over a Riemannian manifold* (M, g) *with* $\dim M \geq 2$. *Let* $(t, q) \in M_I^{a,b}$ *and* $X, Y \in T_{(t,q)}M_I^{a,b}$ *be orthonormal vectors perpendicular to the unit vector* $T := \frac{\partial}{\partial t}(t) \in T_tI \subset T_tI \oplus T_qM = T_{(t,q)}M_I^{a,b}$. *We denote by* R *and* $R^\times$ *the curvature tensor of* $M_I^{a,b}$ *and the Riemannian product* $I \times M$ *at* (t, q), *respectively. Then*

$$R(X, T) = 0 \quad \text{and} \quad R(Y, X)X = R^\times(Y, X)X - \frac{a^2}{(at+b)^2} Y \ .$$

We briefly recall the classification of two-point homogeneous spaces: The Euclidean space $\mathbb{R}^n$ $(n \geq 1)$; the sphere S^n $(n \geq 1)$; the projective spaces $\mathbb{F}P^n$ $(n \geq 2)$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; the Cayley projective plane $\mathbb{O}P^2$; the hyperbolic spaces $\mathbb{F}H^n$

($n \geq 2$) over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; the Cayley hyperbolic plane $\mathbb{O}H^2$. The metric on $\mathbb{R}^n$ is the standard Euclidean metric, on S^1 one may take the metric which is induced from $\mathbb{R}^2$, and the metric on any other space is the unique (up to homothety) Riemannian metric turning it into a Riemannian symmetric space. The main result of this section is

Theorem 2. *Let M be a two-point homogeneous space. Then the cone $M_I^{a,b}$ over M is infinitesimally 2-flat homogeneous if and only if*

(a) $M \in \{\mathbb{R}^n, S^n, \mathbb{R}P^n, \mathbb{R}H^n, \mathbb{C}H^n, \mathbb{H}H^n, \mathbb{O}H^2\}$, or

(b) $M \in \{\mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2\}$ and a^2 is different from the minimum and the maximum of the sectional curvature of M.

Proof. Let M be a two-point homogeneous space. Each cone over a one-dimensional Riemannian manifold is flat and hence infinitesimally 2-flat homogeneous. We therefore assume $\dim M \geq 2$ from now on. With the above notations let $M_I^{a,b}$ be a cone over M and let $(t,q) \in M_I^{a,b}$ be arbitrary.

Lemma 1 shows that the 2-dimensional linear subspaces $\sigma_{T,X}$ spanned by T and some unit vector $X \in T_{(t,q)}(\{t\} \times M) \subset T_{(t,q)}M_I^{a,b}$ are infinitesimal 2-flats. Every isometry k of M with $k(q) = q$ extends to an isometry $\bar{k}$ of $M_I^{a,b}$ by $\bar{k}(s,p) := (s, k(p))$ for $(s,p) \in M_I^{a,b}$. Let $\sigma_{T,X}$ and $\sigma_{T,Y}$ be two infinitesimal 2-flats of $M_I^{a,b}$ at (t,q). Since M is two-point homogeneous there exists an isometry k of M with $k(q) = q$ and $k_*X = Y$. Then $\bar{k}_*$ maps $\sigma_{T,X}$ to $\sigma_{T,Y}$ and we see that the automorphism group $\mathcal{A}_{(t,q)}$ acts transitively on the set of all infinitesimal 2-flats of the form $\sigma_{T,X}$.

Let σ be an arbitrary 2-dimensional linear subspace of $T_{(t,q)}M_I^{a,b}$ which does not contain T. Then there exist $\lambda \in \mathbb{R}$ and orthonormal vectors $X, Y \in T_{(t,q)}(\{t\} \times M) \subset T_{(t,q)}M_I^{a,b}$ such that σ is the span of $\lambda T + X$ and Y. If σ is an infinitesimal 2-flat, Lemma 1 implies

$$0 = R(Y, \lambda T + X)(\lambda T + X) = R(Y, X)X = R^\times(Y, X)X - \frac{a^2}{(at+b)^2}Y \ .$$

The restriction of $R^\times$ to $T_qM \subset T_{(t,q)}(I \times M)$ is the curvature tensor R^M of M at q. The previous equation thus shows that $a^2/(at+b)^2$ is an eigenvalue of the Jacobi operator $T_qM \to T_qM$, $Z \mapsto R^M(Z, X)X$ of M with respect to X. Note that $(at+b)X$ is a unit tangent vector of M. If M is a space of constant curvature κ, the orthogonal complement of $\mathbb{R}X$ in T_qM is an eigenspace of the Jacobi operator of M with respect to $(at+b)X$ with corresponding eigenvalue κ. If $M \in \{\mathbb{R}H^n, \mathbb{C}H^n, \mathbb{H}H^n, \mathbb{O}H^2\}$ then M has negative sectional curvature and hence all eigenvalues of its Jacobi operators are nonpositive. Let $M \in \{\mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2\}$ and denote by κ the maximum of the sectional curvature on M. Then $\kappa/4$ is the minimum of the sectional curvature on M, and the eigenvalues of the Jacobi operator of M with respect to $(at+b)X$ corresponding to eigenvectors perpendicular to X are κ and $\kappa/4$. This discussion shows that every infinitesimal 2-flat in $T_{(t,q)}M_I^{a,b}$ contains T if and only if

(1) $M \in \{\mathbb{R}^n, \mathbb{R}H^n, \mathbb{C}H^n, \mathbb{H}H^n, \mathbb{O}H^2\}$, or

(2) $M \in \{S^n, \mathbb{R}P^n\}$ and a^2 is different from the sectional curvature of M, or

(3) $M \in \{\mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2\}$ and a^2 is different from the minimum and the maximum of the sectional curvature of M.

In all these cases we can now conclude that $M_I^{a,b}$ is infinitesimally 2-flat homogeneous.

If $M \in \{S^n, \mathbb{R}P^n\}$ and a^2 is equal to the sectional curvature of M then, by Lemma 1,

$$0 = R^\times(Y,X)X - \frac{a^2}{(at+b)^2}Y = R(Y,X)X = R(Y,\lambda T + X)(\lambda T + X)$$

for all $\lambda \in \mathbb{R}$ and all orthonormal vectors $X, Y \in T_{(t,q)}(\{t\} \times M) \subset T_{(t,q)}M_I^{a,b}$. Using the fact that the subspaces $\sigma_{T,Y}$ are infinitesimal 2-flats we see that every 2-dimensional linear subspace of $T_{(t,q)}M_I^{a,b}$ is an infinitesimal 2-flat. This shows that $M_I^{a,b}$ is flat, and hence in particular infinitesimally 2-flat homogeneous.

Finally, let $M \in \{\mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2\}$ and assume that a^2 is equal to the minimum or to the maximum of the sectional curvature of M. Let $X, Y \in T_{(t,q)}(\{t\} \times M) \subset T_{(t,q)}M_I^{a,b}$ be orthonormal such that Y is an eigenvector of the Jacobi operator of M with respect to $(at+b)X$ corresponding to the eigenvalue a^2. Then X is an eigenvector of the Jacobi operator of M with respect to $(at+b)Y$ corresponding to the same eigenvalue a^2, and from Lemma 1 we get

$$0 = R^\times(Y,X)X - \frac{a^2}{(at+b)^2}Y = R(Y,X)X$$

and

$$0 = R^\times(X,Y)Y - \frac{a^2}{(at+b)^2}X = R(X,Y)Y\ .$$

Therefore the 2-dimensional linear subspace $\sigma_{X,Y}$ of $T_{(t,q)}M_I^{a,b}$ spanned by X and Y is an infinitesimal 2-flat. On the other hand, if $Z \in T_{(t,q)}(\{t\} \times M) \subset T_{(t,q)}M_I^{a,b}$ is a unit vector which is an eigenvector of the Jacobi operator of M with respect to $(at+b)X$ corresponding to the non-zero eigenvalue different from a^2, we get from Lemma 1

$$0 \neq R^\times(Z,X)X - \frac{a^2}{(at+b)^2}Z = R(Z,X)X\ .$$

This shows that not every 2-dimensional linear subspace of $T_{(t,q)}M_I^{a,b}$ containing X is an infinitesimal 2-flat. Eventually, using again Lemma 1, we get

$$R(Z,\lambda T + X)(\lambda T + X) = R(Z,X)X \neq 0$$

for all $\lambda \in \mathbb{R}$. From this we see that T and $-T$ are the only unit vectors in $T_{(t,q)}M_I^{a,b}$ for which every 2-dimensional linear subspace containing this vector is an infinitesimal 2-flat. This implies that there cannot be an automorphism in $\mathcal{A}_{(t,q)}$ which maps $\sigma_{T,X}$ to $\sigma_{X,Y}$. It follows that $M_I^{a,b}$ is not infinitesimally 2-flat homogeneous. ∎

It can be seen from the classification of semi-symmetric spaces by Szabó in [5] that the cones over $\mathbb{R}^n$, S^n, $\mathbb{R}P^n$ and $\mathbb{R}H^n$ are semi-symmetric spaces, whereas the cones over $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$, $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$ are not semi-symmetric. We therefore conclude from Theorem 2 that there exist infinitesimally 2-flat homogeneous spaces which are not semi-symmetric. Thus the infinitesimal version of the rigidity result by Heintze-Palais-Terng-Thorbergsson and the second author does not hold.

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