On a problem of Horváth concerning barrelled spaces of vector valued continuous functions vanishing at infinity

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Abstract

Let $C_0(\Omega, X)$ be the linear space of all continuous functions from a locally compact normal space Ω into a normed space X vanishing at infinity, equipped with the supremum-norm topology. The main result of the paper says that if X is barrelled, then the space $C_0(\Omega, X)$ is always barrelled. This answers a question posed by J. Horváth.

1 Preliminaries

Over the last twenty years several interesting results have been proved, showing the barrelledness of some normed non complete vector valued function spaces. Recall that much of the importance of barrelledness comes from its connections with closed graph and open mapping theorems (see, e.g., the results obtained by Pták [8, Section 34], Saxon [11] and Valdivia's [12] among others) as well as with the Banach-Steinhaus theorem. For instance, it is shown in [5] that:

The space $\ell_0^\infty(\Sigma, X)$ of Σ-simple functions with values in a Hausdorff locally convex space (lcs) X, where Σ is an infinite algebra of subsets of Ω, endowed with the uniform convergence topology, is barrelled if and only if $\ell_0^\infty(\Sigma)$ and X are barrelled and X is nuclear.

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The following very interesting result can be found in [2]:

The space $\ell_\infty(\Omega, X)$ of all bounded functions on a set Ω with values in a normed space X, endowed with the supremum norm topology, is barrelled provided that the cardinal number $|\Omega|$ or $|X|$ is nonmeasurable.

This fact covers several partial results already obtained by Ferrando (see Section 9.5 of [3], Notes and Remarks) and Kąkol and Roelcke [7]. Another interesting result of this type was obtained in [4]:

If Γ is a nonempty set and X is a normed space, the linear space $c_0(\Gamma, X)$ over $\mathbb{K}$ of all functions $f : \Gamma \to X$ such that for each $\epsilon > 0$ the set $\{\omega \in \Gamma : \|f(\omega)\| > \epsilon\}$ is finite, endowed with the supremum norm, is barrelled (ultrabornological) if and only if X is barrelled (resp. ultrabornological).

By $C_0(\Omega, X)$ we denote the space over $\mathbb{K}$ of all continuous functions $f : \Omega \to X$ from a locally compact Hausdorff topological space Ω to a normed space X vanishing at infinity, *i.e.* such that for each $\epsilon > 0$ there exists a compact set $K_{f,\epsilon}$ in Ω with the property that $\|f(\omega)\| < \epsilon$ for each $\omega \in \Omega \setminus K_{f,\epsilon}$, provided with the supremum norm. When X is a Banach space, $C_0(\Omega, X)$ is also a Banach space since it can be seen as a closed subspace of the Banach space $\ell_\infty(\Omega, X)$. Since each compact set in a discrete topological space is finite, $c_0(\Gamma, X) = C_0(\Gamma, X)$ whenever Γ is equipped with the discrete topology. Finally, as in [6, 2.10], by $\mathcal{M}_0(\Omega, X)$ we denote the linear subspace of $\ell_\infty(\Omega, X)$ of functions vanishing at the infinity and equipped with the supremum norm. Note that $C_0(\Omega, X)$ may be seen as a linear subspace of $\mathcal{M}_0(\Omega, X)$.

Professor John Horváth from the University of Maryland asked the first named author about conditions under which the space $C_0(\Omega, X)$ is barrelled. For a compact Hausdorff space K the barrelledness of $C_0(K, X)$ is a direct consequence of the following result due to Mendoza [9], which will be used later on:

Let X be a normed space and Ω be a locally compact Hausdorff topological space. The space $C(\Omega, X)$ of vector valued continuous functions $f : \Omega \to X$ endowed with the compact-open topology is barrelled if and only if $C(\Omega)$ and X are barrelled [10, 11.10.1].

In this work we prove that if Ω is normal, then $C_0(\Omega, X)$ *is barrelled if and only if X is barrelled*. Moreover, we study the barrelledness of the space $\mathcal{M}_0(\Omega, X)$ and extend some results of [2] to this space.

Throughout this paper K denotes a compact Hausdorff topological space, Ω will be a nonempty locally compact Hausdorff topological space and, unless otherwise stated, X will denote a normed space over the field $\mathbb{K}$ of the real or complex numbers. Although our notation is standard, we refer the reader to references [6, 8, 10] for the necessary background. Finally, recall that a lcs X is *barrelled* if every barrel (*i.e.*, absolutely convex closed and absorbing subset of X) is a neighborhood of zero. We refer also the reader to the monographs [3] and [10] for more results, details and the required definitions.

2 Barrelled spaces $C_0(\Omega, X)$ and $\mathcal{M}_0(\Omega, X)$

We start with the following simple technical observation which will be used in the sequel. If U is an open subset of Ω, then $C_0(U, X)$ can be considered as a subspace of $C_0(\Omega, X)$. In fact, given $f \in C_0(U, X)$, define $f(\xi) = 0$ for each $\xi \in \Omega \setminus U$ and take a fixed point $\omega \in \Omega \setminus U$. Then, given $\epsilon > 0$ choose a compact set $K_{f,\epsilon} \subseteq U$ such that $\|f(\xi)\| < \epsilon$ for all $\xi \in U \setminus K_{f,\epsilon}$ and put $U_\epsilon := \Omega \setminus K_{f,\epsilon}$. Clearly U_ϵ is an open neighborhood of ω in Ω such that $\|f(\zeta)\| < \epsilon$ for each $\zeta \in U_\epsilon$. This shows that the definition $f(\zeta) = 0$ for each $\zeta \in \Omega \setminus U$ makes f continuous in Ω and $\|f(\xi)\| < \epsilon$ for each $\xi \in \Omega \setminus K_{f,\epsilon}$.

We shall need the following key lemma.

Lemma 1. *If Ω is a non compact locally compact topological space and T is a barrel in $C_0(\Omega, X)$, then there exists a compact set Δ in Ω such that T absorbs the closed unit ball of $C_0(\Omega \setminus \Delta, X)$.*

Proof. Firstly observe that the linear subspace of $C_0(\Omega, X)$ formed by all functions of compact support is dense in $C_0(\Omega, X)$. Indeed, given $f \in C_0(\Omega, X)$ and $\epsilon > 0$, choose (i) a compact set $K_{f,\epsilon}$ in Ω such that $\|f(\omega)\| < \epsilon$ if $\omega \in \Omega \setminus K_{f,\epsilon}$, (ii) an open neighborhood U of $K_{f,\epsilon}$ in Ω with compact closure $\overline{U}$ and (iii) a continuous function $\varphi : \Omega \to [0, 1]$ such that $K_{f,\epsilon} \prec \varphi \prec U$, that is, such that $\varphi(\omega) = 1$ for each $\omega \in K_{f,\epsilon}$ and $\operatorname{supp}\varphi \subseteq U$. Then the pointwise product $g := \varphi f$ is continuous on Ω and has compact support and also satisfies the inequality $\|f(\omega) - g(\omega)\| < \epsilon$ for each $\omega \in \Omega$. Indeed, one has $\|f(\omega) - g(\omega)\| = 0$ if $\omega \in K_{f,\epsilon}$, $\|f(\omega) - g(\omega)\| = |1 - \varphi(\omega)| \|f(\omega)\| \leq \|f(\omega)\| < \epsilon$ if $\omega \in U \setminus K_{f,\epsilon}$ and $\|f(\omega) - g(\omega)\| = \|f(\omega)\| < \epsilon$ if $\omega \in \Omega \setminus U$.

Assume the lemma is not true. Let $f_1 \in C_0(\Omega, X)$ be a function with compact support Δ_1 contained in Ω such that $\|f_1\|_\infty = 1$ and $f_1 \notin T$. Since T does not absorb the closed unit ball of $C_0(\Omega \setminus \Delta_1, X)$, there exists a function $f_2 \in C_0(\Omega \setminus \Delta_1, X)$ with compact support contained in $\Omega \setminus \Delta_1$ such that $\|f_2\|_\infty = 1$ and $f_2 \notin 2T$. Put $\Delta_2 :=$ supp f_2. Note that $\Delta_1 \cup \Delta_2$ is a compact set in Ω and choose a function $f_3 \in C_0(\Omega \setminus (\Delta_1 \cup \Delta_2), X)$ with compact support contained in $\Omega \setminus (\Delta_1 \cup \Delta_2)$ such that $\|f_3\|_\infty = 1$ and $f_3 \notin 3T$. We continue in this way to obtain a bounded sequence $\{f_n\}$ in $C_0(\Omega, X)$ and a pairwise disjoint sequence $\{\Delta_n\}$ of compact sets, with $\Delta_n :=$ supp f_n and $f_n \notin nT$ for all $n \in \mathbb{N}$.

Given the functions f_i normalized and disjointly supported, note that for $\xi_1, \ldots, \xi_n \in \mathbb{K}$ one has $\|\sum_{i=1}^n \xi_i f_i\| = \sup_{1\leq i\leq n} |\xi_i|$. This means that $\{f_i\}$ is a normalized basic sequence in the Banach space $\widehat{C_0(\Omega, X)} = C_0\left(\Omega, \widehat{X}\right)$ which is equivalent to the unit vector basis of c_0 (see for instance [1, Chapter 5, Theorem 1]).

Take $\xi \in c_0$. Since $\sum_{i=1}^\infty \xi_i f_i$ converges in $C\left(\Omega, \widehat{X}\right)$ and the series $\sum_{i=1}^\infty \xi_i f_i(\omega)$ contains at most one non-zero term for each $\omega \in \Omega$, its pointwise limit $f_\xi(\omega) := \sum_{i=1}^\infty \xi_i f_i(\omega)$ belongs to X for each $\omega \in \Omega$. Moreover, for $\epsilon > 0$, choosing $j \in \mathbb{N}$ such that $|\xi_i| < \epsilon$ for each $i > j$, the set $K_\epsilon := \bigcup_{i=1}^j \Delta_i$ is compact and such that $\|f_\xi(\omega)\| < \epsilon$ for each $\omega \in \Omega \setminus K_\epsilon$. Hence $f_\xi \in C_0(\Omega, X)$ and, consequently, the closed linear span $[f_i]$ of $\{f_i\}$ is a copy of c_0 contained in $C_0(\Omega, X)$. Thus, there exists $k \in \mathbb{N}$ such that $f_\xi \in kT$ for each $\xi \in c_0$, $\|\xi\|_\infty \leq 1$. In particular $f_k \in kT$, a contradiction. ∎

Theorem 2. *If Ω is a locally compact normal space and X a normed barrelled space, then $C_0(\Omega, X)$ is barrelled.*

Proof. Since X is a normed barrelled space, Mendoza's result [9] quoted above tell us that if K is a compact subset of Ω, then $C_0(K, X) = C(K, X)$ is barrelled. Assume, without loss of generality, that Ω is a non compact locally compact Hausdorff topological space.

By Lemma 1, if T is a barrel in $C_0(\Omega, X)$ there is a compact set Δ in Ω such that T absorbs the closed unit ball of $C_0(\Omega \setminus \Delta, X)$; note that the last space is considered as a subspace of $C_0(\Omega, X)$. Let U be an open neighborhood of Δ in Ω whose closure $\overline{U}$ is compact and such that $\Omega \supseteq \overline{U} \supseteq U \supseteq \Delta$. Choose a partition of unity $\{h_1, h_2\}$ subordinated to the nonempty open covering $\{U, \Omega \setminus \Delta\}$ of Ω, *i.e.* $h_1, h_2 \in C(\Omega)$ with $h_1, h_2 : \Omega \to [0, 1]$, $h_1(\omega) + h_2(\omega) = 1$ for each $\omega \in \Omega$, $h_1(\omega) = 0$ outside of a compact set contained in U and $h_2(\omega) = 0$ outside of a compact set contained in $\Omega \setminus \Delta$. Then $fh_1 \in C_0(U, X)$ and $fh_2 \in C_0(\Omega \setminus \Delta, X)$ for each $f \in C_0(\Omega, X)$.

Since $fh_2 \in C_0(\Omega \setminus \Delta, X)$ for each $f \in C_0(\Omega, X)$ and $\|fh_2\|_\infty \leq \|f\|_\infty$, the barrel T absorbs the set $\{fh_2 : f \in C_0(\Omega \setminus \Delta, X), \|f\|_\infty \leq 1\}$. Consequently there is $j > 0$ such that $fh_2 \in jT$ for each $f \in C_0(\Omega, X)$ with $\|f\|_\infty \leq 1$.

On the other hand, it is clear that the map $J : C\left(\overline{U}, X\right) \to C_0(U, X)$ defined by $(Jg)(\omega) = h_1(\omega) g(\omega)$ for $\omega \in U$ is a bounded linear operator. Since, as we mentioned at the beginning of the proof, $C\left(\overline{U}, X\right)$ is barrelled because $\overline{U}$ is compact, it follows that the barrel $J^{-1}(T \cap C_0(U, X))$ absorbs the closed unit ball of $C\left(\overline{U}, X\right)$. This implies that T absorbs the set $\{fh_1 : f \in C_0(\Omega, X), \|f\|_\infty \leq 1\}$ because $fh_1 = J(f|_{\overline{U}})$ for each $f \in C_0(\Omega, X)$. Thus there is $k \in \mathbb{N}$, $k \geq j$, such that $fh_1 \in kT$ for each $f \in C_0(\Omega, X)$ with $\|f\|_\infty \leq 1$. So we have that $f = fh_1 + fh_2 \in 2kT$ for each $f \in C_0(\Omega, X)$ with $\|f\|_\infty \leq 1$, which shows that T is a neighborhood of the origin in $C_0(\Omega, X)$. Therefore $C_0(\Omega, X)$ is a barrelled space. ∎

The following result is analogous to the Lemma 1.

Lemma 3. *Assume that Ω is a non compact locally compact topological space. If T is a barrel in $\mathcal{M}_0(\Omega, X)$, there exists a compact set Δ in Ω such that T absorbs the closed unit ball of $\mathcal{M}_0(\Omega \setminus \Delta, X)$ when we consider $\mathcal{M}_0(\Omega \setminus \Delta, X)$ as a subspace of $\mathcal{M}_0(\Omega, X)$.*

Proof. Since the set of all X-valued bounded functions with compact support is dense in $\mathcal{M}_0(\Omega, X)$, the proof is identical to that of Lemma 1. ∎

Theorem 4. *Let X be a barrelled space. If either $|\Delta|$ is non-measurable for each compact set $\Delta \subseteq \Omega$ or $|X|$ is non-measurable, then $\mathcal{M}_0(\Omega, X)$ is barrelled.*

Proof. Let T be a barrel in $\mathcal{M}_0(\Omega, X)$. According to Lemma 3 there exists a compact set $\Delta \subseteq \Omega$ such that T absorbs the closed unit ball of $\mathcal{M}_0(\Omega \setminus \Delta, X)$. Since $\mathcal{M}_0(\Omega, X) = \ell_\infty(\Delta, X) \oplus_t \mathcal{M}_0(\Omega \setminus \Delta, X)$ and either $|\Delta|$ or $|X|$ is non-measurable, $\ell_\infty(\Delta, X)$ is barrelled [2]. Hence T absorbs both the closed unit ball of $\ell_\infty(\Delta, X)$

and of $\mathcal{M}_0(\Omega \setminus \Delta, X)$ and, consequently, T is a neighborhood of zero in $\mathcal{M}_0(\Omega, X)$. This shows that $\mathcal{M}_0(\Omega, X)$ is barrelled. ∎

Remark 5. Since every discrete topological space Ω is locally compact and then each bounded function $f : \Omega \to X$ is continuous, we have $C_0(\Omega, X) = \mathcal{M}_0(\Omega, X)$ when Ω is endowed with the discrete topology. So, if X is barrelled, the previous theorem ensures that $C_0(\Omega, X)$ is barrelled when Ω is equipped with the discrete topology. However, as we have seen in Theorem 2, a better result holds, since $C_0(\Omega, X)$ is always barrelled whenever X is barrelled and Ω is normal. If Ω is equipped with the discrete topology, then $C_0(\Omega, X) = c_0(\Omega, X)$. Thus the barrelledness of $C_0(\Omega, X)$ in this particular case is also a consequence of [4, Proposition 2.2].

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