On complex extrapolated successive overrelaxation (esor) : some theoretical results

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Abstract

In this paper we discuss the complex theory of the extrapolated successive overrelaxation (ESOR) method for the numerical solution of large sparse linear systems $A \cdot x = b$ of complex algebraic equations. Some subsets of convergence for this method are obtained through an application of conformal mapping techniques. We also study the choice of the involved complex parameters giving an arbitrarily "good" convergence behavior for the method. Among other results, it is shown that in general there is no value of the complex parameters maximizing the asymptotic rate of convergence and we investigate the conditions under which the complex extrapolated Gauss-Seidel (EGS) method converges as soon as possible..

1 General formulation

Let us consider a complex system of linear equations

$$A \cdot x = b \tag{1.1}$$

where A is a consistently ordered complex $m \times m$ matrix with non-vanishing diagonal elements and b is a given complex m-vector. By splitting A into $A = D - C_L - C_U$, where D is a diagonal matrix possessing the same diagonal elements as A and $-C_L, -C_U$ are the strictly lower and upper triangular parts of A respectively, we define the general extrapolated successive overrelaxation (ESOR) by

$$x^{(n+1)} = (1-\tau) \cdot x^{(n)} + \omega \cdot L x^{(n+1)} + (\tau - \omega) \cdot L x^{(n)} + \tau \cdot U x^{(n)} + \tau \cdot c \quad (n = 0, 1, 2, \ldots) \tag{1.2}$$

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where $L = D^{-1} \cdot C_L$, $U = D^{-1} \cdot C_U$, $c = D^{-1} \cdot b$ and $\omega, \tau (\neq 0)$ are complex parameters. By putting

$$L_{\tau,\omega} = (I - \omega \cdot L)^{-1} \cdot [(1-\tau) \cdot I + (\tau - \omega) \cdot L + \tau \cdot U] = I - \tau \cdot (I - \omega \cdot L)^{-1} \cdot D^{-1} \cdot A = I - \tau \cdot \Lambda_\omega$$

with

$$\Lambda_\omega = (I - \omega \cdot L)^{-1} \cdot D^{-1} \cdot A$$

we can write the ESOR method as

$$x^{(n+1)} = L_{\tau,\omega} x^{(n)} + \tau \cdot (I - \omega \cdot L)^{-1} \cdot c. \tag{1.3}$$

Obviously, when $\omega = 0$ or $\omega = 1$, (1.2) yields the JOR (:Jacobi overrelaxation) or EGS method, respectively; if $\omega = \tau$, (1.2) or (1.3) gives the SOR method. One may note, with the exception of the first iteration, the amount of work involved for the computation of one complete ESOR iteration is equivalent to that of an SOR one.

It is well known that the spectral radius $\rho(L_{\tau,\omega})$ for the $L_{\tau,\omega}$-matrix can be viewed as an asymptotic measure of how rapidly the sequence of the error vectors tends to 0. In what follows, we will study how to choose the complex numbers ω and τ in order to get "good convergence". To put it more precisely, we pose the following two problems:

PROBLEM 1. Assuming known distribution of the eigenvalues for the Jacobi iteration matrix

$$B = L + U,$$

determine the domain Ω in $\mathbb{C}^2$ into which

$$\rho(L_{\tau,\omega}) < 1 \qquad \text{for all } (\omega, \tau) \in \Omega$$

(Ω is called the convergence domain for (1.2) or (1.3).)

PROBLEM 2. Determine the values for ω and τ, if they exist, which are optimum, in the sense of minimising the spectral radius $\rho(L_{\tau,\omega})$.

In the real case, that is when the iteration matrix B possesses only real eigenvalues and the parameters ω and τ are in $\mathbb{R}$, many authors independently presented interesting results ([1], [2], [3], [4], [7], [8], [9]). However, the detailed analysis was not presented since a tremendous number of cases had to be examined.

The purpose of this paper is to study the complex case, that is when B possesses complex eigenvalues and the parameters ω and τ are in $\mathbb{C}$. In Section 2, we follow step-by-step the analysis in [6] and [5] and we give some answers to the first problem by showing that if $\omega \in \mathbb{C}$, $\tau \in \mathbb{C}$ and if $|\omega - 1| < 1$ and

$$\Delta\left((1/\omega); (1/|\omega|)\right) \subset \Delta\left((1/\tau); (1/|\tau|)\right) \text{ or } \tau \in \omega - \overline{\Delta(1;1)} \cdot \Delta\left(\frac{\omega^2+1}{\omega}; \frac{1}{|\omega|}\right), \quad (\omega \neq 1)$$

then $\rho(L_{\tau,\omega}) < 1$. In particular, $\{(\omega, \tau) \in \mathbb{C}^2 : 0 < |\omega - 1| < 1 \text{ and } \omega = k \cdot \tau (k \geq 1)\} \subset \Omega$ (Corollary 2.3). Here, $\Delta(a; r)$ denotes the open planar disk centered at $a \in \mathbb{C}$ and with radius $r > 0$. The case $\omega = 1$ is studied separately. In Section 3, we shall see that in general there is no $(\omega, \tau) \in \mathbb{C}^2$ that minimizes the spectral radius $\rho(L_{\tau,\omega})$ and we investigate the conditions under which for $\varepsilon > 0$ there exists a $(\omega, \tau) \in \mathbb{C}^2$ such that $\rho(L_{\tau,\omega}) < \varepsilon$. Among other results, it is shown that if the spectrum of B is contained in the open interval $(-1, 1)$ and if $0 < \varepsilon < 2$, then for $\omega = 1$ and $\tau = x + i \cdot y$, with $0 < x < 1$ and $0 < y < \sqrt{\varepsilon \cdot (2 - \varepsilon)}$, we have $\rho(L_{\tau,1}) < \varepsilon$ (Corollary 3.8). However, if the Jacobi matrix B has a critical eigenvalue-pair $\pm\tilde{\mu}$, that is a pair which corresponds to the dominant absolute value of the eigenvalues of the $L_{\tau,\omega}$-matrix whenever $(\omega, \tau) \in \mathbb{C}^2$, then for

$$\omega = \frac{2}{1 \mp \sqrt{1 - \tilde{\mu}^2}} \quad \text{and} \quad \tau = \frac{1}{\mp\sqrt{1 - \tilde{\mu}^2}} \qquad (\tilde{\mu} \neq 0, \pm 1)$$

there holds $\rho(L_{\tau,\omega}) = 0$ (see also [7] for the real case). Here, $\sqrt{A}$ (with $A \in \mathbb{C} - \{0\}$) denotes the principal value of $A^{\frac{1}{2}}$, that is $\sqrt{A} = \exp\left(\frac{1}{2} \cdot \log A\right) = \exp\left(\frac{1}{2} \cdot [\ln |A| + i \cdot \arg A]\right)$, where $\log A$ is the principal logarithmic value of A and $\arg A$ is the principal argument of A.

2 On the convergence properties of the ESOR method

A sufficient and necessary condition for ESOR to converge is $\rho(L_{\tau,\omega}) < 1$. To determine a large subset D of $\mathbb{C}^2$, so that $\rho(L_{\tau,\omega}) < 1$ for any $(\omega, \tau) \in D$, we shall consider a geometric interpretation of the relations between the eigenvalues of B, Λ_ω and $L_{\tau,\omega}$.

Let us introduce the following notations:

$$\begin{aligned}
\sigma(B) &= \{\mu : \text{ eigenvalue of } B\}, \\
\sigma(\Lambda_\omega) &= \left\{\lambda : \text{ eigenvalue of } \Lambda_\omega = (I - \omega \cdot L)^{-1} \cdot D^{-1} \cdot A\right\}, \\
\sigma(L_{\tau,\omega}) &= \left\{\zeta : \text{ eigenvalue of } L_{\tau,\omega} = I - \tau \cdot (I - \omega \cdot L)^{-1} \cdot D^{-1} \cdot A\right\}.
\end{aligned}$$

First, observe that the identity $L_{\tau,\omega} = I - \tau \cdot \Lambda_\omega$ implies a linear relation between $\sigma(\Lambda_\omega)$ and $\sigma(L_{\tau,\omega})$: if ζ is an eigenvalue of $L_{\tau,\omega}$, then

$$\zeta = 1 - \tau \cdot \lambda \tag{2.1}$$

where λ is an eigenvalue of Λ_ω. Next, a Young-type result is well known. Its proof is analogous to that of Theorem 5-2.2 in [12] and is therefore omitted.

Theorem 2.1. *If $\mu \in \sigma(B)$ and λ satisfies*

$$(1 - \lambda)^2 = \mu^2 \cdot (1 - \lambda \cdot \omega) \tag{2.2}$$

then $\lambda \in \sigma(\Lambda_\omega)$; conversely, if $\lambda \in \sigma(\Lambda_\omega)$ and μ satisfies (2.2) then $\mu \in \sigma(B)$.

Due to this Theorem, the B-matrix has always eigenvalues in pairs: if μ is an eigenvalue of B, then $-\mu$ is also an eigenvalue of B.

Suppose now

$$\omega \in \mathbb{C} - \{0, 1\}.$$

The above Theorem describes a mapping between the complex μ- and λ-planes and is studied by means of successive elementary transformations. Evidently, (2.2) is equivalent to

$$\mu \cdot \frac{\omega}{\sqrt{\omega-1}} = \sqrt{\frac{\omega-1}{1-\lambda\cdot\omega}} + \sqrt{\frac{1-\lambda\cdot\omega}{\omega-1}}.$$

With

$$2\cdot z = \frac{\omega}{\sqrt{\omega-1}}\cdot\mu \quad \text{and} \quad \xi = \sqrt{\frac{\omega-1}{1-\lambda\cdot\omega}} \tag{2.3}$$

this becomes

$$z = \frac{1}{2}\cdot\left(\xi + \frac{1}{\xi}\right) \quad \text{and} \quad \xi = z + \sqrt{z^2-1}. \tag{2.4}$$

The map defined by (2.4) is the well known Joukowski function. Putting $a := \sqrt{\omega-1}$ (2.4) gives

$$z = \frac{1}{2}\cdot\left(a + \frac{1}{a}\right)\cdot\mu. \tag{2.5}$$

Now, let $\pm\mu$ be an eigenvalue-pair for the B-matrix. By (2.5), these correspond to the points z^+ and z^- in Figure 1 below. There is now an ellipse $\mathcal{E}_{|\alpha|}$ such that z^+ and z^- are two points interior to $\mathcal{E}_{|\alpha|}$. By (2.4), this ellipse is mapped on two circles $|\xi| = |a|$ and $|\xi| = \frac{1}{|a|}$ in the ξ-plane and its interior is mapped on to the annulus $|a| < |\xi| < \frac{1}{|a|}$ in the ξ-plane.

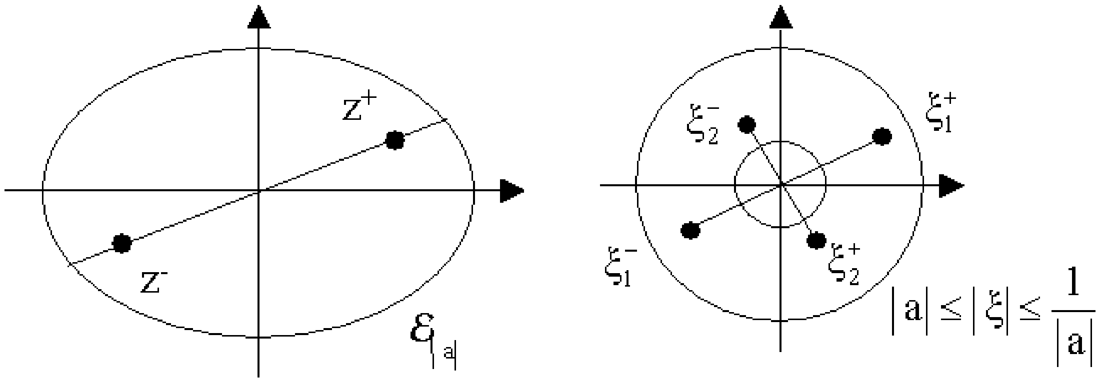

Figure 1

Since (2.3) is equivalent to $\xi^2 - \xi^2\cdot\lambda\cdot\omega = \omega - 1$, it follows that

$$\lambda = \frac{1}{\omega} - \frac{\omega-1}{\omega}\cdot\frac{1}{\xi^2} \tag{2.6}$$

and thus

$$|a| \le |\xi| \le \frac{1}{|a|} \overset{(2.6)}{\Longleftrightarrow} \frac{|\omega-1|^2}{|\omega|} \le \left|\lambda - \frac{1}{\omega}\right| \le \frac{1}{|\omega|}$$

which means that the two circles $|\xi| = |a|$ and $|\xi| = \frac{1}{|a|}$ in the ξ-plane are mapped on the two circles $\left|\lambda - \frac{1}{\omega}\right| = \frac{1}{|\omega|}$ and $\left|\lambda - \frac{1}{\omega}\right| = \frac{|\omega-1|^2}{|\omega|}$ in the λ-plane respectively, and that the annulus $|a| < |\xi| < \frac{1}{|a|}$ in the ξ-plane is mapped on to the annulus

$$\frac{|\omega-1|^2}{|\omega|} < \left|\lambda - \frac{1}{\omega}\right| < \frac{1}{|\omega|} \tag{2.7}$$

in the λ-plane.

We may now formulate the main results of this Section.

Theorem 2.2. *If the relaxation factors ω and τ fulfill*

$$0 < |\omega - 1| < 1 \quad \textit{and} \quad \Delta\left((1/\omega);(1/|\omega|)\right) \subset \Delta\left((1/\tau);(1/|\tau|)\right),$$

then the ESOR method converges (under the assumptions that all eigenvalues of the B-matrix belong to the interior of the ellipse $\mathcal{E}(\sqrt{\omega-1})$ which has the semi-axes

$$\left(\frac{1}{|\sqrt{\omega-1}|} \pm |\sqrt{\omega-1}|\right) \cdot \left(\left|\frac{1}{\sqrt{\omega-1}} \pm \sqrt{\omega-1}\right|\right)^{-1}$$

and the larger semi-axis of which forms an angle

$$\phi = \arg tg \left\{ \frac{1 - |\sqrt{\omega-1}|^2}{1 + |\sqrt{\omega-1}|^2} \cdot tg(\arg \sqrt{\omega-1}) \right\}$$

with the real axis in the μ-plane).

Proof. By (2.5), every point interior to the ellipse $\mathcal{E}(\sqrt{\omega-1})$ in the μ-plane is mapped on the interior of the ellipse $\mathcal{E}_{|\sqrt{\omega-1}|}$ in the z-plane which has the semi axes $\frac{1}{2} \cdot \left| |\sqrt{\omega-1}| \pm |\sqrt{\omega-1}|^{-1} \right|$. By (2.4), every point interior to $\mathcal{E}_{|\sqrt{\omega-1}|}$ in the z-plane is mapped on two points in the annulus $(|\omega-1|^2/|\omega|) < |\lambda - (1/\omega)| < (1/|\omega|)$ in the λ-plane. Thus, if $\sigma(B) \subseteq \mathcal{E}(\sqrt{\omega-1})$, the inequalities (2.7) hold for all $\lambda \in \sigma(\Lambda_\omega)$. In particular, there holds $\sigma(\Lambda_\omega) \subseteq \Delta\left((1/\omega);(1/|\omega|)\right) \subset \Delta\left((1/\tau);(1/|\tau|)\right)$, which implies $|1 - \tau \cdot \lambda| < 1$, for all $\lambda \in \sigma(\Lambda_\omega)$. From (2.1), if follows that $|\zeta| < 1$ for any $\zeta \in \sigma(L_{\tau,\omega})$, that is $\rho(L_{\tau,\omega}) < 1$.

Since the relationship $\omega = k \cdot \tau$ $(k \geq 1)$ can also be regarded as an inclusion $\Delta\left((1/\omega);(1/|\omega|)\right) \subset \Delta\left((1/\tau);(1/|\tau|)\right)$, we immediately get the

Corollary 2.3. *If the relaxation factor ω is as in the above Theorem, and if $\tau = \frac{\omega}{k}$ for some $k \in [1, +\infty]$, then $\rho(L_{\tau,\omega}) < 1$.*

In what follows, for any subset U of $\mathbb{C} - \{0\}$, we will denote by U^{-1} the set $\{z \in \mathbb{C} - \{0\} : z^{-1} \in U\}$. Further, if A and B are two subsets of $\mathbb{C}$, then we will denote by $A \cdot B$ the set $\{a \cdot b : a \in A \text{ and } b \in B\}$.

Theorem 2.4. *If the relaxation factors ω and τ fulfill*

$$0 < |\omega - 1| < 1 \quad \textit{and} \quad \tau \in \omega \cdot \overline{\Delta(2;1)} - \overline{\Delta(1;1)} \cdot \Delta^{-1}\left((1/\omega);(1/|\omega|)\right),$$

the ESOR method converges (under the assumption that all eigenvalues of B belong to the interior of the ellipse $\mathcal{E}(\sqrt{\omega-1})$). In particular, if

$$0 < |\omega - 1| < 1 \quad \textit{and} \quad |\tau - 2\omega| \leq |\omega|$$

then $\rho(L_{\tau,\omega}) < 1$.

Proof. As in the proof of Theorem 2.2, one can show that

$$|1-\omega\cdot\lambda|<1 \quad \text{for all} \quad \lambda\in\sigma(\Lambda_\omega). \tag{2.8}$$

Choose any point $(\zeta, u^{-1}) \in \overline{\Delta(1,1)}\times\Delta^{-1}\left(\frac{1}{\omega};\frac{1}{|\omega|}\right)$. Setting $\tau=\omega\cdot(1+\zeta)-\zeta\cdot u^{-1}$, it is easily seen that if $\zeta=0$ then $\tau=\omega$, otherwise the point $s:=1+(\omega-\tau)\cdot(u^{-1}-\omega)^{-1}$ lies in $\overline{\Delta(0;1)}$, which implies that $s=(1-\tau\cdot u)\cdot(1-\omega\cdot u)\in\overline{\Delta(0;1)}$. In particular, there holds

$$|1-\tau\cdot\lambda|\le|1-\omega\cdot\lambda| \quad \text{for all} \quad \lambda\in\sigma(\Lambda_\omega).$$

Combination with (2.8) shows that $|\zeta|<1$ for any $\zeta=(1-\tau\cdot\lambda)\in\sigma(L_{\tau,\omega})$, which completes the proof.

Next, we shall see how the investigation for the domain of convergence Ω can be cleared of its dependence on the theory of conformal mappings and reconnected to an elementary algebraic treatment so that a suitable determination for Ω is obtained.

Theorem 2.5. *For any $\omega\in\mathbb{C}$, put $\lambda_\omega:=\max\{|\lambda|^{-2}\cdot(-|\lambda|-Im\lambda):\lambda\in\sigma(\Lambda_\omega)\}$ and $\widetilde{\lambda}_\omega:=\min\{|\lambda|^{-2}\cdot(|\lambda|-Im\lambda):\lambda\in\sigma(\Lambda_\omega)\}$ and consider the open set*

$$S_\omega:=\left\{y\in\mathbb{R}:\lambda_\omega<y<\widetilde{\lambda}_\omega\right\}$$

and its subset

$$F_\omega:=\left\{y\in S_\omega:\max_{\lambda\in\sigma(\Lambda_\omega)}\frac{Re\lambda-\sqrt{|\lambda|^2-(|\lambda|^2\cdot y+Im\lambda)^2}}{|\lambda|^2}<\min_{\lambda\in\sigma(\Lambda_\omega)}\frac{Re\lambda+\sqrt{|\lambda|^2-(|\lambda|^2\cdot y+Im\lambda)^2}}{|\lambda|^2}\right\}.$$

If $G:=\{\omega\in\mathbb{C}:F_\omega\neq\varnothing\}$, the domain of convergence ω for the ESOR method is

$$\Omega=G\times\Big\{x+iy\in\mathbb{C}:y\in F_\omega,\quad \omega\in G \quad \text{and} \quad \max_{\lambda\in\sigma(\Lambda_\omega)}\frac{Re\lambda-\sqrt{|\lambda|^2-(|\lambda|^2\cdot y+Im\lambda)^2}}{|\lambda|^2}<x<\min_{\lambda\in\sigma(\Lambda_\omega)}\frac{Re\lambda+\sqrt{|\lambda|^2-(|\lambda|^2\cdot y+Im\lambda)^2}}{|\lambda|^2}\Big\}.$$

Proof. Let $(\omega,\tau)\in\Omega$. As mentioned above, this is equivalent to $|1-\tau\cdot\lambda|<1$ for $\lambda\in\sigma(\Lambda_\omega)$. Putting $\tau=x+iy$ and $\lambda=a+ib$, we have $(a^2+b^2)x^2-(2a)\cdot x+(y^2\cdot(a^2+b^2)+2y\cdot b)<0$ or equivalently

$$a^2-(a^2+b^2)\cdot\left[(a^2+b^2)y^2+2yb\right]>0 \tag{2.9}$$

and

$$\frac{a-\sqrt{a^2-(a^2+b^2)\cdot[(a^2+b^2)y^2+2yb]}}{a^2+b^2}<x<\frac{a+\sqrt{a^2-(a^2+b^2)\cdot[(a^2+b^2)y^2+2yb]}}{a^2+b^2}. \tag{2.10}$$

It is easy to verify that (2.9) holds for any $(a+ib) \in \sigma(\Lambda_\omega)$ if and only if $y \in S_\omega$ and that (2.10) is fulfilled for every $(a+ib) \in \sigma(\Lambda_\omega)$ if and only if $y \in F_\omega$ and

$$\max_{\lambda\in\sigma(\Lambda_\omega)} \frac{Re\lambda - \sqrt{|\lambda|^2 - (|\lambda|^2 \cdot y + Im\lambda)^2}}{|\lambda|^2} < x < \min_{\lambda\in\sigma(\Lambda_\omega)} \frac{Re\lambda + \sqrt{|\lambda|^2 - (|\lambda|^2 \cdot y + Im\lambda)^2}}{|\lambda|^2}.$$

The assumptions of the above Theorem seem to be very technical, but, on the other hand, its proof generalizes to the context of the problem of optimum values (see Theorem 3.4). For instance let us give a direct consequence of this Theorem.

Corollary 2.6. *If $\omega \in \mathbb{C}$ is chosen so that $Re\lambda > 0$ for any $\lambda \in \sigma(\Lambda_\omega)$ and if $\tau = x + iy$ satisfies*

$$0 < y < \min_{\lambda\in\sigma(\Lambda_\omega)} \frac{|\lambda|^2 - Im\lambda}{|\lambda|^2} \quad \textit{and} \quad 0 < x < \min_{\lambda\in\sigma(\Lambda_\omega)} \frac{Re\lambda + \sqrt{|\lambda|^2 - (|\lambda|^2 \cdot y + Im\lambda)^2}}{|\lambda|^2}$$

then $(\omega, \tau) \in \Omega$.

Let us finally turn to the special cases $\omega = 0$ and $\omega = 1$.

If $\omega = 0$, the (1.2) yields the JOR method:

$$x^{(n+1)} = L_{\tau,0}x^{(n)} + \tau \cdot c = [I - \tau \cdot \Lambda_0] \cdot x^{(n)} + \tau \cdot c. \tag{2.11}$$

From (2.2), it follows that if $\mu \in \sigma(B)$ then $\lambda = (1 \pm \mu) \in \sigma(\Lambda_0)$, and conversely, if $\lambda \in \sigma(\Lambda_0)$ then $\mu = \pm(1-\lambda) \in \sigma(B)$. By (2.1), we therefore have: $\rho(L_{\tau,0}) < 1$ iff $|1 - \tau \cdot (1 \pm \mu)|^2 < 1$ for all $\mu \in \sigma(B)$ or iff $[1 - \tau \cdot (1 \pm \mu)] \cdot [1 - \overline{\tau} \cdot (1 \pm \overline{\mu})] = 1 - \overline{\tau} \cdot (1 \pm \overline{\mu}) - \tau \cdot (1 \pm \mu) + |\tau|^2 \cdot |1 \pm \mu|^2 = 1 - 2 \cdot Re[\tau \cdot (1 \pm \mu)] + \{Re[\tau \cdot (1 \pm \mu)]\}^2 + \{Im[\tau \cdot (1 \pm \mu)]\}^2 = \{Re[\tau \cdot (1 \pm \mu)] - 1\}^2 + \{Im[\tau \cdot (1 \pm \mu)]\}^2 < 1$ for all $\mu \in \sigma(B)$. Hence

Theorem 2.7. *If $\pm 1 \notin \sigma(B)$, then a necessary and sufficient condition for the JOR to converge is the validity of the following inequality*

$$\{Re[\tau \cdot (1 \pm \mu)] - 1\}^2 < 1 - \{Im[\tau \cdot (1 \pm \mu)]\}^2 \quad \textit{for any} \quad \mu \in \sigma(B).$$

Corollary 2.8. ([11]) *Suppose $\sigma(B) \subset (-1, 1)$. If*

$$0 < \tau < \frac{2}{1 \pm \mu} \quad \textit{for any} \quad \mu \in \sigma(B),$$

the JOR method converges.

If $\omega = 1$, the (1.2) gives the EGS method:

$$x^{(n+1)} = L_{\tau,1}x^{(n)} + \tau \cdot (I-L)^{-1} \cdot c = [I - \tau \cdot \Lambda_1] \cdot x^{(n)} + \tau \cdot (I-L)^{-1} \cdot c. \tag{2.12}$$

If μ is any eigenvalue of B, then $\lambda = (1-\mu^2) \in \sigma(\Lambda_1)$, because of (2.2). Conversely, if $\lambda \in \sigma(\Lambda_1)$, then, by Theorem 2.1, there exists a $\mu \in \sigma(B)$ such that $\lambda = 1 - \mu^2$. From (2.1), it follows that the inequality $\rho(L_{\tau,1}) < 1$ holds iff $|1 - \tau \cdot (1-\mu^2)|^2 < 1$ for all $\mu \in \sigma(B)$. Since

$$\begin{aligned}\left|1 - \tau \cdot (1-\mu^2)\right|^2 &= \left[1 - \tau \cdot (1-\mu^2)\right] \cdot \left[1 - \overline{\tau} \cdot (1-\overline{\mu}^2)\right] = \\ &1 - 2 \cdot Re\left[\tau \cdot (1-\mu^2)\right] + \left\{Re\left[\tau \cdot (1-\mu^2)\right]\right\}^2 + \left\{Im\left[\tau \cdot (1-\mu^2)\right]\right\}^2 = \\ &\qquad \left\{Re\left[\tau \cdot (1-\mu^2)\right] - 1\right\}^2 + \left\{Im\left[\tau \cdot (1-\mu^2)\right]\right\}^2,\end{aligned}$$

we immediately establish the

Theorem 2.9. *A necessary and sufficient condition for EGS to converge is the validity of the following inequality*

$$\left\{Re[\tau\cdot(1-\mu^2)]-1\right\}^2<1-\left\{Im[\tau\cdot(1-\mu^2)]\right\}^2 \quad \textit{for any} \quad \mu\in\sigma(B),$$

of course under the assumption $\pm 1 \notin \sigma(B)$.

Corollary 2.10. ([11]) *Suppose* $\sigma(B)\subset(-1,1)$. *If*

$$0<\tau<\frac{2}{1-\mu^2} \quad \textit{for any} \quad \mu\in\sigma(B),$$

the EGS method converges.

3 On the existence of optimum values for the ESOR method

We first consider a general optimization problem. Let $U\neq\varnothing$ be an open connected subset of $\mathbb{C}^n$ and let $u_1,u_2,\ldots,u_m,\nu_1,\nu_2,\ldots,\nu_m$ be non constant holomorphic functions in U, such that $u_j^2(t)\neq 4\nu_j(t)$ for any $t\in U$ $(j=1,2,\ldots,m)$. If each equation $z^2+u_j(t)\cdot z+\nu_j(t)=0$ $(t\in U,\ j=1,2,\ldots,m)$ has roots $z_j(u_j(t),\nu_j(t))$ and $z_j'(u_j(t),\nu_j(t))$, we shall prove that the value t_0 of $t\in U$ (if it exists) for which the function

$$f(t):=\max\Big\{|z_1(u_1(t),\nu_1(t))|,\left|z_1'(u_1(t),\nu_1(t))\right|,\ldots,\\ |z_m(u_m(t),\nu_m(t))|,\left|z_m'(u_m(t),\nu_m(t))\right|\Big\}$$

is minimized is characterized by the fact that the absolute values of the roots are equal.

More precisely, we have the following

Theorem 3.1. *If there is a* $t_0\in U$ *fulfilling*

$$f(t_0)\leq f(t)$$

for any $t\in U$, *then there holds*

$$|z_1(u_1(t_0),\nu_1(t_0))|=\left|z_1'(u_1(t_0),\nu_1(t_0))\right|=\ldots=\\ |z_m(u_m(t_0),\nu_m(t_0))|=\left|z_m'(u_m(t_0),\nu_m(t_0))\right|.$$

Proof. Obviously, the functions

$$h_{2j-1}\left(u_j(t),\nu_j(t)\right)=z_j\left(u_j(t),\nu_j(t)\right) \quad\text{and}\quad h_{2j}\left(u_j(t),\nu_j(t)\right)=z_j'\left(u_j(t),\nu_j(t)\right)$$

are holomorphic in $t\in U$, for any j. Suppose there is a $t_0\in U$ satisfying

$$f(t_0)\leq f(t) \tag{3.1}$$

for any $t\in U$. Further, assume that

$$|h_\beta\left(u_b(t_0),\nu_b(t_0)\right)|<|h_\alpha\left(u_a(t),\nu_a(t)\right)| \tag{3.2}$$

for some $\alpha \neq \beta$ $(\alpha, \beta = 1, 2, \ldots, 2m)$, $b \in \{(\beta/2), (\beta+1)/2\}$, $a \in \{(\alpha/2), (\alpha+1)/2\}$. By continuity, it is clear that there is an open neighborhood $V_{t_0} \subset U$ of t_0 into which we have

$$|h_\beta\left(u_b(t), \nu_b(t)\right)| < |h_\alpha\left(u_a(t), \nu_a(t)\right)| \tag{3.3}$$

for every $t \in V_{t_0}$. We shall prove that there exists a $\widetilde{t} \in V_{t_0}$ such that $\left|h_\alpha\left(u_a(\widetilde{t}), \nu_a(\widetilde{t})\right)\right|$ $< |h_\alpha\left(u_a(t_0), \nu_a(t_0)\right)|$. To do so, suppose

$$|h_\alpha\left(u_a(t_0), \nu_a(t_0)\right)| < |h_\alpha\left(u_a(t), \nu_a(t)\right)| \tag{3.4}$$

for all $t \in V_{t_0}$. By (3.2), $h_\alpha(u_a(t_0), \nu_a(t_0)) \neq 0$, and, by (3.4), $h_\alpha(u_a(t), \nu_a(t)) \neq 0$ for any $t \in V_{t_0}$. From the minimum principle for holomorphic functions and from (3.4), it follows that the function $h_\alpha(u_a(t), \nu_a(t))$ is constant in V_{t_0}. By the identity theorem, the holomorphic function $h_\alpha(u_a(t), \nu_a(t))$ must be constant in the open connected set U. This is an absurdity. Consequently, there exist a point $\widetilde{t} \in V_{t_0}$ satisfying

$$\left|h_\alpha\left(u_a(\widetilde{t}), \nu_a(\widetilde{t})\right)\right| < |h_\alpha\left(u_a(t_0), \nu_a(t_0)\right)|. \tag{3.5}$$

By (3.3) and (3.5), we thus obtain the inequalities

$$\left|h_\beta\left(u_b(\widetilde{t}), \nu_b(\widetilde{t})\right)\right| < \left|h_\alpha\left(u_a(\widetilde{t}), \nu_a(\widetilde{t})\right)\right| < |h_\alpha\left(u_a(t_0), \nu_a(t_0)\right)|,$$

which contradict (3.1). Hence, the point $t_0 \in U$ must be such that $|h_\beta\left(u_b(t_0), \nu_b(t_0)\right)|$ $= |h_\alpha\left(u_a(t_0)\ ,\ \nu_a(t_0)\right)|$, for any $\alpha, \beta = 1, 2, \ldots, 2m$ and $b \in \{(\beta/2), (\beta+1)/2\}$, $a \in \{(\alpha/2), (\alpha+1)/2\}$. The proof is complete.

We shall now study the problem of determination of optimum values for the parameters ω and τ. Assume that (ω, τ) is a fixed point in the convergence domain Ω for (1.2) (or (1.3)), i.e. $\rho(L_{\tau,\omega}) < 1$. Further, suppose

$$\omega \notin \left\{\{0, 2\} \cup \frac{1}{1-\mu^2} \cdot (-\infty, -4] : \mu \in \sigma(B) - \{\pm 1\}\right\}.$$

This, in particular, implies that $(\omega \cdot \mu^2 - 4\omega + 4) \notin (-\infty, 0]$ for any $\mu \in \sigma(B)$ and therefore the expression

$$\sqrt{(2-\omega\cdot\mu^2)^2 - 4(1-\mu^2)} = \sqrt{\omega\cdot\mu^2 - 4\cdot\omega + 4} = \exp\left(\frac{1}{2}\cdot\log[\omega\cdot\mu^2 - 4\omega + 4]\right)$$

is well defined and holomorphic in ω for all $\mu \in \sigma(B)$.

Let $\zeta \in \sigma(L_{\tau,\omega})$. Since $L_{\tau,\omega} = I - \tau\cdot\Lambda_\omega$, the complex number

$$\lambda = \frac{(1-\zeta)}{\tau} \tag{3.6}$$

is an eigenvalue of Λ_ω, and, by Theorem 2.1, any root μ of the equation

$$(1-\lambda)^2 = \mu^2 \cdot (1 - \lambda\cdot\omega) \tag{3.7}$$

is an eigenvalue of B. In view of (3.6), the equation (3.7) becomes

$$\zeta^2 - \left(\tau\cdot\omega\cdot\mu^2 - 2\tau + 2\right)\cdot\zeta + \left(\tau\cdot\omega\cdot\mu^2 - 2\tau + 1 + \tau^2 - \tau^2\cdot\mu^2\right) = 0. \tag{3.8}$$

Thus, if $\{\mu_1, \ldots, \mu_m\}$ is the set $\sigma(B)$ of all eigenvalues of B, then for any $j = 1, 2, \ldots, m$ there holds

$$\zeta^2 + u_j(\omega, \tau) \cdot \zeta + \nu_j(\omega, \tau) = 0 \tag{3.9}$$

with

$$u_j(\omega, \tau) = -\left(\tau \cdot \omega \cdot \mu_j^2 - 2\tau + 2\right) \text{ and } \nu_j(\omega, \tau) = \left(\tau \cdot \omega \cdot \mu_j^2 - 2\tau + 1 + \tau^2 - \tau^2 \cdot \mu_j^2\right).$$

The roots of each equation (3.9) are

$$\zeta_j(u_j(\omega, \tau), \nu_j(\omega, \tau)) = 1 - \tau \cdot \left(\frac{2 - \omega \cdot \mu_j^2 - \mu_j \sqrt{\omega^2 \cdot \mu_j^2 - 4\omega + 4}}{2}\right)$$

and

$$\zeta_j'(u_j(\omega, \tau), \nu_j(\omega, \tau)) = 1 - \tau \cdot \left(\frac{2 - \omega \cdot \mu_j^2 + \mu_j \sqrt{\omega^2 \cdot \mu_j^2 - 4\omega + 4}}{2}\right).$$

An application of Theorem 3.1 in each open connected component U of

$$\Omega - \left\{(\omega, \tau) \in \mathbb{C}^2 : \omega \in \{0, 2\} \cup \left\{\frac{1}{1 - \mu^2} \cdot (-\infty, -4] : \mu \in \sigma(B) - \{\pm 1\}\right\}\right\}$$

leads to the following

Theorem 3.2. *If the point*

$$(\omega_0, \tau_0) \in \Omega - \left\{(\omega, \tau) \in \mathbb{C}^2 : \omega \in \{0, 2\} \cup \left\{\frac{1}{1 - \mu^2} \cdot (-\infty, -4] : \mu \in \sigma(B) - \{\pm 1\}\right\}\right\}$$

minimizes the spectral radius $\rho(L_{\tau,\omega})$, then there holds

$$\left|1 - \tau_0 \cdot \left(\frac{2 - \omega_0 \cdot \mu_j^2 \pm \sqrt{\omega_0^2 \cdot \mu_j^2 - 4\omega_0 + 4}}{2}\right)\right| = \left|1 - \tau_0 \cdot \left(\frac{2 - \omega_0 \cdot \mu_i^2 \pm \sqrt{\omega_0^2 \cdot \mu_i^2 - 4\omega_0 + 4}}{2}\right)\right|$$

for any $\mu_j, \mu_i \in \sigma(B)$.

This result is purely theoretic. However, in the sequel, by using this result, we will study the possibility of existence of an optimum value $(\omega, \tau) \in \Omega$ which minimizes $\rho(L_{\tau,\omega})$.

Let

$$\lambda_{2j-1}(\omega) = \left(\frac{2 - \omega \cdot \mu_j^2 - \sqrt{\omega^2 \cdot \mu_j^2 - 4\omega + 4}}{2}\right),$$

$$\lambda_{2j}(\omega) = \left(\frac{2 - \omega \cdot \mu_j^2 + \sqrt{\omega^2 \cdot \mu_j^2 - 4\omega + 4}}{2}\right) \qquad (1 \leq j \leq m).$$

Suppose (ω, τ) is an optimum value for the ESOR method. According to Theorem 3.2, it must hold

$$\left|\frac{1-\tau\cdot\lambda_\alpha(\omega)}{1-\tau\cdot\lambda_\beta(\omega)}\right| = 1 \Leftrightarrow \left|\frac{(1/\lambda_\alpha(\omega))-\tau}{(1/\lambda_\beta(\omega))-\tau}\right| = \left|\frac{\lambda_\beta(\omega)}{\lambda_\alpha(\omega)}\right| \tag{3.10}$$

for every $\alpha, \beta = 1, 2, \ldots, 2m$.
Each equation (3.10) represents a circle $C(H_{\alpha,\beta}(\omega); R_{\alpha,\beta}(\omega))$ with

- center $$H_{\alpha,\beta}(\omega) = \frac{\dfrac{1}{\lambda_\alpha(\omega)} - \left|\dfrac{\lambda_\beta(\omega)}{\lambda_\alpha(\omega)}\right|^2 \cdot \dfrac{1}{\lambda_\beta(\omega)}}{1 - \left|\dfrac{\lambda_\beta(\omega)}{\lambda_\alpha(\omega)}\right|^2} = \frac{|\lambda_\alpha(\omega)|^2\cdot\lambda_\beta(\omega) - |\lambda_\beta(\omega)|^2\cdot\lambda_\alpha(\omega)}{|\lambda_\alpha(\omega)|^2\cdot\lambda_\beta(\omega)\cdot\lambda_\alpha(\omega) - |\lambda_\beta(\omega)|^2\cdot\lambda_\alpha(\omega)\cdot\lambda_\beta(\omega)}$$

and

- radius $$R_{\alpha,\beta}(\omega) = \frac{\left|\dfrac{\lambda_\beta(\omega)}{\lambda_\alpha(\omega)}\right| \cdot \left|\dfrac{1}{\lambda_\alpha(\omega)} - \dfrac{1}{\lambda_\beta(\omega)}\right|}{\left|1 - \left|\dfrac{\lambda_\beta(\omega)}{\lambda_\alpha(\omega)}\right|^2\right|} = \frac{|\lambda_\alpha(\omega) - \lambda_\beta(\omega)|}{||\lambda_\alpha(\omega)|^2 - |\lambda_\beta(\omega)|^2|}.$$

In other words, we have the

Theorem 3.3. *The point $(\omega, \tau) \in \mathbb{C}^2$ is an optimum value for the ESOR method if and only if τ lies in the intersection of the circles $C(H_{\alpha,\beta}(\omega); R_{\alpha,\beta}(\omega))$:*

$$\tau \in \bigcap_{\alpha,\beta=1}^{2m} C\left(H_{\alpha,\beta}(\omega); R_{\alpha,\beta}(\omega)\right).$$

Notice that the explicit algebraic form of the equation (3.10) for the circle $C(H_{\alpha,\beta}(\omega); R_{\alpha,\beta}(\omega))$ is

$$\tau_1^2 - 2\cdot\tau_1\{ReH_{\alpha,\beta}(\omega)\} + \tau_2^2 - 2\cdot\tau_2\{ImH_{\alpha,\beta}(\omega)\} + |H_{\alpha,\beta}(\omega)|^2 = R^2_{\alpha,\beta}(\omega) \quad (\tau = \tau_1 + i\tau_2). \tag{3.11}$$

Following Theorem 3.3, the investigation of the optimum values (ω, τ) for the ESOR method requires the knowledge of the conditions on ω which guarantee that the common intersection of all the circles $C(H_{\alpha,\beta}(\omega); R_{\alpha,\beta}(\omega))$ is not empty.

The last two Theorems allow us to suspect that the existence of an optimum value depends upon how the eigenvalues in the μ-plane are located and that in case of a general distribution such a value may not exist: The next Theorem shows how the complex parameters involved can give an arbitrarily good convergence behavior for the ESOR method; its proof is completely analogous to that of Theorem 2.5.

Theorem 3.4. *Let $\varepsilon > 0$. For any*

$$\omega \in \mathbb{C} - \left\{\frac{1}{1-\mu^2}\cdot(-\infty, -4] : \mu \in \sigma(B) - \{\pm\}\right\} - \{0, 2\},$$

put

$$\lambda_{\omega}^{(\varepsilon)} := \max\left\{|\lambda|^{-2}\cdot(-|\lambda| - \varepsilon\cdot Im\lambda) : \lambda\in\sigma(\Lambda_{\omega})\right\}$$

and

$$\widetilde{\lambda}_{\omega}^{(\varepsilon)} := \min\left\{|\lambda|^{-2}\cdot(-|\lambda| + \varepsilon\cdot Im\lambda) : \lambda\in\sigma(\Lambda_{\omega})\right\},$$

and consider the open set

$$S_{\omega}^{(\varepsilon)} := \left\{y\in\mathbb{R} : \lambda_{\omega}^{(\varepsilon)} < y < \widetilde{\lambda}_{\omega}^{(\varepsilon)}\right\}$$

and its subset

$$\begin{aligned} F_{\omega}^{(\varepsilon)} &:= \left\{ y\in S_{\omega}^{(\varepsilon)} : \max_{\lambda\in\sigma(\Lambda_{\omega})} \frac{Re\lambda - \sqrt{|\lambda|^2 - \left[|\lambda|^2\cdot y + Im\lambda\right]^2 - |\lambda|^2\cdot(1-\varepsilon)^2}}{|\lambda|^2}\right. \\ &\quad \left. < \min_{\lambda\in\sigma(\Lambda_{\omega})} \frac{Re\lambda + \sqrt{|\lambda|^2 - \left[|\lambda|^2\cdot y + Im\lambda\right]^2 - |\lambda|^2\cdot(1-\varepsilon)^2}}{|\lambda|^2}\right\}. \end{aligned}$$

If

$$G^{(\varepsilon)} := \left\{\omega\in\mathbb{C} - \left\{\frac{1}{1-\mu^2}\cdot(-\infty,-4] : \mu\in\sigma(B) - \{\pm 1\}\right\} - \{0,2\} : F_{\omega}^{(\varepsilon)} \neq \varnothing\right\}$$

then, for any

$$(\omega,\tau)\in G^{(\varepsilon)}\times\left\{x+iy\in\mathbb{C} : y\in F_{\omega}^{(\varepsilon)},\ \omega\in G^{(\varepsilon)} \quad \textit{and}\right.$$

$$\left.\begin{aligned} &\max_{\lambda\in\sigma(\Lambda_{\omega})} \frac{Re\lambda - \sqrt{|\lambda|^2 - \left[|\lambda|^2\cdot y + Im\lambda\right]^2 - |\lambda|^2\cdot(1-\varepsilon)^2}}{|\lambda|^2} < x < \\ &\min_{\lambda\in\sigma(\Lambda_{\omega})} \frac{Re\lambda + \sqrt{|\lambda|^2 - \left[|\lambda|^2\cdot y + Im\lambda\right]^2 - |\lambda|^2\cdot(1-\varepsilon)^2}}{|\lambda|^2}\end{aligned}\right\},$$

we have

$$\rho(L_{\tau,\omega}) < \varepsilon.$$

Corollary 3.5. *Let* $0<\varepsilon<2$. *If* $\omega\in\mathbb{C}$ *is chosen so that*

$$\sigma(\Lambda_{\omega})\subset\mathbb{R}^+\times\mathbb{R}^+ \quad \textit{and} \quad \left(1+\left[\frac{Re\lambda}{Im\lambda}\right]^2\right)\cdot\varepsilon\cdot(2-\varepsilon) > 1 \quad (\lambda\in\sigma(\Lambda_{\omega}))$$

and if $\tau = x+iy\in\mathbb{C}$ *is chosen so that*

$$\begin{aligned} &0 < y < \min_{\lambda\in\sigma(\Lambda_{\omega})} \frac{-Im\lambda + |\lambda|\cdot\sqrt{\varepsilon(2-\varepsilon)}}{|\lambda|^2} \quad \textit{and} \\ &0 < x < \min_{\lambda\in\sigma(\Lambda_{\omega})} \frac{Re\lambda + \sqrt{|\lambda|^2 - \left[|\lambda|^2\cdot y + Im\lambda\right]^2 - |\lambda|^2\cdot(1-\varepsilon)^2}}{|\lambda|^2}, \end{aligned}$$

then

$$\rho(L_{\tau,\omega}) < \varepsilon.$$

Corollary 3.6. *Let $0 < \varepsilon < 2$. If $\omega \in \mathbb{C}$ is chosen so that*

$$\sigma(\Lambda_\omega) \subset \mathbb{R}^+,$$

and if $\tau = x + iy \in \mathbb{C}$ satisfies

$$0 < x < \min_{\lambda \in \sigma(\Lambda_\omega)} \frac{1}{\lambda} \quad \text{and} \quad 0 < y < \min_{\lambda \in \sigma(\Lambda_\omega)} \frac{\sqrt{\varepsilon(2-\varepsilon)}}{\lambda},$$

then

$$\rho(L_{\tau,\omega}) < \varepsilon.$$

If, in particular, $\omega = 1$, then, by (3.7),

$$\sigma(\Lambda_\omega) \subset \mathbb{R}^+ \Leftrightarrow \sigma(B) \subset (-1, 1)$$

and

$$\min_{\lambda \in \sigma(\Lambda_\omega)} \frac{1}{\lambda} = \min_{m \in \sigma(B)} \frac{1}{1 - \mu^2}.$$

Letting

$$\underline{\mu} = \min\{\mu : \mu \in \sigma(B) \subset (-1, 1)\},$$

we immediately have the following:

Corollary 3.7. *Let $0 < \varepsilon < 2$. If $\sigma(B) \subset (-1, 1)$ and if $\tau = x + iy \in \mathbb{C}$ satisfies*

$$0 < x < \frac{1}{1 - \underline{\mu}^2} \qquad \text{and} \qquad 0 < y < \frac{\sqrt{\varepsilon(2-\varepsilon)}}{1 - \underline{\mu}^2},$$

then

$$\rho(L_{\tau,1}) < \varepsilon.$$

Corollary 3.8. *Let $0 < \varepsilon < 2$. If $\sigma(B) \subset (-1, 1)$ and if $\tau = x + iy \in \mathbb{C}$ satisfies*

$$0 < x < 1 \qquad \text{and} \qquad 0 < y < \sqrt{\varepsilon(2-\varepsilon)},$$

then

$$\rho(L_{\tau,1}) < \varepsilon.$$

According to Corollary 3.8 (or 3.7), the complex EGS method may have an arbitrarily "good" convergence behavior.

The difficulty of the investigation in practice for the assumptions of Theorem 3.4 and of Corollaries 3.5 and 3.6 forces us to seek for another confronting of the problem.

In what follows, we will assume that the B-matrix has a critical eigenvalue - pair $\pm\widetilde{\mu}$. By definition, the critical eigenvalue - pair $\pm\widetilde{\mu}$ is that pair which corresponds to the dominant absolute value of eigenvalue for the $L_{\tau,\omega}$-matrix whenever $(\omega, \tau) \in \mathbb{C}^2$. Under this strong condition we have

$$\min_{(\omega,\tau) \in \mathbb{C}^2} \rho(L_{\tau,\omega}) = \min_{(\omega,\tau) \in \mathbb{C}^2} \left\{ \max \left\{ |\widetilde{\zeta}|, |\widetilde{\zeta}'| \right\} \right\},$$

where $\widetilde{\zeta}$ and $\widetilde{\zeta}'$ are the roots of the equation

$$\zeta^2 - (\tau \cdot \omega \cdot \widetilde{\mu}^2 - 2\tau + 2) \cdot \zeta + (\tau \cdot \omega \cdot \widetilde{\mu}^2 - 2\tau + 1 + \tau^2 - \tau^2 \cdot \widetilde{\mu}^2) = 0. \tag{3.12}$$

By Theorem 3.1, the value (ω_0, τ_0) of (ω, τ) for which $\max\left\{|\widetilde{\zeta}|, |\widetilde{\zeta}'|\right\}$ is minimized is characterized by the fact that $|\widetilde{\zeta}| = |\widetilde{\zeta}'|$. Setting

$$\omega_0 = \frac{2 \pm 2\sqrt{1-\mu^2}}{\widetilde{\mu}^2} \quad \text{and} \quad \tau_0 = \frac{1}{\mp\sqrt{1-\widetilde{\mu}^2}},$$

it is readily seen that $(\tau_0 \cdot \omega_0 \cdot \widetilde{\mu}^2 - 2\tau_0 + 2) = (\tau_0 \cdot \omega_0 \cdot \widetilde{\mu}^2 - 2\tau_0 + 1 + \tau_0^2 - \tau_0^2 \cdot \widetilde{\mu}^2) = 0$ and therefore, in such a case $\widetilde{\zeta} = \widetilde{\zeta}' = 0$, which implies that $\min\limits_{(\omega,\tau)\in\mathbb{C}^2} \rho(L_{\tau,\omega}) = \rho(L_{\tau_0,\omega_0}) = 0$. We have thus proved the following

Theorem 3.9. *Assume that the B-matrix has a critical eigenvalue - pair $\pm\widetilde{\mu} \neq 0, \pm 1$. The optimum values of (ω, τ) that minimize the spectral radius for the $L_{\tau,\omega}$-matrix and therefore maximize the asymptotic rate of convergence for the ESOR method are*

$$\omega_0 = \frac{2}{1-\sqrt{1-\widetilde{\mu}^2}}\ , \quad \tau_0 = \frac{1}{-\sqrt{1-\widetilde{\mu}^2}}$$

and

$$\omega_0 = \frac{2}{1+\sqrt{1-\widetilde{\mu}^2}}\ , \quad \tau_0 = \frac{1}{\sqrt{1-\widetilde{\mu}^2}}$$

and the minimum of $\rho(L_{\tau,\omega})$ is $\rho(L_{\tau_0,\omega_0}) = 0$.

Example 3.10. Let us consider the following system of linear equations

$$\begin{aligned} 2x_1 - x_2 &= i \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= i. \end{aligned}$$

The general successive overrelaxation method is given by

$$\begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \\ x_3^{(n+1)} \end{pmatrix} = \begin{pmatrix} [1-\tau]x_1^{(n)} + \frac{\tau}{2}x_2^{(n)} + \frac{\tau}{2}i \\ -\frac{\tau\omega}{4}x_1^{(n)} + x_2^{(n)} - \frac{\tau\omega}{4}x_3^{(n)} + \frac{\tau\omega}{2}i \\ -\frac{\tau\omega^2}{8}x_1^{(n)} + \left[1 - \frac{\tau\omega^2}{8}\right]x_3^{(n)} + \frac{\tau\omega^2}{4}i \end{pmatrix}$$

and the Jacobi iteration matrix

$$B = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

has the eigenvalues $0,\ \pm\frac{\sqrt{2}}{2}$. If we choose

$$\omega_0 = \frac{2}{1+\sqrt{1-\left(\pm\frac{\sqrt{2}}{2}\right)^2}} = 1.1715728 \quad \text{and} \quad \tau_0 = \frac{1}{\sqrt{1-\left(\pm\frac{\sqrt{2}}{2}\right)^2}} = 1.4142135$$

our method becomes

$$\begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \\ x_3^{(n+1)} \end{pmatrix} = \begin{pmatrix} -0.4142135x_1^{(n)} + 0.7071067x_2^{(n)} + 0.7071067i \\ -0.4142135x_1^{(n)} + x_2^{(n)} - 0.4142135x_3^{(n)} + 0.828427i \\ -0.2426406x_1^{(n)} + 0.7573593x_3^{(n)} + 0.4852812i \end{pmatrix}$$

and indicatively we have

$$\begin{array}{lll}
x_1^{(0)} = 0.2928932i & x_2^{(0)} = 0.828427i & x_3^{(0)} = 0.9999993i \\
x_1^{(1)} = 1.1715726i & x_2^{(1)} = 1.1213202i & x_3^{(1)} = 1.1715727i \\
x_1^{(2)} = 1.0147186i & x_2^{(2)} = 1.0624459i & x_3^{(2)} = 0.9361075i \\
x_1^{(3)} = 1.0380592i & x_2^{(3)} = 1.0828144i & x_3^{(3)} = 0.948039i \\
x_1^{(4)} = 1.0427939i & x_2^{(4)} = 1.0885727i & x_3^{(4)} = 0.951412i \\
x_1^{(5)} = 1.0449044i & x_2^{(5)} = 1.0909727i & x_3^{(5)} = 0.9528178i \\
x_1^{(6)} = 1.0457273i & x_2^{(6)} = 1.0919162i & x_3^{(6)} = 0.9533704i
\end{array}$$

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