# New near polygons from Hermitian varieties

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**Abstract**

We define a new class of dense near polygons. The unique near $2n$-gon, $n \geq 0$, of this class will be denoted by $\mathbb{G}_n$. We will study the geodetically closed sub near polygons of $\mathbb{G}_n$. We will also determine the complete automorphism group and all spreads of symmetry. New glued near polygons can be constructed from these spreads of symmetry.

## 1 Definitions and Overview

### 1.1 Basic definitions

A *near polygon* is a partial linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $p \in \mathcal{P}$ and for every line $L \in \mathcal{L}$ there exists a unique point on $L$ nearest to $p$. Here distances $\mathrm{d}(\cdot, \cdot)$ are measured in the collinearity graph. If $n$ is the maximal distance between two points, then the near polygon is called a near $2n$-gon. A near 0-gon consists of one point, a near 2-gon is a line, and the class of near quadrangles coincides with the class of generalized quadrangles (GQ's) which were introduced by Tits in [13]. Near polygons themselves were introduced by Shult and Yanushka in [12] because of their relationship with certain line systems in Euclidean spaces. Generalized $2n$-gons ([14]) and dual polar spaces ([4]) form two important classes of near polygons.

A set $X$ of points in a near polygon $\mathcal{S}$ is called a *subspace* if every line meeting $X$ in at least two points is completely contained in $X$. A subspace $X$ is called *geodetically closed* if every point on a shortest path between two points of $X$ is as

*Postdoctoral Fellow of the Fund for Scientific Research - Flanders (Belgium)
Received by the editors August 2002.
Communicated by H. Van Maldeghem.
1991 *Mathematics Subject Classification* : 05B20, 51E12, 51E20.
*Key words and phrases* : near polygon, hermitean variety, generalized quadrangle.

well contained in $X$. Having a subspace $X$, we can define a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ by considering only those points and lines of $\mathcal{S}$ which are completely contained in $X$. If $X$ is geodetically closed, then $\mathcal{S}_X$ clearly is a sub near polygon of $\mathcal{S}$. A geodetically closed sub near polygon $\mathcal{S}_X \neq \mathcal{S}$ is called *big* if every point outside $\mathcal{S}_X$ is collinear with a unique point of $\mathcal{S}_X$. If a geodetically closed sub near polygon $\mathcal{S}_X$ is a nondegenerate generalized quadrangle, then $X$ (and often also $\mathcal{S}_X$) will be called a *quad*. Sufficient conditions for the existence of quads were given in [12]. Every set $X$ of points is contained in a unique minimal geodetically closed sub near polygon $\mathcal{C}(X)$, namely the intersection of all geodetically closed sub near polygons through $X$. We call $\mathcal{C}(X)$ the *geodetic closure* of $X$. If $X_1, \ldots, X_k$ are sets of points, then $\mathcal{C}(X_1 \cup \cdots \cup X_k)$ is also denoted by $\mathcal{C}(X_1, \ldots, X_k)$. If one of the arguments of $\mathcal{C}$ is a singleton $\{x\}$, we will often omit the braces and write $\mathcal{C}(\cdots, x, \cdots)$ instead of $\mathcal{C}(\cdots, \{x\}, \cdots)$.

A near polygon is said to have *order* $(s, t)$ if every line is incident with exactly $s+1$ points and if every point is incident with exactly $t+1$ lines. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. Dense near polygons satisfy several nice properties. By Lemma 19 of [2], every point of a dense near polygon $\mathcal{S}$ is incident with the same number of lines; we denote this number by $t_{\mathcal{S}}+1$. If $x$ and $y$ are two points of a dense near polygon, then by Theorem 4 of [2] $\mathcal{C}(x, y)$ is the unique geodetically closed sub near $[2 \cdot d(x, y)]$-gon through $x$ and $y$. Geodetically closed sub near hexagons of a dense near polygon are called *hexes*.

## 1.2 Sub near polygons of dual polar spaces

For every polar space $P$ of rank at least 2 a *dual polar space* $P^D$ can be defined. The points, respectively lines, of $P^D$ are the maximal, respectively next-to-maximal, totally isotropic subspaces of $P$ with reverse containment as incidence relation. Dual polar spaces are near polygons, see e.g. [4]. If $\pi$ is a totally isotropic subspace of $P$, then the set $U_\pi$ of all maximal totally isotropic subspaces through $\pi$ is a geodetically closed subspace of $P^D$. Conversely, every geodetically closed subspace of $P^D$ is obtained this way. We have noticed earlier that every geodetically closed subspace induces a sub near polygon. The converse however is not necessarily true. By Section 3 of [1], there exist sets $U$, not of the form $U_\pi$, whose elements are maximal totally isotropic subspaces of a polar space $P$ such that $(P^D)_U$ is a near polygon. The sets $U$ considered in [1] have one property in common: they consist of all maximal totally isotropic subspaces having nonempty intersection with a given set $A$ of points of the polar space. Despite this restriction, the authors were able to construct several new near polygons. E.g., by considering the set $A$ of all points of weight 2 on the Hermitian variety $H(5, 4)$ a new dense near hexagon $\mathbb{J}_3$ was found. There is now an obvious way to generalize this construction: take $A$ as the set of all points of weight 2 on the Hermitian variety $H(2n-1, 4)$. Again a near polygon $\mathbb{J}_n$ is obtained, but for $n \geq 4$ $\mathbb{J}_n$ is never dense. In Section 3.2 we will generalize the construction of $\mathbb{J}_3$ in such a way that an infinite class $\mathbb{G}_n$, $n \geq 0$, of dense near polygons is obtained. The near $2n$-gon $\mathbb{G}_n$ is still a sub near polygon of $H^D(2n-1, 4)$ since it is determined by the set $U_n$ of all generators of $H(2n-1, 4)$ which contain exactly $n$ points of weight 2. Notice that in the case $n = 3$, the condition "exactly three points of weight 2" is equivalent to "at least one point of weight 2".

### 1.3 Overview

After we have introduced the near polygon $\mathbb{G}_n$, $n \geq 0$, in Section 3.2, we will study the geodetically closed sub near polygons of $\mathbb{G}_n$ in Sections 3.3, 3.4 and 3.5. It turns out that with every geodetically closed sub near polygon there corresponds a subspace on $H(2n-1,4)$ with special properties. These "good subspaces" of $H(2n-1,4)$ are studied in Section 3.1. Using the geodetically closed sub near polygons, we are able to determine $\mathrm{Aut}(\mathbb{G}_n)$ in Section 4. In Section 5 we determine all spreads of symmetry of $\mathbb{G}_n$. In Section 6 we will show that these spreads of symmetry give rise to new glued near polygons. The study of $\mathbb{G}_n$ performed in the present paper will allow us in [10] to determine all dense near $2(n+1)$-gons which have $\mathbb{G}_n$ as a big geodetically closed sub near polygon.

## 2 Some notions regarding near polygons

Before defining $\mathbb{G}_n$, we recall some relevant notions and results from the literature.

### 2.1 Direct product

Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two near polygons. A new near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ can be derived from $\mathcal{S}_1$ and $\mathcal{S}_2$. It is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ and is denoted by $\mathcal{S}_1 \times \mathcal{S}_2$. We have: $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$, $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$, the point $(x, y)$ of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $y \; \mathrm{I}_2 \; L$, the point $(x, y)$ of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $x \; \mathrm{I}_1 \; M$ and $y = u$. If $\mathcal{S}_i$, $i \in \{1, 2\}$, is a near $2n_i$-gon then the direct product $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(n_1 + n_2)$-gon. Since $\mathcal{S}_1 \times \mathcal{S}_2 \cong \mathcal{S}_2 \times \mathcal{S}_1$ and $(\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3 \cong \mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3)$, also the direct product of $k \geq 1$ near polygons $\mathcal{S}_1, \ldots, \mathcal{S}_k$ is well-defined.

### 2.2 Big geodetically closed sub near polygons

Let $\mathcal{S}$ be a dense near $2n$-gon. Recall that a geodetically closed sub near $2(n-1)$-gon $\mathcal{F}$ of $\mathcal{S}$ is called *big* if every point $x$ outside $\mathcal{F}$ is collinear with a unique point $\pi(x)$ of $\mathcal{F}$. If $x \in \mathcal{F}$, then we put $\pi(x)$ equal to $x$. The map $\pi$ is called the *projection on* $\mathcal{F}$. Suppose now that every line of $\mathcal{S}$ is incident with exactly three points. For every big geodetically closed sub near $2(n-1)$-gon $\mathcal{F}$ of $\mathcal{S}$, we then can define the following permutation $\mathcal{R}_\mathcal{F}$ on the point set of $\mathcal{S}$: if $x \in \mathcal{F}$, then we put $\mathcal{R}_\mathcal{F}(x) := x$; if $x \notin \mathcal{F}$, then we put $\mathcal{R}_\mathcal{F}(x)$ equal to unique third point of the line $x\,\pi(x)$. By Section 4 of [1], $\mathcal{R}_\mathcal{F}$ is an automorphism of order 2 of $\mathcal{S}$. We call $\mathcal{R}_\mathcal{F}$ the *reflection around* $\mathcal{F}$.

The following lemma provides a method for recognizing big geodetically closed sub near polygons.

**Lemma 1 (Lemma 5 of [9])** *Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geq 2$, let $\mathcal{F}$ denote a geodetically closed sub near $2(n-1)$-gon of $\mathcal{S}$ and let $x$ denote an arbitrary point of $\mathcal{F}$. Then $\mathcal{F}$ is big in $\mathcal{S}$ if and only if every quad through $x$ either is contained in $\mathcal{F}$ or intersects $\mathcal{F}$ in a line.*

### 2.3 GQ's with three points on every line

If $\mathcal{S}$ is a generalized quadrangle with only lines of size 3, then one of the following possibilities occurs, see e.g. [11].

- $\mathcal{S}$ is degenerate: $\mathcal{S}$ consists of $k \geq 2$ lines of size 3 through a point.
- $\mathcal{S}$ is isomorphic to the $(3 \times 3)$-grid (i.e. the direct product of two lines of size 3). The $(3 \times 3)$-grid has order $(2, 1)$.
- $\mathcal{S}$ is isomorphic to $W(2)$. The points and lines of $W(2)$ are the totally isotropic points and lines of a symplectic polarity in $\mathrm{PG}(3, 2)$. The generalized quadrangle $W(2)$ has order $(2, 2)$, or shortly order 2.
- $\mathcal{S}$ is isomorphic to $Q(5, 2)$. The points and lines of $Q(5, 2)$ are the points and lines, respectively, lying on a nonsingular elliptic quadric in $\mathrm{PG}(5, 2)$. The generalized quadrangle $Q(5, 2)$ has order $(2, 4)$. Its point-line dual is $H(3, 4)$, the GQ of the points and lines of a nonsingular Hermitian variety in $\mathrm{PG}(3, 4)$.

In the sequel, a quad which is isomorphic to a grid, $W(2)$ or $Q(5, 2)$ will be called a grid-quad, a $W(2)$-quad or a $Q(5, 2)$-quad.

### 2.4 The near polygons $\mathbb{H}_n$

The following incidence structure $\mathbb{H}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ can be constructed from a set $V$ of size $2n + 2$:

- $\mathcal{P}$ is the set of all partitions of $V$ in $n + 1$ sets of order 2;
- $\mathcal{L}$ is the set of all partitions of $V$ in $n - 1$ sets of order 2 and 1 set of order 4;
- a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if and only if the partition determined by $p$ is a refinement of the partition determined by $L$.

It was noticed in [1] that $\mathbb{H}_n$ is a near $2n$-gon. Every line of $\mathbb{H}_n$ is incident with three points and every point is incident with $\binom{n+1}{2}$ lines. The near polygon $\mathbb{H}_0$ is a point, $\mathbb{H}_1$ is the line of size 3 and $\mathbb{H}_2$ is isomorphic to $W(2)$. The near polygon $\mathbb{H}_n$, $n \geq 2$, has no $Q(5, 2)$-quads.

## 3 The near polygons $\mathbb{G}_n$

Let the vector space $V(2n, 4)$, $n \geq 1$, with base $\{\bar{e}_0, \ldots, \bar{e}_{2n-1}\}$ be equiped with the nonsingular Hermitian form $(\bar{x}, \bar{y}) = x_0 y_0^2 + x_1 y_1^2 + \ldots + x_{2n-1} y_{2n-1}^2$, and let $H = H(2n - 1, 4)$ denote the corresponding Hermitian variety in $\mathrm{PG}(2n - 1, 4)$. In the sequel we will often consider subspaces on $H$ and the dimensions which we will use for these subspaces are always projective.

### 3.1 Good subspaces on $H$

The *support* $S_p$ of a point $p = \langle \bar{x} \rangle$ of $\mathrm{PG}(2n-1,4)$ is the set of all $i \in \{0,\ldots,2n-1\}$ for which $(\bar{x},\bar{e}_i) \neq 0$. The number $|S_p|$ is called the *weight* of $p$. Since $\bar{x} = \sum (\bar{x},\bar{e}_i)\, \bar{e}_i$, $|S_p|$ is equal to the number of nonzero coordinates. A point of $\mathrm{PG}(2n-1,4)$ belongs to $H$ if and only if its weight is even. A subspace $\pi$ on $H$ is said to be *good* if it is generated by a (possibly empty) set $\mathcal{G}_\pi \subseteq H$ of points whose supports are two by two disjoint. If $\pi$ is good, then $\mathcal{G}_\pi$ is uniquely determined. If $\mathcal{G}_\pi$ contains $k_{2i}$ points of weigth $2i$, $i \in \mathbb{N} \setminus \{0\}$, then $\pi$ is said to be of *type* $(2^{k_2}, 4^{k_4}, \ldots)$. Let $Y$, respectively $Y'$, denote the set of all good subspaces of dimension $n-1$, respectively $n-2$. Every element of $Y$ has type $(2^n)$. Every element of $Y'$ has type $(2^{n-1})$ or $(2^{n-2}, 4^1)$.

**Lemma 2** *If $\pi$ is a good subspace on $H$, then there exist $\pi_1, \pi_2 \in Y$ such that $\pi = \pi_1 \cap \pi_2$.*

*Proof.* For every point $p = \langle \bar{x} \rangle$ of $\mathcal{G}_\pi$ we take two partitions $P_p^1$ and $P_p^2$ of $S_p$ into $\frac{|S_p|}{2}$ sets of size 2 in such a way that the graph $(S_p, P_p^1 \cup P_p^2)$ is a cycle of length $|S_p|$ if $|S_p| \geq 4$. If we define $A_p^k := \{\langle (\bar{x},\bar{e}_i)\bar{e}_i + (\bar{x},\bar{e}_j)\bar{e}_j \rangle | \{i,j\} \in P_p^k\}$, $k \in \{1,2\}$, then clearly $\langle A_p^1 \rangle \cap \langle A_p^2 \rangle = \{p\}$. If we define $A^k := \bigcup_{p \in \mathcal{G}_\pi} A_p^k$, $k \in \{1,2\}$, then $\langle A^1 \rangle \cap \langle A^2 \rangle = \langle \mathcal{G}_\pi \rangle = \pi$. Now, let $N$ be the complement of $\bigcup_{p \in \mathcal{G}_\pi} S_p$ in $\{0,\ldots,2n-1\}$. Clearly $|N|$ is even. If $|N| = 0$, then we put $B^1 = B^2 = \emptyset$. If $|N| \neq 0$, then we consider a partition $P$ of $N$ into $\frac{|N|}{2}$ sets of size 2 and an element $\alpha \in \mathrm{GF}(4)^* \setminus \{1\}$. We put $B^1 := \{\langle \bar{e}_i + \bar{e}_j \rangle | \{i,j\} \in P\}$ and $B^2 := \{\langle \bar{e}_i + \alpha \bar{e}_j \rangle | \{i,j\} \in P \text{ and } i < j\}$. Clearly $\langle B^1 \rangle \cap \langle B^2 \rangle = \emptyset$. If $\pi_k := \langle A^k \cup B^k \rangle$, $k \in \{1,2\}$, then $\pi_1, \pi_2 \in Y$ and $\pi_1 \cap \pi_2 = \pi$. ■

**Lemma 3** *The intersection of two good subpaces $\pi_1$ and $\pi_2$ is again a good subspace.*

*Proof.* Consider the following graph $\Gamma$ on the vertex set $\{0,\ldots,2n-1\}$. Two vertices $i$ and $j$ are adjacent if and only if there exists a $p \in \mathcal{G}_{\pi_1} \cup \mathcal{G}_{\pi_2}$ such that $\{i,j\} \subseteq S_p$. Let $C_1,\ldots,C_f$ denote the connected components of $\Gamma$. For every $i \in \{1,\ldots,f\}$, there is at most one point $p \in \pi_1 \cap \pi_2$ with $S_p = C_i$. We can always label the components of $\Gamma$ such that the following holds for a certain $f' \in \{0,\ldots,f\}$:

(i) for every $i$ with $1 \leq i \leq f'$, there exists a unique point $p_i \in \pi_1 \cap \pi_2$ with $S_{p_i} = C_i$;

(ii) for every $i$ with $f' < i \leq f$, there exists no point $p \in \pi_1 \cap \pi_2$ with $S_p = C_i$.

It is now easily seen that $\pi_1 \cap \pi_2$ is good with $\mathcal{G}_{\pi_1 \cap \pi_2} = \{p_i | 1 \leq i \leq f'\}$. ■

### 3.2 Description of $\mathbb{G}_n$

Let $X \subseteq H$ denote the set of all points of weight 2.

**Lemma 4** *If $\pi$ is a generator of $H$, then $n-2 \neq |\pi \cap X| \neq n-1$.*

*Proof.* We use induction on $n$. For $n \in \{1,2\}$, it is easily seen that every generator of $H$ contains exactly $n$ points of weight 2. Suppose therefore that $n \geq 3$ and let $\pi$ be a generator containing the point $\langle \bar{a} \rangle = \langle (a_0, a_1, 0, 0, \ldots, 0) \rangle$. The points of $\pi \cap X$ different from $\langle \bar{a} \rangle$ are all contained in the space $\alpha \leftrightarrow X_0 = X_1 = 0$. The intersection $H' := H \cap \alpha$ is a nonsingular Hermitian variety in $\alpha$ and $\pi' := \pi \cap \alpha$ is a generator of $H'$. By induction, $n-3 \neq |\pi' \cap X| \neq n-2$; hence $n-2 \neq |\pi \cap X| \neq n-1$. ■

Let $H^D(2n-1,4)$ denote the dual polar space correponding to $H(2n-1,4)$. The distance $\mathrm{d}(\pi_1,\pi_2)$ between two points $\pi_1$ en $\pi_2$ of $H^D(2n-1,4)$ is equal to $n-1-\dim(\pi_1 \cap \pi_2)$, see e.g. [4]. The incidence structure $(Y, Y', \mathrm{I})$, again with reverse containment as incidence relation I, is a substructure of $H^D(2n-1,4)$, which we denote by $\mathbb{G}_n$. By Lemma 4, every generator through an element of $Y'$ belongs to $Y$. Hence, every line of $\mathbb{G}_n$ is incident with three points.

**Lemma 5** *Let $\pi_1, \pi_2 \in Y$. The distance between $\pi_1$ and $\pi_2$ in $\mathbb{G}_n$ is equal to $d(\pi_1,\pi_2)$.*

*Proof.* The proof is by induction. If $\mathrm{d}(\pi_1,\pi_2)=1$, then $\pi_1 \cap \pi_2$ is a good subspace of dimension $n-2$ and hence belongs to $Y'$. As a consequence also the $\mathbb{G}_n$-distance between $\pi_1$ and $\pi_2$ is equal to 1. Suppose therefore that $\mathrm{d}(\pi_1,\pi_2) \geq 2$. Take an $x \in X \cap (\pi_1 \setminus (\pi_1 \cap \pi_2))$ and let $\pi_3$ be the unique generator through $x$ intersecting $\pi_2$ in an $(n-2)$-dimensional subspace. Since there are at least $n-2$ elements in $X \cap \pi_2$ $H$-collinear with $x$, $|X \cap \pi_3| \geq n-1$. By Lemma 4, $\pi_3 \in Y$. Since $\mathrm{d}(\pi_1,\pi_3) = \mathrm{d}(\pi_1,\pi_2)-1$, the distance between $\pi_1$ and $\pi_3$ in $\mathbb{G}_n$ is equal to $\mathrm{d}(\pi_1,\pi_2)-1$. Since $\pi_2$ and $\pi_3$ are collinear in $\mathbb{G}_n$, the distance between $\pi_1$ and $\pi_2$ in $\mathbb{G}_n$ is at most $\mathrm{d}(\pi_1,\pi_2)$. Since $\mathbb{G}_n$ is embedded in $H^D(2n-1,4)$, this distance is at least $\mathrm{d}(\pi_1,\pi_2)$. This proves our lemma. ■

**Corollary 1** *$\mathbb{G}_n$ is a sub near $2n$-gon of $H^D(2n-1,4)$.*

*Proof.* Let $x$ be a point and $L$ a line of $\mathbb{G}_n$, then $x$ and $L$ are also objects of $H^D(2n-1,4)$. In the near polygon $H^D(2n-1,4)$, $L$ contains a unique point nearest to $x$. By the previous lemma, this property also holds in $\mathbb{G}_n$. Hence $\mathbb{G}_n$ is also a near polygon. Since $\mathrm{d}(\pi_1,\pi_2) = n-1-\dim(\pi_1 \cap \pi_2)$ for all $\pi_1,\pi_2 \in Y$ and since there exist $\pi_1,\pi_2 \in Y$ such that $\pi_1 \cap \pi_2 = \emptyset$, see Lemma 2, it follows that $\mathbb{G}_n$ is a near $2n$-gon. ■

The near polygon $\mathbb{G}_1$ is the unique line of size 3. The points, respectively lines, of $\mathbb{G}_2$ are all the maximal, respectively next-to maximal, subspaces of $H(3,4)$. Hence $\mathbb{G}_2 \cong H^D(3,4) \cong Q(5,2)$. We define $\mathbb{G}_0$ as the unique near 0-gon.

### 3.3 Geodetically closed sub near polygons in $\mathbb{G}_n$

**Theorem 1** *The near polygon $\mathbb{G}_n$ is dense. For every two points $\pi_1$ and $\pi_2$ of $\mathbb{G}_n$, $\mathcal{C}(\pi_1, \pi_2)$ is the unique geodetically closed sub near $[2 \cdot d(\pi_1, \pi_2)]$-gon through $\pi_1$ and $\pi_2$. Moreover, $\mathcal{C}(\pi_1, \pi_2)$ consists of all elements of $Y$ through $\pi_1 \cap \pi_2$.*

*Proof.* We noticed earlier that every line of $\mathbb{G}_n$ is incident with three points. Now, let $\pi_1, \pi_2 \in Y$ such that $\mathrm{d}(\pi_1, \pi_2) = 2$, or equivalently $\dim(\pi_1 \cap \pi_2) = n - 3$. Choose an $x_3 \in X \cap (\pi_2 \setminus (\pi_1 \cap \pi_2))$ and an $x_4 \in X \cap \pi_1$ not $H$-collinear with $x_3$. Let $\pi_i$, $i \in \{3, 4\}$, denote the unique generator through $x_i$ intersecting $\pi_{i-2}$ in an $(n-2)$-dimensional subspace. By the proof of Lemma 5, we know that $\pi_3$ and $\pi_4$ are common neighbours of $\pi_1$ and $\pi_2$. Hence $\mathbb{G}_n$ is dense. By theorem 4 of [2], we then know that $\pi_1$ and $\pi_2$ are contained in a unique geodetically closed sub near $[2 \cdot \mathrm{d}(\pi_1, \pi_2)]$-gon which necessarily coincides with $\mathcal{C}(\pi_1, \pi_2)$. Now, let $\mathcal{F}$ denote the set of all generators of $Y$ through $\pi_1 \cap \pi_2$. Clearly $\mathcal{F}$ is a subspace of $\mathbb{G}_n$. If $\gamma$ denotes a shortest path in $\mathbb{G}_n$ between two points of $\mathcal{F}$, then by Lemma 5, $\gamma$ is also a shortest path in $H^D(2n-1, 4)$ and hence every point of it contains $\pi_1 \cap \pi_2$. As a consequence every point on $\gamma$ is contained in $\mathcal{F}$ and $\mathcal{F}$ is geodetically closed. If $\pi$ and $\pi'$ are two arbitrary elements of $\mathcal{F}$, then $\pi \cap \pi'$ contains $\pi_1 \cap \pi_2$ and hence $\mathrm{d}(\pi, \pi') = n - 1 - \dim(\pi \cap \pi') \leq n - 1 - \dim(\pi_1 \cap \pi_2) = \mathrm{d}(\pi_1, \pi_2)$. As a consequence the diameter of $\mathcal{F}$ is at most $\mathrm{d}(\pi_1, \pi_2)$. Since $\mathcal{F}$ contains $\pi_1$ and $\pi_2$, the diameter is precisely $\mathrm{d}(\pi_1, \pi_2)$. Since $\mathcal{F}$ is a geodetically closed sub near $[2 \cdot \mathrm{d}(\pi_1, \pi_2)]$-gon through $\pi_1$ and $\pi_2$, it coincides with $\mathcal{C}(\pi_1, \pi_2)$. ∎

For every geodetically closed subspace $\mathcal{F}$ of $\mathbb{G}_n$, let $\pi_{\mathcal{F}}$ denote the intersection of all points of $\mathcal{F}$ regarded as generators of $H$. Since there exist elements $\pi_1, \pi_2 \in Y$ such that $\pi_1 \cap \pi_2 = \emptyset$, $\pi_{\mathbb{G}_n} = \emptyset$.

**Lemma 6** (*a*) *There is a one-to-one correspondence between the geodetically closed subspaces of $\mathbb{G}_n$ and the good subspaces on $H$.*

(*b*) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two geodetically closed sub near polygons, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if $\pi_{\mathcal{F}_2} \subseteq \pi_{\mathcal{F}_1}$.*

*Proof.* Let $\mathcal{F}$ denote an arbitrary geodetically closed sub near polygon of $\mathbb{G}_n$. If $\pi_1$ and $\pi_2$ denote two points of $\mathcal{F}$ at maximal distance from each other, then $\mathcal{F} = \mathcal{C}(\pi_1, \pi_2)$. By Theorem 1, $\pi_{\mathcal{F}} = \pi_1 \cap \pi_2$. Hence $\pi_{\mathcal{F}}$ is good by Lemma 3. Conversely, suppose that $\pi$ is a good subspace on $H$. If $\pi = \pi_{\mathcal{F}}$, then $\mathcal{F}$ necessarily consists of all elements of $Y$ through $\pi$. Hence, the equation $\pi_{\mathcal{F}} = \pi$ has at most one solution for $\mathcal{F}$. It suffices to show that this equation has at least one solution. By Lemma 2, there exist elements $\pi_1, \pi_2 \in Y$ such that $\pi = \pi_1 \cap \pi_2$. If we put $\mathcal{F}$ equal to $\mathcal{C}(\pi_1, \pi_2)$, then by Theorem 1, $\pi_{\mathcal{F}} = \pi_1 \cap \pi_2 = \pi$. This proves part (a). Part (b) follows from the fact that the points of a geodetically closed sub near polygon $\mathcal{F}$ are precisely the generators of $Y$ through $\pi_{\mathcal{F}}$. ∎

**Corollary 2** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two geodetically closed sub near polygons of $\mathbb{G}_n$ and let $\mathcal{F}_3 = \mathcal{C}(\mathcal{F}_1, \mathcal{F}_2)$. Then $\pi_{\mathcal{F}_3} = \pi_{\mathcal{F}_1} \cap \pi_{\mathcal{F}_2}$.*

*Proof.* Since $\mathcal{F}_3$ is the smallest geodetically closed sub near polygon through $\mathcal{F}_1$ and $\mathcal{F}_2$, $\pi_{\mathcal{F}_3}$ is the biggest good subspace contained in $\pi_{\mathcal{F}_1}$ and $\pi_{\mathcal{F}_2}$. The result now easily follows from Lemma 3. ∎

**Lemma 7** *Let $p$ denote an arbitrary point of weight $2n$ in $\mathrm{PG}(2n-1, 4)$, then $p \in H$ and the set of all generators of $Y$ through $p$ determines a geodetically closed sub near $2(n-1)$-gon isomorphic to $\mathbb{H}_{n-1}$.*

*Proof.* Put $p = \langle \alpha_0 \bar{e}_0 + \cdots + \alpha_{2n-1} \bar{e}_{2n-1} \rangle$. The set $\{p\}$ is a good subspace of $H$ and hence, by Lemma 6, the set of all generators of $Y$ through $p$ determines a geodetically closed sub near $2(n-1)$-gon $\mathcal{B}$. The set $\{0, \ldots, 2n-1\}$ has size $2n$ and hence, by Section 2.4, a near $2(n-1)$-gon $\mathcal{A} \cong \mathbb{H}_{n-1}$ can be constructed from this set. For every point $P$ of $\mathcal{A}$, i.e. for every partition $P$ of $\{0, \ldots, 2n-1\}$ into $n$ sets of size 2, we put $\phi(P) := \langle \{ \langle \alpha_i \bar{e}_i + \alpha_j \bar{e}_j \rangle | \{i, j\} \in P \} \rangle$. Clearly $\phi(P)$ is a generator of $Y$ through $p$. Conversely, every generator of $Y$ through $p$ is of the form $\phi(P)$ for some point $P$ of $\mathcal{A}$. We will now show that $\phi$ determines an isomorphism between the collinearity graphs of $\mathcal{A}$ and $\mathcal{B}$. If $P_1$ and $P_2$ are two collinear points of $\mathcal{A}$, then $\phi(P_1) \cap \phi(P_2)$ is a good subspace of type $(2^{n-2}, 4^1)$; hence $\phi(P_1)$ and $\phi(P_2)$ are collinear in $\mathcal{B}$. Conversely, suppose that $\phi(P_1)$ and $\phi(P_2)$ are collinear in $\mathcal{B}$, then $\phi(P_1) \cap \phi(P_2)$ is a good subspace of type $(2^{n-1})$ or $(2^{n-2}, 4^1)$. If $\phi(P_1) \cap \phi(P_2)$ has type $(2^{n-1})$, then $|P_1 \cap P_2| \geq n-1$ and hence $P_1 = P_2$, a contradiction. As a consequence $\phi(P_1) \cap \phi(P_2)$ has type $(2^{n-2}, 4^1)$ and $P_1$ and $P_2$ are collinear in $\mathcal{A}$. Since the collinearity graphs of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, $\mathcal{A}$ and $\mathcal{B}$ themselves are isomorphic. (Notice that the lines of a near polygon correspond with the maximal cliques in its collinearity graph.) ∎

**Theorem 2** *The geodetically closed sub near $(n-k)$-gons, $k \in \{0, \ldots, n\}$, of $\mathbb{G}_n$ are of the form $\mathbb{H}_{n_1-1} \times \cdots \times \mathbb{H}_{n_k-1} \times \mathbb{G}_{n_{k+1}}$ with $n_1, \ldots, n_k \geq 1$, $n_{k+1} \geq 0$ and $n_1 + \cdots + n_{k+1} = n$.*

*Proof.* Let $\mathcal{F}$ denote an arbitrary geodetically closed sub near $(n-k)$-gon, $k \in \{0, \ldots, n\}$, and put $\mathcal{G}_{\pi_\mathcal{F}} = \{p_1, \ldots, p_k\}$. Let $S_i$, $i \in \{1, \ldots, k\}$, denote the support of $p_i$, and let $S_{k+1} = \{0, \ldots, 2n-1\} \setminus (S_1 \cup \cdots \cup S_k)$. For every $i \in \{1, \ldots, k+1\}$, we put $|S_i| = 2n_i$ and $\alpha_i := \langle \bar{e}_j | j \in S_i \rangle$. Clearly, $n_1, \ldots, n_k \geq 1$, $n_{k+1} \geq 0$ and $n_1 + \cdots + n_{k+1} = n$. Also $\alpha_i \cap H$ is a nonsingular Hermitian variety of type $H(2n_i - 1, 4)$. If $\pi$ is an arbitrary point of $\mathcal{F}$, or equivalently an arbitrary generator of $Y$ through $\pi_\mathcal{F}$, then $\pi = \langle \pi \cap \alpha_1, \cdots, \pi \cap \alpha_k, \pi \cap \alpha_{k+1} \rangle$. Moreover, $\pi \cap \alpha_i$ is a generator of $\alpha_i \cap H$ containing $n_i$ points of weight 2, and $p_i \in \pi \cap \alpha_i$ if $i \neq k+1$. Conversely, if $\beta_i$, $i \in \{1, \ldots, k+1\}$, is a generator of $\alpha_i \cap H$ containing $n_i$ vertices of weight 2 such that $p_i \in \beta_i$ if $i \leq k$, then $\langle \beta_1, \ldots, \beta_{k+1} \rangle$ is a generator of $\mathcal{F}$ through $\pi_\mathcal{F}$. Hence, by Lemma 7, the map $\pi \rightarrow (\pi \cap \alpha_1, \cdots, \pi \cap \alpha_k, \pi \cap \alpha_{k+1})$ determines a bijection between the point sets of the near polygons $\mathcal{F}$ and $\mathbb{H}_{n_1-1} \times \cdots \times \mathbb{H}_{n_k-1} \times \mathbb{G}_{n_{k+1}}$. Now, two points $\pi_1$ and $\pi_2$ of $\mathcal{F}$ are collinear if and only if $\dim(\pi_1 \cap \pi_2) = n-2$. This happens if and only if there exists a $j \in \{1, \ldots, k+1\}$ such that $\dim(\pi_1 \cap \pi_2 \cap \alpha_j) = n_j - 2$ and $\dim(\pi_1 \cap \pi_2 \cap \alpha_i) = n_i - 1$ for every $i \in \{1, \ldots, k+1\} \setminus \{j\}$. These conditions are equivalent with $\dim((\pi_1 \cap \alpha_j) \cap (\pi_2 \cap \alpha_j)) = n_j - 2$ and $\pi_1 \cap \alpha_i = \pi_2 \cap \alpha_i$. Hence $\pi_1$ and $\pi_2$ are collinear in $\mathcal{F}$ if and only if $(\pi_1 \cap \alpha_1, \cdots, \pi_1 \cap \alpha_{k+1})$ and $(\pi_2 \cap \alpha_1, \cdots, \pi_2 \cap \alpha_{k+1})$ are collinear in $\mathbb{H}_{n_1-1} \times \cdots \times \mathbb{H}_{n_k-1} \times \mathbb{G}_{n_{k+1}}$. Since the collinearity graphs of $\mathcal{F}$ and $\mathbb{H}_{n_1-1} \times \cdots \times \mathbb{H}_{n_k-1} \times \mathbb{G}_{n_{k+1}}$ are isomorphic, the near polygons themselves are isomorphic. ■

### 3.4 Lines and quads in $\mathbb{G}_n$

Let $n \geq 3$. If $L$ is a line of $\mathbb{G}_n$, then there are two possibilities for $\pi_L$ $(= L)$:

(a) $\pi_L$ has type $(2^{n-1})$;

(b) $\pi_L$ has type $(2^{n-2}, 4^1)$.

If $\mathcal{Q}$ is a quad of $\mathbb{G}_n$, then there are four possibilities for $\pi_\mathcal{Q}$.

(i) $\pi_\mathcal{Q}$ has type $(2^{n-2})$.
By Lemma 4, each of the 27 generators through $\pi_\mathcal{Q}$ belongs to $Y$, proving that $\mathcal{Q}$ is a $Q(5,2)$-quad. The quad $\mathcal{Q}$ has 18 lines of type (a) and 27 lines of type (b). The 18 lines of type (a) define three grids which partition the point set of $\mathcal{Q}$.

(ii) $\pi_\mathcal{Q}$ has type $(2^{n-3}, 6^1)$.
From the 27 generators through $\pi_\mathcal{Q}$, 15 are contained in $Y$, proving that $\mathcal{Q}$ is a $W(2)$-quad. Clearly $\mathcal{Q}$ contains only lines of type (b).

(iii) $\pi_\mathcal{Q}$ has type $(2^{n-3}, 4^1)$.
From the 27 generators through $\pi_\mathcal{Q}$, nine are contained in $Y$, proving that $\mathcal{Q}$ is a grid. The quad $\mathcal{Q}$ contains three lines of type (a) and three lines of type (b). Three lines of the same type partition the point set of $\mathcal{Q}$.

(iv) $\pi_\mathcal{Q}$ has type $(2^{n-4}, 4^2)$.
This type of quad only exists if $n \geq 4$. From the 27 generators through $\pi_\mathcal{Q}$, nine are contained in $Y$, proving that $\mathcal{Q}$ is a grid. All six lines of $\mathcal{Q}$ have type (b).

By Lemma 6, it then easily follows:

**Lemma 8** *Consider the near polygon $\mathbb{G}_n$ with $n \geq 3$. Then*

- *each point is contained in $n$ lines of type (a) and $3\frac{n(n-1)}{2}$ lines of type (b);*
- *each line of type (a) is contained in exactly $n-1$ $Q(5,2)$-quads, 0 $W(2)$-quads and $3\frac{(n-1)(n-2)}{2}$ grid-quads;*
- *each line of type (b) is contained in a unique $Q(5,2)$-quad, $3(n-2)$ $W(2)$-quads and $\frac{(n-2)(3n-7)}{2}$ grid-quads;*
- *each line is contained in exactly $\frac{(n-1)(3n-4)}{2}$ quads.*

In the sequel lines of type (a) in $\mathbb{G}_n$, $n \geq 3$, will be called *special*, while lines of type (b) are called *ordinary*. Clearly, a line is special if and only if it is not contained in a $W(2)$-quad. For every permutation $\sigma$ of $\{0, \ldots, 2n-1\}$ and for every $\lambda_0, \ldots, \lambda_{2n-1} \in \mathrm{GF}(4)^*$, the linear transformation of $V(2n, 4)$ defined by $\bar{e}_i \mapsto \lambda_i \bar{e}_{\sigma(i)}$, $i \in \{0, \ldots, 2n-1\}$, determines an automorphism of $\mathbb{G}_n$. Using these automorphisms it is easily seen that any two lines of the same type are in the same $\mathrm{Aut}(\mathbb{G}_n)$-orbit. Similarly, any two quads of the same type are contained in the same $\mathrm{Aut}(\mathbb{G}_n)$-orbit. Since a special line can never be mapped to an ordinary line, $\mathrm{Aut}(\mathbb{G}_n)$ has two orbits on the set of lines and three or four orbits on the set of quads depending on whether $n = 3$ or $n \geq 4$. In Section 4 we will determine $\mathrm{Aut}(\mathbb{G}_n)$.

**Remark.** The above remarks on the orbits of $\mathrm{Aut}(\mathbb{G}_n)$, $n \geq 3$, do not hold for $\mathbb{G}_2$. Since $\mathbb{G}_2 \cong Q(5, 2)$ all lines are in the same orbit.

### 3.5 Some properties of $\mathbb{G}_n$

**Lemma 9** *The near $2n$-gon $\mathbb{G}_n$, $n \geq 1$, has order $(s, t) = (2, \frac{3n^2-n-2}{2})$ and $v = \frac{3^n \cdot (2n)!}{2^n \cdot n!}$ points.*

*Proof.* Clearly, the lemma holds if $n \in \{1, 2\}$. So suppose that $n \geq 3$. $H(2n-1, 4)$ has exactly $\frac{3^n \cdot (2n)!}{2^n \cdot n!}$ good subspaces of type $(2^n)$. We noticed earlier that every line is incident with exactly $s + 1 = 3$ points, and by Lemma 8, it follows that $t + 1 = n + 3\frac{n(n-1)}{2}$. ■

**Lemma 10** *Let $\mathcal{F}$ be a geodetically closed sub near polygon of $\mathbb{G}_n$ isomophic to $\mathbb{G}_k$, $k \geq 2$, and let $x$ denote an arbitrary point of $\mathcal{F}$. Then $\pi_{\mathcal{F}}$ has type $(2^{n-k})$ and precisely $k$ from the $n$ special lines through $x$ are contained in $\mathcal{F}$.*

*Proof.* Recall that no near polygon of type $\mathbb{H}_l$, $l \geq 0$, has a $Q(5, 2)$-quad. If $\mathcal{F} \cong \mathbb{H}_l \times \mathcal{A}$ for some $l \geq 1$ and some dense near $2(k-l)$-gon $\mathcal{A}$, then $\mathcal{F}$ has a line that is not contained in a $Q(5, 2)$-quad, contradicting Lemma 8. By the proof of Theorem 2, it then follows that that $\pi_{\mathcal{F}}$ has type $(2^{n-k})$. Lemma 6 now allows us to count the number of special lines through $x$ which are also contained in $\mathcal{F}$. It is easily seen that this number equals $k$. ■

**Lemma 11** *If $L_1, \ldots, L_k$ are different special lines of $\mathbb{G}_n$, $n \geq 3$, through a fixed point $x$, then $\mathcal{C}(L_1, \ldots, L_k) \cong \mathbb{G}_k$.*

*Proof.* Put $\mathcal{F} = \mathcal{C}(L_1, \ldots, L_k)$. By Corollary 2, $\pi_{\mathcal{F}} = \pi_{L_1} \cap \cdots \cap \pi_{L_k}$. Every $\pi_{L_i}$, $i \in \{1, \ldots, k\}$, is a good subspace of type $(2^{n-1})$ contained in the good subspace of type $(2^n)$ associated with $x$. Hence $\pi_{\mathcal{F}} = \pi_{L_1} \cap \cdots \cap \pi_{L_k}$ is a good subspace of type $(2^{n-k})$. By Theorem 2, $\mathcal{F} \cong \underbrace{\mathbb{H}_0 \times \cdots \times \mathbb{H}_0}_{n-k} \times \mathbb{G}_k \cong \mathbb{G}_k$. ■

**Lemma 12** *Let $\mathcal{F}$ be a geodetically closed sub near $2(n-1)$-gon of $\mathbb{G}_n$, $n \geq 3$.*

(*a*) *If $\mathcal{F} \cong \mathbb{G}_{n-1}$, then $\mathcal{F}$ is big in $\mathbb{G}_n$.*

(*b*) *If $\mathcal{F}$ is big in $\mathbb{G}_n$, then $\mathcal{F} \cong \mathbb{G}_{n-1}$ and $\pi_{\mathcal{F}}$ has type $(2^1)$.*

*Proof.*

(a) If $\mathcal{F} \cong \mathbb{G}_{n-1}$, then the total number of points at distance at most 1 from $\mathcal{F}$ is equal to $|\mathcal{F}| \cdot (1 + 2(t - t_{\mathcal{F}}))$ which is exactly the total number of points in $\mathbb{G}_n$. Hence $\mathcal{F}$ is big in $\mathbb{G}_n$.

(b) Take a line $L$ intersecting $\mathcal{F}$ in a point, then $L$ is contained in precisely $\frac{(n-1)(3n-4)}{2}$ quads, see Lemma 8. Since $\mathcal{F}$ is big, each of these $\frac{(n-1)(3n-4)}{2}$ quads meets $\mathcal{F}$ in a line. Hence $t_{\mathcal{F}} + 1 = \frac{(n-1)(3n-4)}{2}$. Since $\mathcal{F}$ is a geodetically closed sub near $2(n-1)$-gon, $\pi_{\mathcal{F}}$ has type $((2k)^1)$ for a certain $k \in \{1, \ldots, n\}$. By Theorem 2, $\mathcal{F} \cong \mathbb{H}_{k-1} \times \mathbb{G}_{n-k}$. Hence $\frac{(n-1)(3n-4)}{2} = t_{\mathcal{F}} + 1 = \frac{k(k-1)}{2} + \frac{(n-k)(3n-3k-1)}{2}$ or $(k-1)(6n-4k-4) = 0$. Now $6n - 4k - 4 = 4(n-k) + 2n - 4 > 0$ since $n \geq 3$. Hence $k = 1$, $\mathcal{F} \cong \mathbb{G}_{n-1}$ and $\pi_{\mathcal{F}}$ has type $(2^1)$. ■

## 4 Determination of $\mathrm{Aut}(\mathbb{G}_n)$, $n \geq 3$

Let $n \geq 3$ and let $B$ denote the set of all big geodetically closed sub near $2(n-1)$-gons of $\mathbb{G}_n$ isomorphic to $\mathbb{G}_{n-1}$, or equivalently, the set of all geodetically closed sub near polygons $\mathcal{F}$ for which $\pi_{\mathcal{F}}$ has type $(2^1)$. Consider the following relation $R$ on the elements of $B$: $(\mathcal{F}_1, \mathcal{F}_2) \in R \Leftrightarrow (\mathcal{F}_1 = \mathcal{F}_2)$ or $(\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ and every line meeting $\mathcal{F}_1$ and $\mathcal{F}_2$ is special).

**Lemma 13** *The relation $R$ is an equivalence relation and each equivalence class contains exactly three elements.*

*Proof.* For every element $\mathcal{F}$ of $B$, let $C_{\mathcal{F}}$ denote the set of all elements $\mathcal{F}' \in B$ satisfying $(\mathcal{F}, \mathcal{F}') \in R$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two elements of $B$ such that $\pi_{\mathcal{F}_1} = \langle \bar{e}_i + \alpha_1 \bar{e}_j \rangle$ and $\pi_{\mathcal{F}_2} = \langle \bar{e}_i + \alpha_2 \bar{e}_j \rangle$, then one readily verifies that $(\mathcal{F}_1, \mathcal{F}_2) \in R$. Hence $|C_{\mathcal{F}}| \geq 3$ for every $\mathcal{F} \in B$. It now suffices to prove that $|C_{\mathcal{F}}| \leq 3$. Let $L$ denote an arbitrary special line intersecting $\mathcal{F}$ in a point. If $\mathcal{F}'$ is an element of $C_{\mathcal{F}}$, then $\mathcal{F}'$ intersects $L$ in a point. Now, each point $x$ on $L$ is contained in at most one element of $C_{\mathcal{F}}$, namely the element of $B$ generated by the $n-1$ special lines through $x$ different from $L$. Hence $|C_{\mathcal{F}}| \leq 3$. This proves our lemma. ■

Clearly the equivalence classes are in bijective correspondence with the pairs $\{i, j\} \subseteq \{0, \ldots, 2n-1\}$. Consider now the graph $\Gamma$ whose vertices are the equivalence classes, with two classes $C_1$ and $C_2$ adjacent if and only if $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ for every $\mathcal{F}_1 \in C_1$ and every $\mathcal{F}_2 \in C_2$. Clearly two vertices are adjacent if and only if the corresponding pairs have one element in common. Hence, $\Gamma$ is a triangular graph.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two elements of $B$ satisfying $\pi_{\mathcal{F}_1} = \langle \bar{e}_0 + r\bar{e}_1 \rangle$ and $\pi_{\mathcal{F}_2} = \langle \bar{e}_0 + s\bar{e}_1 \rangle$, $r \neq s$, then $\mathcal{F}_3 := \mathcal{R}_{\mathcal{F}_2}(\mathcal{F}_1)$ (recall the definition of $\mathcal{R}_{\mathcal{F}_2}$ given in Section 2.2) is the unique element of $C_{\mathcal{F}_1}$ different from $\mathcal{F}_1$ and $\mathcal{F}_2$; hence $\pi_{\mathcal{F}_3} = \langle \bar{e}_0 + (r+s)\bar{e}_1 \rangle$.

**Lemma 14** *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two elements of $B$ satisfying $\pi_{\mathcal{F}_1} = \langle \bar{e}_0 + r\bar{e}_1 \rangle$ and $\pi_{\mathcal{F}_2} = \langle \bar{e}_0 + s\bar{e}_2 \rangle$, then $\mathcal{F}_3 := \mathcal{R}_{\mathcal{F}_2}(\mathcal{F}_1)$ satisfies $\pi_{\mathcal{F}_3} = \langle \bar{e}_1 + r^{-1}s\bar{e}_2 \rangle$.*

*Proof.* Every point $p$ of $\mathcal{F}_1$ is of the form $\langle \bar{e}_0 + r\bar{e}_1, \bar{e}_2 + t\bar{e}_i, \bar{v}_3, \cdots, \bar{v}_n \rangle$ for some $i \in \{3, \ldots, 2n-1\}$, some $t \in \mathrm{GF}(4)^*$ and some vectors $\bar{v}_j$, $j \in \{3, \ldots, n\}$, of weight 2. The unique line $L$ through $p$ intersecting $\mathcal{F}_2$ is then equal to $\langle \bar{e}_0 + r\bar{e}_1 + s\bar{e}_2 + st\bar{e}_i, \bar{v}_3, \cdots, \bar{v}_n \rangle$. The point $\langle \bar{e}_0 + st\bar{e}_i, \bar{e}_1 + r^{-1}s\bar{e}_2, \bar{v}_3, \cdots, \bar{v}_n \rangle$ of $L$ is not contained in $\mathcal{F}_1 \cup \mathcal{F}_2$ and hence belongs to $\mathcal{F}_3$. Considering all possibilities for $i$, $t$ and $\bar{v}_j$, $j \in \{3, \ldots, 2n-1\}$, we easily see that $\pi_{\mathcal{F}_3} = \langle \bar{e}_1 + r^{-1}s\bar{e}_2 \rangle$. ∎

**Theorem 3** *For every permutation $\phi$ of $\{0, \ldots, 2n-1\}$, every automorphism $\theta$ of $\mathrm{GF}(4)$, and all $\lambda_0, \ldots, \lambda_{2n-1} \in \mathrm{GF}(4)^*$, the semilinear map $V(2n,4) \to V(2n,4) : \sum \alpha_i \bar{e}_i \mapsto \sum \lambda_i \alpha_i^\theta \bar{e}_{\phi(i)}$ induces an automorphism of $\mathbb{G}_n$. Conversely, every automorphism of $\mathbb{G}_n$, $n \geq 3$, is obtained in this way.*

*Proof.* Clearly every semilinear map $V(2n,4) \to V(2n,4) : \sum \alpha_i \bar{e}_i \mapsto \sum \lambda_i \alpha_i^\theta \bar{e}_{\phi(i)}$ induces an automorphism of $\mathbb{G}_n$. We will now prove that every $\mu \in \mathrm{Aut}(\mathbb{G}_n)$ is derived from a semilinear map. The action of $\mu$ on the set $B$ determines an action on the vertices of $\Gamma$. Clearly, that action permutes the $2n$ maximal cliques of size $2n-1$ in $\Gamma$. Thus, there exists a permutation $\phi$ of $\{0, \ldots, 2n-1\}$ such that, if $C$ is the equivalence class corresponding to the pair $\{i,j\}$, then $\mu(C)$ is the class corresponding to $\{\phi(i), \phi(j)\}$. Now, fix $i, j \in \{0, \ldots, 2n-1\}$ with $i \neq j$. For all $r \in \mathrm{GF}(4)^*$, $\mu$ maps the element $\langle \bar{e}_i + r\bar{e}_j \rangle$ of $B$ to an element of the form $\langle \bar{e}_{\phi(i)} + r'\bar{e}_{\phi(j)} \rangle$ (notice that we identify each element $\mathcal{F} \in B$ with $\pi_{\mathcal{F}}$); hence there exists an $\epsilon_{ij} \in \{1,2\}$ and a $\lambda_{ij} \in \mathrm{GF}(4)^*$ such that $\mu(\langle \bar{e}_i + r\bar{e}_j \rangle) = \langle \bar{e}_{\phi(i)} + \lambda_{ij} r^{\epsilon_{ij}} \bar{e}_{\phi(j)} \rangle$ for all $r \in GF(4)^*$. Clearly, $\lambda_{ji} = \lambda_{ij}^{-1}$ and $\epsilon_{ji} = \epsilon_{ij}$ for all $i, j \in \{0, \ldots, 2n-1\}$ with $i \neq j$. Put $\lambda_{ii}$ equal to 1 for all $i \in \{0, \ldots, 2n-1\}$. Now take mutually distinct $i, j, k \in \{0, \ldots, 2n-1\}$. For all $r, s \in \mathrm{GF}(4)^*$, the reflection of $\langle \bar{e}_i + r\bar{e}_j \rangle$ around $\langle \bar{e}_i + s\bar{e}_k \rangle$ equals $\langle \bar{e}_j + r^{-1}s\bar{e}_k \rangle$. Since $\mu \in \mathrm{Aut}(\mathbb{G}_n)$, the reflection of $\langle \bar{e}_{\phi(i)} + \lambda_{ij} r^{\epsilon_{ij}} \bar{e}_{\phi(j)} \rangle$ around $\langle \bar{e}_{\phi(i)} + \lambda_{ik} s^{\epsilon_{ik}} \bar{e}_{\phi(k)} \rangle$ equals $\langle \bar{e}_{\phi(j)} + \lambda_{jk} (r^{-1}s)^{\epsilon_{jk}} \bar{e}_{\phi(k)} \rangle$, or equivalently, $\lambda_{ij}^{-1} r^{-\epsilon_{ij}} \lambda_{ik} s^{\epsilon_{ik}} = \lambda_{jk} r^{-\epsilon_{jk}} s^{\epsilon_{jk}}$. Since this holds for all $r, s \in \mathrm{GF}(4)^*$, $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$, $\epsilon_{ij} = \epsilon_{jk}$ and $\epsilon_{ik} = \epsilon_{jk}$. It now easily follows that $\epsilon_{ij} = \epsilon_{01} = \epsilon$ and $\lambda_{ij} = \lambda_{0i}^{-1}\lambda_{0j}$ for all $i, j \in \{0, \ldots, 2n-1\}$ with $i \neq j$. For all $r \in \mathrm{GF}(4)^*$ and all $j, k \in \{0, \ldots, 2n-1\}$ with $j \neq k$, $\mu(\langle \bar{e}_j + r\bar{e}_k \rangle) = \langle \bar{e}_{\phi(j)} + \lambda_{0j}^{-1}\lambda_{0k} r^\epsilon \bar{e}_{\phi(k)} \rangle = \langle \lambda_{0j}\bar{e}_{\phi(j)} + \lambda_{0k} r^\epsilon \bar{e}_{\phi(k)} \rangle$. The action of $\mu$ on the elements of $B$ completely determines the action of $\mu$ on the points of $\mathbb{G}_n$. For, if $p$ is a point of $\mathbb{G}_n$, then $\mu(p) = \bigcap \mu(\mathcal{F})$ where $\mathcal{F}$ ranges over all the $n$ elements of $B$ through $p$. Hence $\mu$ is induced by the semilinear map $\sum \alpha_i \bar{e}_i \mapsto \sum \lambda_{0i} \alpha_i^\epsilon \bar{e}_{\phi(i)}$. ∎

**Remark.** We have $|\mathrm{Aut}(\mathbb{G}_n)| = 2 \cdot 3^{2n-1} \cdot (2n)!$. The condition $n \geq 3$ in Theorem 3 is necessarily. For $n = 2$, the natural distinction between lines of type (a) and lines of type (b) disappears, see Section 3.4. Since $\mathbb{G}_2 \cong Q(5,2)$, $|\mathrm{Aut}(\mathbb{G}_2)| = |P\Gamma U(4,4)| = 103680$, while $2 \cdot 3^3 \cdot 4! = 1296$.

## 5 Spreads in $\mathbb{G}_n$

For two lines $K$ and $L$ of a near polygon, let $\mathrm{d}(K,L)$ denote the minimal distance between a point of $K$ and a point of $L$. By Lemma 1 of [2], one of the following possibilities occurs:

(a) there exist unique points $k \in K$ and $l \in L$ such that $\mathrm{d}(K,L) = \mathrm{d}(k,l)$;

(b) for every point $k \in K$ there exists a unique point $l \in L$ such that $\mathrm{d}(K,L) = \mathrm{d}(k,l)$.

If condition (b) is satisfied, then $K$ and $L$ are called *parallel*. A *spread* of a near polygon is a set of lines partitioning the point set. A spread is called *admissible* if every two lines of it are parallel. Clearly, every spread of a generalized quadrangle is admissible. A spread $S$ of a near polygon $\mathcal{A}$ is called a *spread of symmetry* if for every line $K$ of $S$ and for every two points $k_1$ and $k_2$ on $K$, there exists an automorphism of $\mathcal{A}$ fixing each line of $S$ and mapping $k_1$ to $k_2$. We easily see that every spread of symmetry is an admissible spread. In this section, we will determine all admissible spreads of $\mathbb{G}_n$, $n \geq 2$. For $n \geq 3$ it will turn out that all admissible spreads are also spreads of symmetry. Suppose first that $n = 2$. The generalized quadrangle $\mathbb{G}_2$ is the dual polar space $H^D(3,4)$ and every spread of $\mathbb{G}_2$ corresponds to a set $M$ of points on the Hermitian variety $H = H(3,4)$. By [3], there are two types of spreads in $H^D(3,4)$.

(i) If $\pi$ is a nontangent plane of $\mathrm{PG}(3,4)$, then $M := \pi \cap H$ defines a spread of $H^D(3,4)$.

(ii) Let $\zeta$ denote the Hermitian polarity associated with $H(3,4)$, let $L$ be a line of $\mathrm{PG}(3,4)$ intersecting $H$ in three points and let $\pi$ be a nontangent plane through $l$. Then $M := [(\pi \cap H) \cup (L^\zeta \cap H)] \setminus (L \cap H)$ defines a spread of $H^D(3,4)$.

As remarked earlier both spreads are admissible, but by [5] only the spreads of type (i) are spreads of symmetry. We now determine all admissible spreads in $\mathbb{G}_n$, $n \geq 3$.

For every $i,j \in \{0,\ldots,2n-1\}$ with $i \neq j$, let $A_{i,j}$ denote the set of all good subspaces $\alpha$ on $H = H(2n-1,4)$ that satisfy the following properties:

- $\alpha$ has type $(2^{n-1})$;
- $\langle\langle \bar{e}_i + r\bar{e}_j\rangle, \alpha\rangle$ is a generator of $H$ for every $r \in \mathrm{GF}(4)^*$.

Clearly, $\bigcup_{0\leq i<j\leq 2n-1} A_{i,j}$ is the set of all special lines of $\mathbb{G}_n$. For every $i \in \{0,\ldots,2n-1\}$, we put $B_i := \bigcup_{j\neq i} A_{i,j}$. Obviously $B_i$ consists of all good subspaces of type $(2^{n-1})$ contained in $\langle \bar{e}_i\rangle^\zeta \cap H$. Here $\zeta$ denotes the Hermitian polarity associated with $H$.

**Lemma 15** *Let $n \geq 2$. For every $i \in \{0, \ldots, 2n-1\}$, $B_i$ is a spread of symmetry of $\mathbb{G}_n$. As a consequence $B_i$ is also an admissible spread.*

*Proof.* If $\pi$ is a point of $\mathbb{G}_n$, i.e. a good subspace of type $(2^n)$, then $\pi$ contains a unique point of the form $\langle \bar{e}_i + r\bar{e}_j \rangle$. Clearly $\langle (X \cap \pi) \setminus \{\langle \bar{e}_i + r\bar{e}_j \rangle\} \rangle$ is the unique line of $B_i$ incident with $\pi$. This proves that $B_i$ is a spread. For every $\lambda \in \mathrm{GF}(4)^*$, the linear map $\bar{e}_i \mapsto \lambda \bar{e}_i$, $\bar{e}_j \mapsto \bar{e}_j$ for all $j \neq i$, induces an automorphism $\theta_\lambda$ of $\mathbb{G}_n$ which fixes each line of $S$. Clearly, $\{\theta_\lambda | \lambda \in \mathrm{GF}(4)^*\}$ acts regularly on every line of $B_i$, proving that $B_i$ is a spread of symmetry. ■

**Lemma 16 (Theorem 5 of [8])** *Let $S$ be an admissible spread of a near polygon $\mathcal{A}$, let $L \in S$ and let $\mathcal{F}$ be a geodetically closed sub near polygon of $\mathcal{A}$ through $L$. Then every line of $S$ which meets $\mathcal{F}$ is completely contained in $\mathcal{F}$. As a consequence, the set of lines of $S$ contained in $\mathcal{F}$ is an admissible spread of $\mathcal{F}$.*

**Lemma 17** *An admissible spread $S$ of $\mathbb{G}_n$, $n \geq 3$, only contains special lines.*

*Proof.* Suppose that $S$ has an ordinary line $L$ and let $x$ denote an arbitrary point of $L$. By Lemmas 8 and 10, there exists a unique pair $\{L_1, L_2\}$ of special lines through $x$ such that $L \in \mathcal{C}(L_1, L_2)$. Let $L_3$ denote a special line through $x$ different from $L_1$ and $L_2$ and let $\mathcal{H}$ denote the hex $\mathcal{C}(L_1, L_2, L_3)$. By Lemma 11, $\mathcal{H} \cong \mathbb{G}_3$. By Lemma 16, the spread $S$ induces an admissible spread $S'$ in $\mathcal{H}$. By Lemma 8, there exist two $W(2)$-quads $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in $\mathcal{H}$ through the line $L$. Let $S_i$, $i \in \{1, 2\}$, denote the spread of $\mathcal{Q}_i$ induced by $S'$. Let $L'$ be an element of $S_2$ different from $L$, let $\mathcal{Q}_3$ denote a $Q(5, 2)$-quad of $\mathcal{H}$ through $L'$ and let $S_3$ denote the spread of $\mathcal{Q}_3$ induced by $S'$. Now, $\mathcal{Q}_1$ and $\mathcal{Q}_3$ are disjoint, and since $\mathcal{Q}_3$ is big in $\mathcal{H}$, every point of $\mathcal{Q}_1$ has distance one to a unique point of $\mathcal{Q}_3$. As a consequence $\mathcal{Q}_1$ projects to a subGQ $\mathcal{Q}_4$ of $\mathcal{Q}_3$ isomorphic to $W(2)$. If $y \in \mathcal{Q}_4$ then $y$ is collinear with a unique point $y'$ of $\mathcal{Q}_1$ and $y'$ is contained in a unique line $M$ of $S_1$. The unique line of $S_3$ through $y$ is contained in the quads $\mathcal{C}(M, y)$ and $\mathcal{Q}_3$ and hence coincides with the line $\mathcal{C}(M, y) \cap \mathcal{Q}_3$ which is precisely the projection of $M$ on $\mathcal{Q}_3$. As a consequence the spread $S_1$ projects to a spread $S_4$ of $\mathcal{Q}_4$ and $S_4 \subseteq S_3$. Let $z$ be a point of $\mathcal{Q}_3 \setminus \mathcal{Q}_4$. Through $z$ there is a line of $S_3$ and five lines intersecting an element of $S_4$. (Notice that $|S_4| = 5$ since $\mathcal{Q}_4 \cong W(2)$.) Hence, the point $z$ of $\mathcal{Q}_3$ is contained in at least six lines, contradicting $\mathcal{Q}_3 \cong \mathbb{G}_2$. ■

**Lemma 18** *Let $S$ be a spread of $\mathbb{G}_n$, $n \geq 3$, satisfying*

(*a*) *every line of $S$ is special,*

(*b*) *if a grid-quad contains one line of $S$, then it contains exactly three lines of $S$.*

*Then $S = B_i$ for a certain $i \in \{0, \ldots, 2n-1\}$.*

*Proof.* Suppose that $S$ contains a special line $K$ of the set $A_{2n-2,2n-1}$, e.g. let $K = \langle \langle \alpha_0 \bar{e}_0 + \alpha_1 \bar{e}_1 \rangle, \langle \alpha_2 \bar{e}_2 + \alpha_3 \bar{e}_3 \rangle, \ldots, \langle \alpha_{2n-4} \bar{e}_{2n-4} + \alpha_{2n-3} \bar{e}_{2n-3} \rangle \rangle$ for certain $\alpha_0, \ldots, \alpha_{2n-3} \in \mathrm{GF}(4)^*$. Now, for every $\lambda \in \mathrm{GF}(4)^*$, the grid-quad $\mathcal{Q}$ for which $\pi_{\mathcal{Q}} = \langle \langle \alpha_0 \bar{e}_0 + \alpha_1 \bar{e}_1 + \lambda \alpha_2 \bar{e}_2 + \lambda \alpha_3 \bar{e}_3 \rangle, \ldots, \langle \alpha_{2n-4} \bar{e}_{2n-4} + \alpha_{2n-3} \bar{e}_{2n-3} \rangle \rangle$ contains $K$. Hence, the two other lines in $\mathcal{Q}$ disjoint from $K$ are also contained in $S$, or equivalently,

$\langle\langle \alpha_0\bar{e}_0 + \lambda\alpha_2\bar{e}_2\rangle, \langle \alpha_1\bar{e}_1 + \lambda\alpha_3\bar{e}_3\rangle, \ldots, \langle \alpha_{2n-4}\bar{e}_{2n-4} + \alpha_{2n-3}\bar{e}_{2n-3}\rangle\rangle \in S$ and $\langle\langle \alpha_0\bar{e}_0 + \lambda\alpha_3\bar{e}_3\rangle, \langle \alpha_1\bar{e}_1 + \lambda\alpha_2\bar{e}_2\rangle, \ldots, \langle \alpha_{2n-4}\bar{e}_{2n-4} + \alpha_{2n-3}\bar{e}_{2n-3}\rangle\rangle \in S$. Applying this several times, we see that every line of $A_{2n-2,2n-1}$ belongs to $S$. Hence $S$ is a union of sets of the form $A_{i,j}$. Since $S = \frac{|Y|}{3}$, $S$ is the union of $2n-1$ sets of the form $A_{i,j}$. For all $i, j, k, l \in \{0, \ldots, 2n-1\}$ with $i \neq j$, $k \neq l$ and $\{i, j\} \cap \{k, l\} = \emptyset$, $A_{i,j} \cup A_{k,l}$ always contains two intersecting lines. The lemma now easily follows. ∎

**Corollary 3** *The spreads $B_i$, $i \in \{0, \ldots, 2n-1\}$, are the only admissible spreads in $\mathbb{G}_n$, $n \geq 3$.*

*Proof.* This follows immediately from Lemmas 15, 16, 17 and 18. ∎

## 6 Glued near polygons derived from $\mathbb{G}_n$

By "glueing" near polygons it is possible to derive new near polygons. This procedure was described in [6] for generalized quadrangles and in [8] for the general case. We recall the construction.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two near polygons both with constant line size $s+1$, and suppose that their respective diameters $d_1$ and $d_2$ are at least 2. Let $S_i = \{L_1^{(i)}, \ldots, L_{\alpha_i}^{(i)}\}$, $i \in \{1, 2\}$, be an admissible spread of $\mathcal{A}_i$. In $S_i$, a special line $L_1^{(i)}$ is chosen which we will call the *base line*. For every $i \in \{1, 2\}$, for all $j, k \in \{1, \ldots, \alpha_i\}$ and for every $x \in L_j^{(i)}$, let $p_{j,k}^{(i)}(x)$ denote the unique point of $L_k^{(i)}$ nearest to $x$. We put $\Phi_{j,k}^{(i)} := p_{k,1}^{(i)} \circ p_{j,k}^{(i)} \circ p_{1,j}^{(i)}$. For every $i \in \{1, 2\}$, the group $\Pi_{S_i}(L_1^{(i)}) := \langle \Phi_{j,k}^{(i)} | 1 \leq j, k \leq \alpha_i\rangle$ is called the *group of projectivities of $L_1^{(i)}$ with respect to $S_i$.*

For every bijection $\theta$ between $L_1^{(1)}$ and $L_1^{(2)}$, we consider the following graph $\Gamma$ with vertex set $L_1^{(1)} \times S_1 \times S_2$. Two vertices $(x, L_{i_1}^{(1)}, L_{j_1}^{(2)})$ and $(y, L_{i_2}^{(1)}, L_{j_2}^{(2)})$ are adjacent if and only if exactly one of the following three conditions is satisfied:

(A) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $L_{j_1}^{(2)} = L_{j_2}^{(2)}$ and $x \neq y$;

(B) $L_{j_1}^{(2)} = L_{j_2}^{(2)}$, $\mathrm{d}(L_{i_1}^{(1)}, L_{i_2}^{(1)}) = 1$ and $\Phi_{i_1,i_2}^{(1)}(x) = y$;

(C) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $\mathrm{d}(L_{j_1}^{(2)}, L_{j_2}^{(2)}) = 1$ and $\Phi_{j_1,j_2}^{(2)} \circ \theta(x) = \theta(y)$.

By [8], the graph $\Gamma$ has diameter $d_1 + d_2 - 1$ and every two adjacent vertices are contained in a unique maximal clique. Considering these maximal cliques as lines, we obtain a partial linear space $\mathcal{A}_1 \otimes \mathcal{A}_2$. If $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a near polygon, then it is called a *glued near polygon.* This precisely happens when the condition in the following theorem is satisfied.

**Theorem 4 (Theorem 14 of [8])** *The partial linear space $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a glued near polygon if and only if the commutator $[\Pi_{S_1}(L_1^{(1)}), \theta^{-1}\Pi_{S_2}(L_1^{(2)})\theta]$ is the trivial group of permutations of $L_1^{(1)}$.*

If $\mathcal{A}_1 \cong \mathcal{B}_1 \times L$ and if $S_1 = \{L_x | x \text{ is a point of } \mathcal{B}_1\}$ with $L_x := \{(x, y) | y \in L\}$ (we call such a spread a *trivial spread* of $\mathcal{A}_1$), then $\Pi_{S_1}(L_1^{(1)})$ is the trivial group and $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a near polygon. In fact we have $\mathcal{A}_1 \otimes \mathcal{A}_2 \cong \mathcal{B}_1 \times \mathcal{A}_2$. The following theorem shows the importance of the notion "spread of symmetry".

**Theorem 5 (Theorems 11 and 16 of [8])** *Suppose that each line of $\mathcal{A}_1$ and $\mathcal{A}_2$ is incident with three points and that none of the spreads $S_1$ and $S_2$ is trivial. Then $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a near polygon (for an arbitrary choice of the base lines and the bijection $\theta$ between these base lines) if and only if $S_1$ and $S_2$ are spreads of symmetry.*

Now, suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are fixed near polygons with three points on each line and that $S_1$ and $S_2$ are fixed nontrivial spreads of symmetry in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. By [8] every near polygon which can be obtained for a certain choice of the base lines can always be obtained for any other choice of the base lines (by changing the map $\theta$ accordingly). Hence we may also fix base lines $L_1^{(1)} \in S_1$ and $L_1^{(2)} \in S_2$. For every bijection $\theta$ between $L_1^{(1)}$ and $L_1^{(2)}$, there then exists a near polygon $\mathcal{A}_1 \otimes_\theta \mathcal{A}_2$. By reasons of symmetry, all these near polygons are isomorphic if the group of automorphisms of $\mathcal{A}_1$ which fix $S_1$ and the base line $L_1^{(1)} \in S_1$ induces the full group of permutations on this base line.

**Lemma 19** *Let $S$ be a spread of symmetry of $\mathbb{G}_n$, $n \geq 2$, and let $K$ be a line of $S$. Then the group of automorphisms of $\mathbb{G}_n$ fixing $S$ and $K \in S$ induces the full group of permutations on the line $K$.*

*Proof.* Since there is up to an isomorphism only one spread of symmetry in $\mathbb{G}_n$, $n \geq 2$, we may suppose that $S$ is the spread $B_0$ and that $K$ is the line $\langle\langle \bar{e}_2 + \bar{e}_3\rangle, \cdots, \langle \bar{e}_{2n-2} + \bar{e}_{2n-1}\rangle\rangle$. In Theorem 3 we determined all automorphisms of $\mathbb{G}_n$, $n \geq 3$. For $n = 2$, the maps defined there still are automorphisms (but not all automorphisms are of this form). There are now precisely 6 automorphisms if we put $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{2n-1}$ equal to 1 and $\phi$ equal to the trivial permutation of $\{0, \ldots, 2n-1\}$. We easily see that these six automorphisms induce the full group of permutations on the line $K$. ■

By the results of this section and the fact that there is up to an isomorphism only one spread of symmetry in $\mathbb{G}_n$, $n \geq 2$, we then have:

**Corollary 4** *For all positive integers $m, n \geq 2$, there exists a unique glued near polygon of the form $\mathbb{G}_m \otimes \mathbb{G}_n$.*

**Remark**. Also the near polygons $H^D(2n-1, 4)$, $n \geq 3$, and the near hexagon derived from the extended ternary Golay code (see [12]) are known to have spreads of symmetry. Hence, more glued near polygons can be derived from $\mathbb{G}_n$.

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