Holomorphic functions on locally closed convex sets and projective descriptions

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Abstract

Let Q be a bounded, convex and locally closed subset of $\mathbb{C}^N$, let $H(Q)$ be the space of all functions which are holomorphic on an open neighborhood of Q. We endow $H(Q)$ with its projective topology. We show that the topology of the weighted inductive limit of Fréchet spaces of entire functions which is obtained as the Laplace transform of the strong dual to $H(Q)$ can be described be means of canonical weighted seminorms if and only if the intersection of Q with each supporting hyperplane to the closure of Q is compact. We also find conditions under which this (LF)-space of entire functions coincides algebraically with its projective hull.

Introduction

More that 30 years ago, Martineau investigated in [15] the spaces $H(Q)$ of analytic functions on a convex nonpluripolar set Q in $\mathbb{C}^N$, in the case that Q admits a countable fundamental system of compact sets. This setup covers nonpluripolar compact convex sets and convex open sets in $\mathbb{C}^N$ and convex open sets in $\mathbb{R}^N$. In the latter case $H(Q)$ coincides with the space of all real analytic functions on Q.

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The strong dual $H(Q)'_b$ of one of these spaces can be canonically identified, via the Laplace transform, with a weighted (LF)-space of entire functions on $\mathbb{C}^N$, i.e. $H(Q)'_b$ is isomorphic to a Hausdorff countable inductive limit of Fréchet spaces of entire functions defined by weighted sup-seminorms. See the details below. The description of the topology of this type of weighted inductive limits of spaces of holomorphic functions has been investigated thoroughly in recent years, since the work of Ehrenpreis [12] on analytically uniform spaces.

The problem of the projective description of the topology of weighted inductive limits of spaces of holomorphic or continuous functions was investigated systematically in several articles by Bierstedt, Bonet, Meise and Summers; see e.g. [3, 4, 5]. For weighted inductive limits in which the steps are defined by means of weighted sup-seminorms, the aim is to find a projective description of the topology of the inductive limit by weighted sup-seminorms which allow direct computations and estimates as required in the applications. In the theory of Ehrenpreis [12] of analytically uniform spaces, the topology of certain weighted inductive limits of spaces of entire functions which appear as the Fourier Laplace transform of spaces of test functions or ultradistributions was required to have a fundamental system of weighted sup-seminorms. Berenstein and Dostal [1] later used the term "complex representation". This term corresponds with the projective description of such inductive limits in [5] which is the one used and explained below in this article. In [5] it is proved that the projective description holds for weighted inductive limits of Banach spaces of holomorphic functions defined on an arbitrary open subset of $\mathbb{C}^N$ whenever the linking maps between the generating Banach spaces are compact. In general the problem of projective description for weighted inductive limits of Banach spaces of holomorphic functions has a negative answer as was recently shown in the examples to be found in [9, 8, 10].

The case of (LF)-spaces of holomorphic functions is more complicated. Ehrenpreis [11, p. 557-558] showed that the space of real analytic functions $A(\mathbb{R}^N)$ on $\mathbb{R}^N$ is not analytically uniform. This implies that the topology of the weighted (LF)-space of entire functions which is isomorphic to the strong dual of the space of real analytic functions cannot be described by means of the canonical weighted sup-seminorms; see also [1]. This result is related to our present research. Bonet and Meise [6] show that the topological projective description also fails for the natural weighted inductive limits of spaces of entire functions which appear as the Fourier Laplace transform of spaces of ultradistributions of compact support in the non-quasianalytic case. We refer to the recent survey article by K.D. Bierstedt [2] for further details, motivations and open problems.

In this article we continue the investigations on the projective description of weighted (LF)-spaces of entire functions. In the context of the Laplace transform of the spaces $H(Q)'_b$, several necessary and sufficient conditions to ensure that the projective description holds algebraically or topologically are presented when Q is bounded. Our main results are the Theorems 6 and 8 and the Corollary 7. They are based on a new description of the topology of the projective hull in the present case which is given in Lemma 4. The present research will be continued in [7] with a study of the algebraic projective description of the weighted inductive limits which appear as the Fourier Laplace transform of spaces of quasi analytic and real analytic functions on an open convex set in $\mathbb{R}^N$.

Notation and Preliminaries

A subset Q of $\mathbb{C}^N$ is called locally closed if for each $z \in Q$ there is a closed neighborhood U of z in $\mathbb{C}^N$ such that $Q \cap U$ is closed. Every open subset and every compact subset of $\mathbb{C}^N$ is locally closed. If D is a subset of $\mathbb{C}^N$, we denote by $\overline{D}$ and $\mathrm{int}D$ the closure and the interior of D. For a convex set $Q \subset \mathbb{C}^N$ the symbols $\mathrm{int}_r Q$ and $\partial_r Q$ denote the relative interior and the relative boundary of Q *with respect to the affine hull of* Q. Let $\overline{B}(\mu, r) := \{z \in \mathbb{C}^N \mid |z - \mu| \leq r\}$, $\mu \in \mathbb{C}^N$, $r \geq 0$.

Lemma 1. *The following assertions are equivalent for a convex subset Q of $\mathbb{C}^N$:*

(i) The set Q is locally closed.

(ii) Q admits a countable fundamental system of compact subsets.

(iii) Q is the union of the relative interior $\mathrm{int}_r Q$ *of Q and an open (in $\partial_r Q$) subset ω of $\partial_r Q$. In this case, if $(K_n)_n$ and $(\omega_n)_n$ are fundamental sequences of compact sets of* $\mathrm{int}_r Q$ *and ω respectively, the convex hulls Q_n of $K_n \cup \omega_n$, $n \in \mathbb{N}$, define a fundamental system of compact subsets of Q.*

Proof. The equivalence of (ii) and (iii) was proved in [17, 1.2].

(i)$\Rightarrow$(iii): If the set $\omega := Q \cap \partial_r Q$ is not open in $\partial_r Q$, then there is a $z_0 \in \omega$ such that for each n there exists $z_n \in (\partial_r Q \cap \overline{B}(z_0, 1/n)) \backslash Q$. Since for any n the point z_{n+1} is an adherent point of $Q \cap \overline{B}(z_0, 1/n)$ and does not belong to Q the set $Q \cap \overline{B}(z_0, 1/n)$ is not closed. Hence Q is not locally closed.

(iii)$\Rightarrow$(i): Assume that Q is not locally closed. There is a point $z_0 \in \omega$ such that for any n the set $Q \cap \overline{B}(z_0, 1/n)$ is not closed. Consequently for each n there is an adherent point z_n of $Q \cap \overline{B}(z_0, 1/n)$ such that $z_n \notin Q \cap \overline{B}(z_0, 1/n))$. Since $\overline{B}(z_0, 1/n)$ is closed, $z_n \notin Q$ and $z_n \in \partial_r Q$. Thus $z_n \in (\partial_r Q \cap \overline{B}(z_0, 1/n)) \backslash Q$. Hence for any n the neighborhood $\overline{B}(z_0, 1/n) \cap \partial_r Q$ of the point $z_0 \in \omega$ (in $\partial_r Q$) is not contained in ω. Therefore the set $\omega = \partial_r Q \cap Q$ is not open in $\partial_r Q$.

General Assumption: In the rest of this article Q denotes a locally closed convex set and $(Q_n)_n$ is a fixed increasing fundamental sequence of compact convex sets in Q. Without loss of generality we assume that the origin belongs to Q, and we select the first convex compact set Q_1 so that $0 \in Q_1$. We write $\omega := Q \cap \partial_r Q$. By $\partial_r \omega$ we denote the relative boundary of ω with respect to $\partial_r Q$.

According to [17, 1.3], a locally closed convex set Q is called ($\mathbb{C}$-)*strictly convex at* $\partial_r \omega$ if the intersection of Q with each supporting (complex) hyperplane to $\overline{Q}$ is compact. If the interior of Q is empty, the set Q is strictly convex at $\partial_r \omega$ if and only if Q is compact. If the interior of Q is not empty, Q is ($\mathbb{C}$-)strictly convex at $\partial_r \omega$ if and only if each line segment (of which the $\mathbb{C}$-linear affine hull belongs to some supporting hyperplane of $\overline{Q}$) of $\omega = Q \cap \partial_r Q$ is relatively compact in ω. By [15, Lemme 3 of the proof of Thme 1.2] (see also the proof of [17, 1.16]) if Q is $\mathbb{C}$-strictly convex at $\partial_r \omega$, then Q has a neighbourhood basis of linearly convex open sets, hence a basis of domains of holomorphy.

Proposition 2. *A convex locally closed set Q is strictly convex at $\partial_r \omega$ if and only if the collection of all convex open neighborhoods of Q is a neighborhood basis of Q.*

Proof. It was shown in the proof of [17, 1.15] that if Q is strictly convex at $\partial_r\omega$, then the collection of all convex open neighborhoods of Q is a neighborhood basis of Q. Conversely, suppose that Q is not strictly convex at $\partial_r\omega$. There exists a supporting hyperplane P to $\overline{Q}$ such that the intersection of P with Q contains a line interval I which is not relatively compact in $P \cap Q$. After a translation and an orthogonal transformation of $\mathbb{R}^{2N} = \mathbb{C}^N$ we can suppose that P is the hyperplane $\{x \in \mathbb{R}^{2N} \,|\, x_1 = 0\}$, the interval I is contained in the line $\{x \in \mathbb{R}^{2N} \,|\, x_1 = x_3 = ... = x_{2N} = 0\}$, and Q is contained in the half-space $\{x \in \mathbb{R}^{2N} \,|\, x_1 \leq 0\}$. We put $I_1 := Q \cap (\mathbb{R}I)$. The interval I_1 is not compact in the line $\mathbb{R}I$.

Now, $Q \cap \mathbb{C}_1$ has no neighborhood basis of convex domains in the plane $\mathbb{C}_1 := \{x \in \mathbb{R}^{2N} \,|\, x_3 = ... = x_{2N} = 0\}$. Indeed, with the canonical identification, suppose for example that I_1 is the interval (a, b), where $-\infty < a < b \leq +\infty$. The domain $\Omega := \{z \in \mathbb{C} \,|\, \mathrm{Im} z < (\mathrm{Re} z - a)^2\}$ is a neighborhood of $Q \cap \mathbb{C}_1$. Since for each point $t \in \Omega$ with $\mathrm{Im} t > 0$ there is a point $w \in I_1$ which is sufficiently close to a such that the segment $[w, t]$ is not contained in Ω, no convex neighborhood of $Q \cap \mathbb{C}_1$ can be contained in Ω.

Now, we choose an open neighborhood D of $Q \cap \mathbb{C}_1$ in $\mathbb{C}_1$ such that no open convex neighborhood U of $Q \cap \mathbb{C}_1$ in $\mathbb{C}_1$ exists with $U \subset D$. There is a neighborhood Ω of Q in $\mathbb{R}^{2N}$ such that $\Omega \cap \mathbb{C}_1 = D$. By the choice of D, no convex open domain G containing Q can be contained in Ω. The proof is complete.

For an open set $D \subset \mathbb{C}^N$, we denote by $H(D)$ the space of all holomorphic functions on D with its standard Fréchet topology. For a compact subset K of $\mathbb{C}^N$, the space of all functions which are holomorphic on some open neighborhood of K is denoted by $H(K)$ and it is endowed with its natural inductive limit topology. We denote by $H(Q)$ the vector space of all functions which are holomorphic on some open neighborhood of Q. Since the algebraic equality $H(Q) = \cap_{n\in\mathbb{N}} H(Q_n)$ holds, we endow $H(Q)$ with the projective topology of $H(Q) := \mathrm{proj}_n H(Q_n)$. This topology does not depend of the choice of the fundamental system $(Q_n)_n$. See more details in [17, pp. 296-299]. In the case that Q is a convex locally closed subset of $\mathbb{R}^N$, the space $H(Q)$ is a space of real analytic functions. In particular, if Q is an open convex subset of $\mathbb{R}^N$, then $H(Q) = A(Q)$, where $A(Q)$ denotes the space of all the real analytic functions on Q.

Next, we recall the necessary notation for weighted inductive limits; see [5, 6], and we state the problem of projective description. For notation concerning locally convex spaces we refer the reader to [16].

We denote by $V = (v_{n,k})$ a double sequence of strictly positive upper semicontinuous weights on $\mathbb{C}^N$, $N \in \mathbb{N}$, such that

$$v_{n+1,k}(z) \leq v_{n,k}(z) \leq v_{n,k+1}(z), z \in \mathbb{C}^N$$

for each $n, k \in \mathbb{N}$. The *weighted inductive limit of Fréchet spaces $VH(\mathbb{C}^N)$ of entire functions associated with V* is defined by

$$VH(\mathbb{C}^N) := \mathrm{ind}_n \mathrm{proj}_k H(v_{n,k}, \mathbb{C}^N),$$

where the steps $H(v, \mathbb{C}^N)$ are defined, for a positive weight v on $\mathbb{C}^N$, as the Banach space of entire functions

$$H(v, \mathbb{C}^N) := \{f \in H(\mathbb{C}^N) |\ \ ||f||_v := \sup_{z\in\mathbb{C}^N} v(z)|f(z)| < \infty\}.$$

The space $VH(\mathbb{C}^N)$ is a Hausdorff (LF)-space. In order to describe its topology by means of weighted sup-seminorms, Bierstedt, Meise and Summers [5] associated with V the system $\overline{V}$ of all those weights $\overline{v} : \mathbb{C}^N \to [0, \infty[$ which are upper semicontinuous and have the property that for each n there are $\alpha_n > 0$ and $k = k(n)$ such that $\overline{v} \leq \alpha_n v_{n,k}$ on $\mathbb{C}^N$. The projective hull of the weighted inductive limit is defined by

$$H\overline{V}(\mathbb{C}^N) := \{f \in H(\mathbb{C}^N) \mid ||f||_{\overline{v}} := \sup_{z \in \mathbb{C}^N} \overline{v}(z)|f(z)| < \infty \text{ for all } \overline{v} \in \overline{V}\},$$

endowed with the Hausdorff locally convex topology defined by the system of seminorms $\{||.||_{\overline{v}} \mid \overline{v} \in \overline{V}\}$. The projective hull is a complete locally convex space and $VH(\mathbb{C}^N)$ is contained in its projective hull with continuous inclusion.

The problem of projective description is to determine conditions under which
(1) the spaces $VH(\mathbb{C}^N)$ and $H\overline{V}(\mathbb{C}^N)$ coincide algebraically, or
(2) the space $VH(\mathbb{C}^N)$ is a topological subspace of its projective hull $H\overline{V}(\mathbb{C}^N)$.

A positive answer to question (2), i.e. whether $VH(\mathbb{C}^N)$ is a topological subspace of its projective hull, is of particular importance, because when the answer is positive it permits to describe the topology of the weighted (LF)-space of holomorphic functions by means of weighted sup-seminorms.

In case $v_{n,k} = v_{n,k+1} =: v_n$ holds for each n, k, the space $VH(\mathbb{C}^N)$ is in fact an (LB)-space. As a consequence of the main result of Bierstedt, Meise, Summers [5, 1.6], in this case the projective description holds algebraically and topologically if the sequence $V = (v_n)_n$ satisfies the following condition

(S): for each n there is $m > n$ such that v_m/v_n vanishes at infinity on $\mathbb{C}^N$.

This positive result will be used later in this article.

For each convex set $D \subset \mathbb{C}^N$ we denote by $H_D : \mathbb{C}^N \to \mathbb{R} \cup \{\infty\}$ the support function of D, $H_D(z) := \sup_{w \in D} \mathrm{Re}\langle z, w\rangle$, $z \in \mathbb{C}^N$. Here $\langle z, w\rangle := \sum_{j=1}^{N} z_j w_j$. For each $n \in \mathbb{N}$, let $H_n := H_{Q_n}$ be the support functions of the convex compact sets $Q_n, n \in \mathbb{N}$, which constitute a fundamental sequence of convex compact subsets of Q.

In this article we are interested in the weight functions

$$v_{n,k}(z) := \exp(-H_n(z) - |z|/k), \ n, k \in \mathbb{N}, \ z \in \mathbb{C}^N.$$

By [17, 1.10], the Laplace transform

$$\mathcal{F}(\varphi)(z) := \varphi(\exp\langle \cdot, z\rangle), \ z \in \mathbb{C}^N,$$

is a linear topological isomorphism from the strong dual $H(Q)'_b$ of $H(Q)$ onto $VH(\mathbb{C}^N)$.

We denote by $\overline{V}_0$ the set of all weights $\overline{v}$ such that there are unbounded increasing sequences $k(n) \in \mathbb{N}$ and $\alpha_n \geq 0$ with

$$\overline{v}(z) = \inf_{n \in \mathbb{N}} \exp(-H_n(z) - |z|/k(n) + \alpha_n) \ \text{ for all } \ z \in \mathbb{C}^N.$$

It is easy to see that every weight in $\overline{V}_0$ is contained in $\overline{V}$ and that every element in $\overline{V}$ is estimated by a weight in $\overline{V}_0$.

Let $G \subset \mathbb{C}^N$ be open and convex and let $(G_n)_{n\in\mathbb{N}}$ be a fundamental sequence of (convex) compact subsets G_n of G with $G_n \subset \operatorname{int} G_{n+1}$ for all n. A good description of the Laplace transform of the strong dual $H(G)'_b$ of $H(G)$ is needed. We put

$$V_G := (v_{G,n})_n,\ v_{G,n}(z) := \exp(-H_{G_n}(z)),\ z \in \mathbb{C}^N, n \in \mathbb{N}.$$

Again by [17, 1.10], the Laplace transform

$$\mathcal{F}(\varphi)(z) := \varphi(\exp\langle\cdot, z\rangle),\ z \in \mathbb{C}^N,$$

is a linear topological isomorphism from $H(G)'_b$ onto the weighted (LB)-space $V_G H(\mathbb{C}^N)$. Moreover, as a consequence of [5, 1.6], the space $V_G H(\mathbb{C}^N)$ and its projective hull $H\overline{V_G}(\mathbb{C}^N)$ coincide algebraically and topologically. In the case that G is bounded another description of the topology of $V_G H(\mathbb{C}^N)$ can be given. To define the family of weights we use, we denote by $\overline{V}(G)$ the collection of all upper semicontinuous functions $\overline{u}(z) = \exp(-H_G(z) + \gamma(z))$, $z \in \mathbb{C}^N$, where $\gamma : \mathbb{C}^N \to \mathbb{R}$ satisfies $\gamma(z) = o(|z|)$ as $|z| \to \infty$. This type of weights was used by Napalkov [18]. The next lemma will be very useful in our Lemma 4 below and its consequences.

Lemma 3. *If G is a bounded convex open subset of $\mathbb{C}^N$, then the space $V_G H(\mathbb{C}^N)$ coincides algebraically and topologically with the weighted space $H(\overline{V}(G))(\mathbb{C}^N) = \operatorname{proj}_{\overline{u}\in\overline{V}(G)} H(\overline{u}, \mathbb{C}^N)$.*

Proof. We assume without loss of generality that $0 \in G$. Since G is bounded and open, there are $M > 0, m > 0$ such that $m|z| \le H_G(z) \le M|z|$ for all $z \in \mathbb{C}^N$. We select $a_n > 0$ increasing and tending to 1, and we take $G_n := a_n\overline{G}$, $n \in \mathbb{N}$, as a fundamental sequence of convex compact subsets of G. In this case $v_{G,n}(z) := \exp(-H_{G_n}(z)) = \exp(-a_n H_G(z)),\ z \in \mathbb{C}^N, n \in \mathbb{N}$. To complete the proof it is enough to show that $\overline{V}(G)$ and $\overline{V_G}$ are equivalent.

Let $\overline{u}$ be a function in $\overline{V}(G)$, $\overline{u}(z) = \exp(-H_G(z)+\gamma(z))$, $z \in \mathbb{C}^N$. We fix $n \in \mathbb{N}$. For $z \in \mathbb{C}^N$ we have

$$-H_G(z) + \gamma(z) \le -H_{G_n}(z) + \gamma(z) - (1 - a_n)m|z|.$$

Since $\gamma(z) = o(|z|)$ as $|z| \to \infty$, we conclude that there is $C_n > 0$ such that

$$\exp(-H_G(z) + \gamma(z)) \le C_n \exp(-H_{G_n}(z)) = C_n v_{G,n}(z),$$

and $\overline{u}$ belongs to $\overline{V_G}$.

Now suppose that $\overline{u}$ belongs to $\overline{V_G}$. By [5, Proposition 0.2], we may assume that for each n there is $c_n > 1$ such that, for all $z \in \mathbb{C}^N$,

$$-\infty < \log \overline{u}(z) \le -a_n H_G(z) + \log c_n.$$

We set $\gamma(z) = \max\{0, \log \overline{u}(z) + H_G(z)\},\ z \in \mathbb{C}^N$. It is enough to show that $\gamma(z) = o(|z|)$ as $|z| \to \infty$. To see this, we observe that, if $|z| \ge R_n := \log c_{n+1}/((a_{n+1} - a_n)m)$, we have

$$\log \overline{u}(z) \le -a_{n+1} H_G(z) + \log c_{n+1} \le -a_n H_G(z),$$

hence

$$\log \overline{u}(z) + H_G(z) \le (1 - a_n) H_G(z) \le (1 - a_n) M|z|,$$

from where the conclusion follows.

Main results

Our next lemma is of technical character. However, it is essential in our results. It gives a concrete description of the topology of the projective hull which is very suitable for our purposes.

Lemma 4. *Let Q be a bounded convex and locally closed subset of $\mathbb{C}^N$. The following assertions hold:*

(i) For every bounded convex open neighbourhood G of Q and every $\kappa \in \overline{V}(G)$ there is $\overline{v} \in \overline{V}_0$ with $\kappa \leq \overline{v}$.

(ii) For any $\overline{v} \in \overline{V}_0$ there is a bounded convex open neighbourhood G of Q and there is $\kappa \in \overline{V}(G)$ such that $\overline{v} \leq \kappa$.

In particular, the space $H\overline{V}(\mathbb{C}^N)$ coincides algebraically with the intersection of the spaces $V_G H(\mathbb{C}^N) = H(\overline{V}(G))(\mathbb{C}^N)$ as G varies in the set of all convex open neighbourhoods of Q. Moreover

$$H\overline{V}(\mathbb{C}^N) = \mathrm{proj}_G \mathrm{proj}_{\overline{v} \in \overline{V}(G)} H(\overline{v}, \mathbb{C}^N),$$

with G running as before.

Proof. (i) We fix a convex open bounded neighbourhood G of Q and a function $\kappa \in \overline{V}(G)$ such that $\kappa(z) = \exp(-H_G(z) + \gamma(z))$, $z \in \mathbb{C}^N$, with $\gamma(z) = o(|z|)$ as $|z| \to \infty$. Since G is a neighbourhood of Q, for each n there exist $k(n)$ and $\alpha_n \geq 0$ with

$$H_n(z) + |z|/k(n) - \alpha_n \leq H_G(z) - \gamma(z) \text{ for all } z \in \mathbb{C}^N.$$

Moreover, the sequences $(k(n))_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ can be taken increasing and unbounded. We define

$$\overline{v} := \inf_{n\in\mathbb{N}} \exp(-H_n(z) - |z|/k(n) + \alpha_n) \text{ for all } z \in \mathbb{C}^N.$$

Then $\kappa \leq \overline{v}$ holds obviously, so that we proved (i).

(ii) We fix a function $\overline{v} := \inf_{n\in\mathbb{N}} \exp(-H_n - |\cdot|/k(n) + \alpha_n) \in \overline{V}_0$ and put $L(z) := \sup_{n\in\mathbb{N}}(H_n(z) + |z|/(k(n)+1))$ for all $z \in \mathbb{C}^N$. We select inductively increasing unbounded sequences $r(s) > 0$ and $n(s)$, $s \in \mathbb{N}$, as follows.

First observe that, since Q is bounded, $\overline{Q}$ is a convex compact subset whose support functional coincides with H_Q and it is continuous and convex. Moreover, the sequence $(H_n)_n$ is increasing and converges pointwise to H_Q. By Dini's theorem, it converges uniformly on the unit ball of $\mathbb{C}^N$. As the support functionals are positively homogeneous, we conclude that for each $\varepsilon > 0$ there is $n(0)$ such that for all $n \geq n(0)$ and all $z \in \mathbb{C}^N$, we have $H_n(z) \leq H_{n(0)}(z) + \varepsilon|z|$. This fact will be used several times below.

For $s = 1$ we put $r(s) := 1$. Since $\alpha_n \to \infty$ as $n \to \infty$, there is $n(1) > n(0)$ such that

$$H_n(z) + |z|/k(n) - \alpha_n \leq H_1(z) + |z|/k(1) - \alpha_1, \text{ if } |z| \leq 1, \ n > n(1),$$

and, on account of our remark above for $\varepsilon = 1$,

$$\sup_{1\le n\le n(1)} (H_n(z) + |z|/(k(n)+1)) \ge L(z) - |z| \text{ for all } z \in \mathbb{C}^N.$$

Now we put $r(2) := 2$ and choose $n(2) > n(1)$ with

$$H_n(z) + |z|/k(n) - \alpha_n \le H_1(z) + |z|/k(1) - \alpha_1, \text{ if } |z| \le 2,\ n > n(2),$$

and, on account of our remark above for $\varepsilon = 1/2$,

$$\sup_{1\le n\le n(2)} (H_n(z) + |z|/(k(n)+1)) \ge L(z) - |z|/2 \text{ for all } z \in \mathbb{C}^N.$$

If for $s > 1$ we selected $r(1), ..., r(s)$ and $n(1), ..., n(s)$, we choose $r(s+1) > \max(s; r(s))$ with

$$\sup_{1\le n\le n(s)} \alpha_n/r(s+1) < 1/(s+1),$$

$$H_n(z) + |z|/k(n) - \alpha_n \le H_1(z) + |z|/k(1) - \alpha_1, \text{ if } |z| \le r(s+1),\ n > n(s+1),$$

and, on account of our remark above for $\varepsilon = 1/(s+1)$,

$$\sup_{1\le n\le n(s+1)} (H_n(z) + |z|/(k(n)+1)) \ge L(z) - |z|/(s+1) \text{ for all } z \in \mathbb{C}^N.$$

The function L is positive homogeneous of order 1 and convex. Hence there is a convex compact set D in $\mathbb{C}^N$ with $H_D = L$. Since $Q_n + \dfrac{1}{k(n)+1} B \subset D$ for all $n \in \mathbb{N}$, where $B := \{z \in \mathbb{C}^N \mid |z| \le 1\}$, the interior G of D is not empty and $H_G = L$. In addition, $Q_n \subset G$ for all $n \in \mathbb{N}$ and consequently G is an open convex neighborhood of Q.

Clearly, $\overline{v}(z) = \exp(-h(z)),\ z \in \mathbb{C}^N$ with $h(z) := \sup_{n\in\mathbb{N}}(H_n(z) + |z|/k(n) - \alpha_n)$.

We fix $s \ge 3$ and $z \in \mathbb{C}^N$ with $r(s-1) < |z| \le r(s)$. Then

$$h(z) = \sup_{1\le n\le n(s)} (H_n(z) + |z|/k(n) - \alpha_n).$$

We distinguish two cases.

Case 1. Suppose that

$$\sup_{1\le n\le n(s-2)} (H_n(z) + |z|/k(n) - \alpha_n) \ge \sup_{n(s-2)<n\le n(s)} (H_n(z) + |z|/k(n) - \alpha_n).$$

Then

$$h(z) = \sup_{1\le n\le n(s)} (H_n(z) + |z|/k(n) - \alpha_n) = \sup_{1\le n\le n(s-2)} (H_n(z) + |z|/k(n) - \alpha_n) \ge$$

$$\sup_{1\le n\le n(s-2)} (H_n(z) + |z|/(k(n)+1)) - \sup_{1\le n\le n(s-2)} (\alpha_n - |z|/(k(n)(k(n)+1))) \ge$$

$$L(z) - |z|/(s-2) - \sup_{1\le n\le n(s-2)} \alpha_n.$$

For

$$\gamma_s(z) := |z|/(s-2) + \sup_{1\le n\le n(s-2)} \alpha_n$$

we have

$$0 \le \gamma_s(z) \le |z|/(s-2) + r(s-1)/(s-1) \le (1/(s-2) + 1/(s-1))|z|.$$

Case 2. Suppose that

$$\sup_{1\le n\le n(s-2)} (H_n(z) + |z|/k(n) - \alpha_n) \le \sup_{n(s-2)<n\le n(s)} (H_n(z) + |z|/k(n) - \alpha_n).$$

Then there is $\tilde{n}$ with $n(s-2) < \tilde{n} \le n(s)$ and such that

$$h(z) = \sup_{1\le n\le n(s)} (H_n(z) + |z|/k(n) - \alpha_n) = H_{\tilde{n}}(z) + |z|/k(\tilde{n}) - \alpha_{\tilde{n}} =$$

$$H_{\tilde{n}}(z) + |z|/(k(\tilde{n}) + 1) + |z|/((k(\tilde{n}) + 1)k(\tilde{n})) - \alpha_{\tilde{n}}.$$

We have for each n with $1 \le n \le n(s-2)$

$$H_{\tilde{n}}(z) + |z|/(k(\tilde{n}) + 1) = H_{\tilde{n}}(z) + |z|/k(\tilde{n}) - \alpha_{\tilde{n}} -$$

$$|z|/((k(\tilde{n}) + 1)k(\tilde{n})) + \alpha_{\tilde{n}} \ge$$

$$H_n(z) + |z|/k(n) - \alpha_n + \alpha_{\tilde{n}} - |z|/((k(\tilde{n}) + 1)k(\tilde{n})) \ge H_n(z) + |z|/(k(n) + 1) -$$

$$|z|/((k(\tilde{n}) + 1)k(\tilde{n})).$$

Hence

$$H_{\tilde{n}}(z) + |z|/(k(\tilde{n}) + 1) \ge \sup_{1\le n\le n(s-2)} (H_n(z) + |z|/((k(n) + 1)) -$$

$$|z|/(k(n(s-2))(k(n(s-2)) + 1))) \ge L(z) - |z|/(s-2) - \varepsilon_s|z|,$$

where

$$\varepsilon_s := 1/(k(n(s-2))(k(n(s-2)) + 1)) \to 0 \text{ as } s \to \infty.$$

From the inequality

$$H_{\tilde{n}}(z) + |z|/k(\tilde{n}) - \alpha_{\tilde{n}} \ge H_{n(s-2)}(z) + |z|/k(n(s-2)) - \alpha_{n(s-2)},$$

it follows that

$$0 \le \alpha_{\tilde{n}} \le H_{\tilde{n}}(z) - H_{n(s-2)}(z) + (1/k(\tilde{n}) + 1/k(n(s-2)))|z| +$$

$$\alpha_{n(s-2)} \le H_{n(s)}(z) - H_{n(s-2)}(z) + 2|z|/k(n(s-2)) + r(s-1)/(s-1) \le$$

$$(\delta_s + 2/k(s-2) + 1/(s-1))|z|,$$

where, since Q is bounded,

$$\delta_s := \sup_{|t|=1}(H_{n(s)}(t) - H_{n(s-2)}(t)) \to 0 \text{ as } s \to \infty.$$

We define in this case

$$\gamma_s(z) := (\delta_s + 2/k(n(s-2)) + 1/(s-1) + 1/(s-2) + \varepsilon_s)|z|.$$

Then $h(z) \geq L(z) - \gamma_s(z)$. If

$$\gamma(z) := \begin{cases} \sup_{|z|\leq r(1)}(h(z) - L(z)), & |z| \leq r(1), \\ \gamma_s(z), & r(s-1) < |z| \leq r(s),\ s \geq 2, \end{cases}$$

we have $\gamma(z) = o(|z|)$ as $|z| \to \infty$. Consequently, $\kappa(z) := \exp(-L(z) + \gamma^*(z))$, $z \in \mathbb{C}^N$, where γ^* is the upper semicontinuous regularization of γ, belongs to $\overline{V}(G)$ and $\overline{v} \leq \kappa$. The lemma is proved.

For the definition of a carrier of an analytic functional we refer to Hörmander [13]. The definition and properties of the conjugate diagram of an entire function of exponential type in one variable can be seen in [14, Chapter I, 20].

Proposition 5. *Let Q be a bounded convex locally closed set in $\mathbb{C}^N$. An entire function f on $\mathbb{C}^N$ belongs to the projective hull $H\overline{V}(\mathbb{C}^N)$ if and only if f is of exponential type and, for each bounded open convex neighbourhood G of Q, some carrier of the analytic functional $\mathcal{F}^{-1}(f)$ is contained in G.*

In particular, for $N = 1$ an entire function f belongs to $H\overline{V}(\mathbb{C}^N)$ if and only if the conjugate diagram of f is contained in each bounded convex open neighbourhood G of Q.

Proof. By Lemma 4, $f \in H\overline{V}(\mathbb{C}^N)$ if and only if $f \in H(\overline{V}(G))(\mathbb{C}^N)$ for all bounded convex open neighbourhoods G of Q. By [13, 4.7.3] and Lemma 3, $f \in V_G H(\mathbb{C}^N)$ if and only if a carrier of $\mathcal{F}^{-1}(f)$ is contained in G. The rest of the assertion follows from the properties of the conjugate diagram [14].

Theorem 6. *(i) If a convex locally closed set Q in $\mathbb{C}^N$ is bounded and strictly convex at $\partial_r\omega$, then the weighted inductive limit $VH(\mathbb{C}^N)$ coincides with its projective hull $H\overline{V}(\mathbb{C}^N)$ algebraically and topologically.*
(ii) Suppose that a convex locally closed set Q in $\mathbb{C}^N$ is bounded, nonpluripolar, and that it has a neighborhood basis of domains of holomorphy. If $VH(\mathbb{C}^N)$ is a topological subspace of $H\overline{V}(\mathbb{C}^N)$, then Q is strictly convex at $\partial_r\omega$.

Proof. (i) Since the hypotheses on Q and Proposition 2 imply that Q has a neighbourhood basis of bounded convex open sets G, it follows that $H(Q) = \mathrm{ind}_G H(G)$ as G runs over the bounded convex open neighbourhoods of Q. Since every bounded set in $H(Q)$ is bounded in $H(G)$ for some G as above, $H(Q)'_b = \mathrm{proj}_G H(G)'_b$ holds algebraically and topologically. Applying the Laplace transform, we have that $VH(\mathbb{C}^N) = \mathrm{proj}_G V_G H(\mathbb{C}^N)$ holds algebraically and topologically. By Lemma 4, we conclude the algebraic and topological identity $VH(\mathbb{C}^N) = H\overline{V}(\mathbb{C}^N)$.

(ii) Since $VH(\mathbb{C}^N)$ contains the polynomials, it follows from Taylor [19, Theorem 4] that the space $VH(\mathbb{C}^N)$ is dense in $V_G H(\mathbb{C}^N)$ for each bounded convex open neighbourhood G of Q. Therefore $VH(\mathbb{C}^N)$ is dense in $H\overline{V}(\mathbb{C}^N)$ by Lemma 4. This yields the algebraic equality $VH(\mathbb{C}^N)' = (\mathrm{proj}_G V_G H(\mathbb{C}^N))'$. Next fix an open neighbourhood Ω of Q and note that the hypothesis implies that we may assume that Ω is a domain of holomorphy. Hence there exists a function $h \in H(\Omega)$ which cannot be continued analytically beyond Ω (see Hörmander [13, Theorem 2.5]). Since h

can be considered as an element $\tilde{h}$ of $H(Q)$, there is an element $\hat{h}$ in $VH(\mathbb{C}^N)'$ so that $\mathcal{F}^t(\hat{h}) = \tilde{h}$, where $\mathcal{F}$ denotes the Laplace transform. From the projective limit representation for $VH(\mathbb{C}^N)'$ above it follows that there exist a bounded convex open neighbourhood G of Q and $\hat{g} \in V_G H(\mathbb{C}^N))'$ which coincides with $\hat{h}$ in $VH(\mathbb{C}^N)'$. Now let $g := \mathcal{F}^t(\hat{g})$. For each $z \in Q$ we have

$$g(z) = \langle \delta_z, g\rangle = \langle \delta_z, \mathcal{F}^t(\hat{g})\rangle = \langle \mathcal{F}(\delta_z), \hat{g}\rangle = \langle \mathcal{F}(\delta_z), \hat{h}\rangle =$$

$$= \langle \delta_z, \mathcal{F}^t(\hat{h})\rangle = \langle \delta_z, \tilde{h}\rangle = \tilde{h}(z) = h(z).$$

Since Q is not pluripolar, for some domain Ω_1 with $Q \subset \Omega_1$ the functions g and h coincide on Ω_1. By the choice of h we have $G \subset \Omega$. Consequently, Q has a neighborhood basis of bounded convex open sets containing Q. By Proposition 2, Q is strictly convex at $\partial_r\omega$, and the proof is complete.

As a consequence of Theorem 6, we obtain the following corollary.

Corollary 7. *Let Q be a nonpluripolar bounded convex subset of $\mathbb{R}^N$ which is locally closed. The following holds:*

(a) The weighted inductive limit $VH(\mathbb{C}^N)$ is a topological subspace of its projective hull $H\overline{V}(\mathbb{C}^N)$ if and only if Q is compact.

(b) If Q is compact, the spaces $VH(\mathbb{C}^N)$ and $H\overline{V}(\mathbb{C}^N)$ coincide also algebraically.

Proof. If Q is compact, then $VH(\mathbb{C}^N)$ is an (LB)-space, and one implication in part (a) and part (b) follow from [5, 1.6]. The other implication in (a) follows from Theorem 6(ii), since the present hypotheses on Q imply that Q is compact if and only if it is strictly convex at $\partial_r\omega$.

Theorem 8. *Let Q be a bounded convex locally closed subset of $\mathbb{C}^N$.*

(i) Assume that the following condition $(*)$ holds:

There is a supporting hyperplane Π to $\overline{Q}$ such that $\Pi \cap Q \neq \emptyset$ and there exists $z_0 \in (\Pi \cap \overline{Q}) \setminus Q$ which is a smooth point of ∂Q,

then $VH(\mathbb{C}^N) \neq H\overline{V}(\mathbb{C}^N)$.

(ii) $VH(\mathbb{C}) \neq H\overline{V}(\mathbb{C})$ if and only if the condition $(*)$ holds.

Proof. (i): There is $z_1 \in Q$ such that the interval $[z_0, z_1]$ is contained in $\Pi \cap \overline{Q}$. We assume that there exists a convex open neighborhood G of Q with $z_0 \notin G$. As $z_0 \in \overline{Q}$ and $Q \subset G$ we have that $z_0 \in \overline{Q} \cap \partial G$. If Π_1 a supporting hyperplane to $\overline{G}$ at z_0, then $\Pi_1 \neq \Pi$ because $z_1 \in G$ and hence $z_1 \notin \Pi_1$. Moreover Π_1 is a supporting hyperplane to $\overline{Q}$ at z_0. Since z_0 is a smooth point of ∂Q this is a contradiction. Hence $z_0 \in G$ for any convex open neighborhood G of Q. The function $f(z) := \exp\langle z_0, z\rangle$, $z \in \mathbb{C}^N$, belongs to $H\overline{V}(\mathbb{C}^N) = \cap_G V_G H(\mathbb{C}^N)$ but does not belong to $VH(\mathbb{C}^N)$.

(ii): We must prove the implication "$VH(\mathbb{C}) \neq H\overline{V}(\mathbb{C}) \Rightarrow (*)$". From $VH(\mathbb{C}) \neq H\overline{V}(\mathbb{C})$ it follows that there is $f \in \cap_G V_G H(\mathbb{C})$ such that $f \notin VH(\mathbb{C})$. Let K be the conjugate diagram of f. Since the compact set K is not contained in Q_n for each n it is not contained in Q. By Proposition 5, for each convex open neighborhood G of Q the set K is contained in G. We choose $z_0 \in K \setminus Q$. Since $\cap_G G \subset \overline{Q}$ the point

z_0 belongs to $\overline{Q}$. Hence $z_0 \in (K \cap \partial Q) \setminus Q$. Let l be a supporting line to $\overline{Q}$ at z_0. If $l \cap Q = \emptyset$ the set Q is contained in an open half-plane G with the boundary l. In this case $z_0 \notin G$ and we have a contradiction to $K \subset G$. Hence the set $l \cap Q$ is not empty. If z_0 is a corner point of ∂Q there is a supporting line l_1 to $\overline{Q}$ at z_0 such that $l_1 \cap Q = \emptyset$. This contradicts the inclusion $K \subset G$. Thus z_0 is a smooth point of ∂Q.

Using different methods, we deal with the algebraic identity in the case of open (not necessarily bounded) intervals in the real line in [7]. We mention the following result for purposes of comparison.

Proposition 9. *Let Q be a convex open subset of $\mathbb{R}^N$. The weighted (LF)-space $VH(\mathbb{C}^N)$ which is isomorphic to the space $H(Q)'_b$ coincides algebraically with its projective hull $H\overline{V}(\mathbb{C}^N)$.*

References

[1] C.A. Berenstein, M.A. Dostal, *Analytically Uniform Spaces and their Applications to Convolution Equations*, Springer Lecture Notes in Math. **256**, Berlin, 1972.

[2] K.D. Bierstedt, A survey of some results and open problems in weighted inductive limits and projective description for spaces of holomorphic functions, *Bull. Soc. Roy. Sci. Liège* **70** (2001) 167-182.

[3] K.D. Bierstedt, J. Bonet, Weighted (LF)-spaces of continuous functions, *Math. Nachr.* **165** (1994) 25-48.

[4] K.D. Bierstedt, R. Meise, Induktive Limites gewichteter Räume stetiger und holomorpher Funktionen, *J. reine angew. Math.* **282** (1976) 186-220.

[5] K.D. Bierstedt, R. Meise, W.H. Summers, A projective description of weighted inductive limits, *Trans. Amer. Math. Soc.* **272** (1982) 107-160.

[6] J. Bonet, R. Meise, Ultradistributions of Roumieu type and projective descriptions, *J. Math. Anal. Appl.* **255** (2001) 122-136.

[7] J. Bonet, R. Meise, S.N. Melikhov, Projective representations of spaces of quasianalytic functionals, Preprint 2003.

[8] J. Bonet, S.N. Melikhov, Interpolation of entire functions and projective descriptions, *J. Math. Anal. Appl.* **205** (1997) 454-460.

[9] J. Bonet, J. Taskinen, The subspace problem for weighted inductive limits of spaces of holomorphic functions, *Michigan Math. J.* **42** (1995) 259-268.

[10] J. Bonet, D. Vogt, On the topological description of weighted inductive limits of spaces of holomorphic and harmonic functions, *Archiv Math.* **72** (1999) 360-366.

[11] L. Ehrenpreis, Solution of some problem of division, Part IV. Invertible and elliptic operators, *Amer. J. Math.* **82** (1960) 522-588.

[12] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley-Interscience Publ., New York, 1970.

[13] L. Hörmander. *Notions of Convexity*, Birkhäuser, Basel, 1994.

[14] B. Ya. Levin, *Distribution of Zeros of Entire Functions.* Revised Edition. AMS. Providence, Rhode Island, 1980.

[15] A. Martineau. Sur la topologie des espaces de fonctions holomorphes. *Math. Ann.* **163** (1966) 62–88.

[16] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.

[17] S. N. Melikhov, S. Momm. Analytic solutions of convolution equations on convex sets with an obstacle in the boundary. *Math. Scand.* **86** (2000) 293–319.

[18] V.V. Napalkov, On discrete weakly sufficient sets in certain spaces of entire functions, *Math. USSR Izv.* **19** (1982) 349-357.

[19] B.A. Taylor. On weighted polynomial approximation of entire functions. *Pacific J. Math.* **36** (1971) 523–539.

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