Positive solutions for a nonlocal boundary-value problem with vector-valued response

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1 Introduction

We investigate the nonlinear problem:

$$\frac{d}{dt}(k(t)x'(t)) + V_x(t, x(t)) = 0, \text{ a.e. in } [\mathbf{0}, \mathbf{T}] \tag{1.1}$$

$$x(0) = 0,$$

where

H $T > 0$ *is arbitrary,* $V : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ *is Gateaux differentiable in the second variable and measurable in* t *function,* $k : [\mathbf{0}, \mathbf{T}] \to \mathbf{R}^+$, $x = (x_1, ..., x_n)$.

We are looking for solutions of (1.1) being a pair (x, p) of absolutely continuous functions $x, p : [0, T] \longrightarrow \mathbf{R}^n$, $x(0) = 0$ such that

$$\frac{d}{dt}p(t) + V_x(t, x(t)) = 0,$$
$$p(t) = k(t)x'(t)).$$

Of course, if k is an absolutely continuous function, then our solution of (1.1) belongs to $C^{1,+}([0, T], \mathbf{R}^n)$ of continuously differentiable functions x whose derivatives x' are absolutely continuous. In the sequel we will not assume that V_x is superlinear

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or sublinear. In the paper we discuss a general problem with non-local boundary conditions:

$$x'(T) = \int_\eta^T x'(s)dg(s), \tag{1.1a}$$

η is a real number in the open interval $(0,T)$, $g = (g_1, ..., g_n) : [0,T] \longrightarrow \mathbf{R}^n$, g_i, $i = 1, ..., n$, are increasing functions and the integral in (1.1a) is meant in the sense of Riemann-Stieltjes. Precisely speaking (1.1a) is the system of n equations:

$$x_i'(T) = \int_\eta^T x_i'(s)dg_i(s), \text{ for } i = 1, ..., n.$$

It is clear that (1.1) is the Euler - Lagrange equation to the functional

$$J(x) = \int_0^T (-V(t,x(t)) + \frac{k(t)}{2}|x'(t)|^2)dt \tag{1.2}$$

considered on the space A_0 of absolutely continuous functions $x : \mathbf{R} \to \mathbf{R}^n$, $x(0) = 0$. A subspace of A_0 of functions satisfying (1.1a) we shall denote by A_{0b}.

Problem like (1.1), (1,1a) was studied by many authors, mainly, in one dimensional case ($n = 1$). This is widely discussed in [3], where V has the special form: $V_x(t,x) = q(t)f(x)$, for some functions $q : [0,1] \to R$, $f : R \to R$, where q, f are continuous, f is nonnegative for $x > 0$, quiet at infinity and

$$\sup_{x\in[0,v]} f(x) \le \theta v \tag{1.3}$$

for some $v > 0$ and $\theta > 0$. Moreover it is assumed there that $g(\eta+) > 0$. The methods used in [3] are of topological type. They are based on the fixed point theorem in cones due to Krasnosielski. We consider the general case when V satisfies hypothesis (**H**) and a condition analogous to (1.3), so that $V_x(\cdot, x)$ is measurable only and $V_x(t, \cdot)$ is not, in general, quiet at infinity, in consequence, our assumptions are not strong enough to use the above theorem. Moreover in this paper t and x are not separated and $x \in R^n$, $n \ge 1$, in consequence, (1.1) is the system of ODE and we do not assume that $g(\eta+) > 0$. However, we believe that our paper may contribute some new look at this problem. This is because we propose to study (1.1), (1,1a) by duality methods in a way, to some extend, analogous to the methods developed for (1.1) in sublinear cases [5], [6]. Since functional (1.2) is, in general, unbounded in A_0 (especially in superlinear case), therefore it is obvious that we must look for critical points of J of "*minmax*" type. The main difficulties which appear here are: what kind of sets we should choose over which we wish to calculate "*minmax*" of J and then to link this value with critical points of J. Of course, we have the mountain pass theorems, the saddle points theorems, the Morse theory, ... (see e.g. [7], [5]) but because of boundary condition (1.1a) they cannot be applied directly to find critical points of J. It seems to us that variational methods are used to study problem (1.1), (1,1a) for the first time.

Our aim is to find a nonlinear subspace X of A_0 defined by the type of nonlinearity of V. To this effect let us denote by P positive cone in R^n i.e. $P = \{x \in R^n : x_i > 0, i = 1, ..., n\}$ and by $\bar{P} = \{x \in R^n : x_i \ge 0, i = 1, ..., n\}$. We say that $x \ge y$ for $x, y \in R^n$ if $x - y \in \bar{P}$. Let $a := \int_\eta^T \frac{1}{k(t)}dg(t)$ be the vector of integrals $\left[\int_\eta^T \frac{1}{k(t)}dg_i(t)\right]_{i=1,...,n}$ and $\alpha := [(1-a_i)^{-1}]_{i=1,...,n}$. If $b, c \in R^n$ by bc we always mean a vector $[b_i c_i]_{i=1,...,n}$. We set the basic hypothesis we need:

H1 *the function k is absolutely continuous and positive, $V(t,\cdot)$ is convex in some neighborhood of $\bar{P}$ and $V_x(t,\cdot)$ is continuous and nonnegative in P, $t \in [0,T]$, and $\int_0^T V_x(t,0)dt \neq 0$,*

H2 *the functions $g_i : [0,T] \to R$, $i = 1,...n$, are increasing and such that $g(\eta) = 0$,*

H3 $1 - \int_\eta^T \frac{1}{k(t)} dg_i(t) > 0$, for all $i = 1, ..., n$,

H4 *for a given $\theta \in P$, there exists $v \in R^n$, $v \in P$ such that*

$$\int_0^T V_x(t, \beta v)dt \leq \theta v, \tag{1.4}$$

where $\theta v = [\theta_i v_i]_{i=1,...,n}$, $\beta = \int_0^T \frac{1}{k(r)}dr \left(\theta \left(\alpha \int_\eta^T \frac{1}{k(t)}dg(t) + \mathbf{1}\right)\right)$, $\mathbf{1} = [1, ..., 1]$, and there exist l, $l_1 \in L^2([0,T], R)$ such that

$$\left(t \ \to \sup\left\{V(t,x) : x \in \bar{P}, x \leq \beta v\right\}\right) \leq l(t),\ t \in [0,T]$$

$$\left(t \ \to \sup\left\{V_{x_i}(t,x) : x \in \bar{P}, x \leq \beta v\right\}\right) \leq l_1(t),\ t \in [0,T], i = 1, ..., n.$$

Through the paper we shall assume hypotheses (**H**) and (**H1-H4**). We would like to stress that because of (**H1**) each $x_j \to V_{x_i}(t, (x_1, ..., x_j, ...x_n))$, $i = 1, ..., n$, $j = 1, ..., n$, $t \in [0,T]$, is increasing function if $(x_1, ..., x_j, ...x_n)$ lies in $\bar{P}$.

Having the type of nonlinearity of V fixed we are able to define nonlinear subspaces $\overline{X}$, $\tilde{X}$ and X as follows.

Taking into account the structure of the space X we shall study the functional

$$J(x) = \int_0^T (-V(t,x(t)) + \frac{1}{2}k(t)\left|x'(t)\right|^2)dt - < x(T), k(T)x'(T) >$$

on the space X. We shall look for a "min" of J over the set X i.e.

$$\min_{x \in X} J(x).$$

To show that element $\bar{x} \in X$ realizing "min" is a critical point of J we develop a duality theory between J and dual to it J_D, described in the next section, where the functional J_D has the following form

$$J_D(p) = -\int_0^T \frac{1}{2k(t)}\left|p(t)\right|^2 + \int_0^T V^*(t, -p'(t))dt. \tag{1.5}$$

To construct the set X first we put

$$\overline{X} = \{x \in A_{0b} :\ p(t) = k(t)x'(t),\ \ t \in\ [0,T] \text{ belongs to } A,\ x \leq \beta v\}$$

where A is the space of absolutely continuous functions $x : [0,T] \to \mathbf{R}^n$ with $x' \in L^2$. We reduce the space $\overline{X}$ to the set

$$\tilde{X} = \left\{x \in \overline{X} : x \in \bar{P},\ x'(t) \geq 0,\ t \in [0,T]\right\}.$$

Now we construct an operator $\mathbf{A}$ in a way similar as it is done in [3]. Thus we calculate from (1.1)

$$x'(t) = \frac{1}{k(t)} x'(T) + \frac{1}{k(t)} \int_t^T V_x(s, x(s))ds.$$

Taking into account (1.1a) we obtain

$$x'(T) = \alpha \int_\eta^T \frac{1}{k(t)} \int_t^T V_x(s, x(s))dsdg(t) \tag{1.6}$$

where $\int_\eta^T \frac{1}{k(t)} \int_t^T V_x(s, x(s))dsdg(t) := \left[\int_\eta^T \frac{1}{k(t)} \int_t^T V_{x_i}(s, x(s))dsdg_i(t)\right]_{i=1,\dots,n}$.

Then, from (1.1a) we get

$$x(t) = \alpha \int_\eta^T \frac{1}{k(t)} \int_t^T V_x(s, x(s))dsdg(t) \int_0^t \frac{1}{k(s)} ds + \int_0^t \frac{1}{k(r)} \int_r^T V_x(s, x(s))dsdr.$$

Thus the operator $\mathbf{A}$ we define as

$$\mathbf{A}x(t) = \alpha \int_\eta^T \frac{1}{k(t)} \int_t^T V_x(s, x(s))dsdg(t) \int_0^t \frac{1}{k(s)} ds + \int_0^t \frac{1}{k(r)} \int_r^T V_x(s, x(s))dsdr.$$

The set X is defined as a subset of $\tilde{X}$ having the property: $\mathbf{A}X \subset X$ i.e. for each $x \in X$ there exists $w \in X$ such that $w = \mathbf{A}x$.

Lemma 1.1 $\tilde{X}$ has the above property and, in consequence we can take $X = \tilde{X}$.

Proof. We easily see that if $x \in \tilde{X}$ then $\mathbf{A}x(t) \geq 0$ and $(\mathbf{A}x(t))' \geq 0$. We show that $\tilde{X} = X$. Thus to end the proof it is enough to prove that if $x \in \tilde{X}$ then $\mathbf{A}x \leq \beta v$. We have

$$\begin{aligned}
\mathbf{A}x &= \alpha \int_\eta^T \frac{1}{k(t)} \int_t^T V_x(s, x(s))dsdg(t) \int_0^t \frac{1}{k(s)} ds + \int_0^t \frac{1}{k(r)} \int_r^T V_x(s, x(s))dsdr \\
&\leq \alpha \int_\eta^T \frac{1}{k(t)} \int_t^T V_x(s, \beta v)dsdg(t) \int_0^t \frac{1}{k(s)} ds + \int_0^t \frac{1}{k(r)} \int_r^T V_x(s, \beta v)dsdr \\
&\leq \int_0^T \frac{1}{k(r)} dr \left(\theta \left(\alpha \int_\eta^T \frac{1}{k(t)} dg(t) + \mathbf{1}\right)\right) v = \beta v.
\end{aligned}$$

This ends the proof.

It is clear that X depends strongly on the type of nonlinearity of V. We easily see that X is not in general a closed set in A.

As the dual set to X we shall consider the following set

$$X^d = \{p \in A : \text{ there exists } x \in X \text{ such that } p(t) = k(t)x'(t),\ t \in [0, T]\}.$$

Remark 1.1 From the definition of $\mathbf{A}$ and X we derive that for each $x \in X$ there exists $p \in X^d$ such that $p'(\cdot) = -V_x(\cdot, x(\cdot))$ and therefore

$$\int_0^T < -p'(t), x(t) > dt - \int_0^T V^*(t, -p'(t))dt = \int_0^T V(t, x(t))dt.$$

It is clear that $X^d \subset \bar{P}$. Just because of the duality theory we are able to avoid in our proof of an existence of critical points the deformation lemmas, the Ekeland variational principle or PS type conditions. One more advantage of our duality results is obtaining for the first time in the superlinear case a measure of a duality gap between primal and dual functional for approximate solutions to (1.1) (for the sublinear case see [6]).

The main result of our paper is the following:

Theorem (*Main*)

Under hypothesis (H) *and* $(H1)-(H4)$ *there exists a pair* $(\bar{x}, \bar{p})$, $\bar{x} \in X, \bar{p} \in X^d$ *being a solution to (1.1) and such that*

$$J(\bar{x}) = \min_{x \in X} J(x) = \min_{p \in X^d} J_D(p) = J_D(\bar{p}).$$

We see that our hypotheses on V concern only convexity of $V(t, \cdot)$ in $\bar{P}$ and that this function is rather of general type. We do not assume that $V(t, x) \geq 0$.

2 Duality results

To obtain a duality principle we need a kind of perturbation of J and convexity of a function considered on a whole space. To this effect let us define

$$\breve{V}(t, x) = \begin{cases} V(t, x) \text{ if } x \in \bar{P},\ t \in [0, T] \\ \infty \text{ if } x \notin \bar{P},\ t \in [0, T]. \end{cases}$$

As our all investigation reduce to the set X, therefore $\breve{V} = V$ in it. We need this notation only for the purpose of duality and hence we will not change a notation for the functional J containing V or $\breve{V}$. Thus define for each $x \in X$ the perturbation of J as

$$\begin{aligned} J_x(y, a) &= \int_0^T (\breve{V}(t, x(t) + y(t)) - \frac{k(t)}{2} |x'(t)|^2) dt \\ &\quad + < x(T) - a, k(T)x'(T) > \\ &= \int_0^T (\breve{V}(t, x(t) + y(t)) - \frac{k(t)}{2} |x'(t)|^2) dt \\ &\quad + < x(T), k(T)x'(T) > - < a, k(T)x'(T) > \end{aligned}$$

for $y \in L^2$, $a \in R^n$. Of course, $J_x(0) = -J(x)$. For $x \in X$ and $p \in X^d$, we define a type of conjugate of J by

$$\begin{aligned} J_x^{\#}(p) &= \sup_{y \in L^2, a \in R^n} \left(\int_0^T < y(t), p'(t) > dt + < p(T), a > - J_x(y, a) \right) \qquad (2.1) \\ &= \sup_{y \in L^2} \{ \int_0^T < y(t), p'(t) > dt - \int_0^T \breve{V}(t, x(t) + y(t)) dt \} \\ &\quad + \sup_{a \in R^n} \{ < a, p(T) > + < a, k(T)x'(T) > \} \\ &\quad + \int_0^T \frac{k(t)}{2} |x'(t)|^2 dt - < x(T), k(T)x'(T) > . \end{aligned}$$

By a direct calculation we obtain

$$J_x^{\#}(p) = -\int_0^T < x(t), p'(t) > dt + \frac{1}{2}\int_0^T k(t)\left|x'(t)\right|^2 dt \tag{2.2}$$

$$+\int_0^T V^*(t,p'(t))dt - < x(T), k(T)x'(T) > + l(k(T)x'(T)+p(T)) =$$

$$\begin{aligned} = & \int_0^T < x'(t), p(t) > dt - < x(T), p(T) > \\ & + \frac{1}{2}\int_0^T k(t)\left|x'(t)\right|^2 dt + \int_0^T V^*(t,p'(t))dt + \\ - < & \; x(T), k(T)x'(T) > + l(k(T)x'(T)+p(T)), \end{aligned}$$

where $l : R^n \to \{0,+\infty\}$

$$l(b) = \begin{cases} 0, & for \quad b = 0 \\ +\infty, & for \quad b \neq 0 \end{cases}.$$

Now we take "min" from $J_x^{\#}(p)$ with respect to $x \in X$ and calculate it. Because X is not a linear space we need some trick to avoid calculation of the conjugate with respect to a nonlinear space. To this effect we use the special structure of the set X^d. First we observe that for each $p \in X^d$ there exists $x_p \in X$ such that

$$p(t) = k(t)x_p'(t) \text{ for all } t \in [0,T].$$

Therefore:

$$\int_0^T < x_p'(t), p(t) > dt - \int_0^T \frac{k(t)}{2}\left|x_p'(t)\right|^2 dt = \int_0^T \frac{1}{2k(t)}\left|p(t)\right|^2 dt.$$

Next let us note that, on the other hand

$$\int_0^T < x_p'(t), p(t) > dt - \int_0^T \frac{k(t)}{2}\left|x_p'(t)\right|^2 dt$$

$$\leq \sup_{x\in\{z\in X, p(T)=k(T)z'(T)\}} \left\{\int_0^T < x'(t), p(t) > dt - \int_0^T \frac{k(t)}{2}\left|x(t)'\right|^2 dt\right\}$$

$$\leq \sup_{x\in X} \left\{\int_0^T < x'(t), p(t) > dt - \int_0^T \frac{k(t)}{2}\left|x(t)'\right|^2 dt\right\}$$

$$\leq \sup_{x'\in L^2} \left\{\int_0^T < x'(t), p(t) > dt - \int_0^T \frac{k(t)}{2}\left|x'(t)\right|^2 dt\right\} = \int_0^T \frac{1}{2k(t)}\left|p(t)\right|^2 dt$$

and actually all inequalities above are equalities. Therefore we can calculate for $p \in X^d$

$$\sup_{x\in X}\left(-J_x^{\#}(-p)\right) = \sup_{x\in X}\left\{\int_0^T < x'(t), p(t) > dt - \frac{1}{2}\int_0^T k(t)\left|x'(t)\right|^2 dt\right.$$

$$-\int_0^T V^*(t,-p'(t))dt - < x(T), p(T) > +$$

$$< x(T), k(T)x'(T) > -l(k(T)x'(T) - p(T))\}$$

so that

$$\sup_{x \in X} \left(-J_x^{\#}(-p)\right) = -\int_0^T V^*(t, -p'(t))dt + \int_0^T \frac{1}{2k(t)} |p(t)|^2 \, dt = -J_D(p). \tag{2.3}$$

From (2.3) we infer for $p \in X^d$ that

$$\sup_{x \in X} \left(-J_x^{\#}(-p)\right) = -J_D(p), \tag{2.4}$$

where J_D is given by (1.5).

Now we claim that for all $x \in X$

$$\sup_{p \in X^d} \left(-J_x^{\#}(-p)\right) \leq -J(x). \tag{2.5}$$

To prove the above assertion, fix $x \in X$. Then Remark 1.1 yields the existence of $\bar{p} \in X^d$ such that

$$\int_0^T < -\bar{p}'(t), x(t) > dt - \int_0^T V^*(t, -\bar{p}'(t))dt = \int_0^T \breve{V}(t, x(t))dt$$

and further

$$\begin{aligned}
&\int_0^T < -\bar{p}'(t), x(t) > dt - \int_0^T V^*(t, -\bar{p}'(t))dt \\
&\quad \leq \sup_{p \in X^d} \{\int_0^T < -p'(t), x(t) > dt - \int_0^T V^*(t, -p'(t))dt\} \\
&\quad = \sup_{p' \in L^2} \{\int_0^T < -p'(t), x(t) > dt - \int_0^T V^*(t, -p'(t))dt\} \\
&\quad = \int_0^T \breve{V}(t, x(t))dt.
\end{aligned}$$

Hence we see that

$$\begin{aligned}
\sup_{p \in X^d} \left(-J_x^{\#}(-p)\right) &= \sup_{p \in X^d} \left\{\int_0^T < x(t), -p'(t) > dt - \int_0^T V^*(t, -p'(t))dt \right. \\
&\quad \left. -l(k(T)x'(T) - p(T))\right\} + < x(T), k(T)x'(T) > \\
&\quad -\frac{1}{2}\int_0^T k(t) |x'(t)|^2 \, dt \leq \sup_{p \in X^d} \{-l(k(T)x'(T) - p(T))\} \\
&\quad + \sup_{p \in X^d} \left\{\int_0^T < x(t), -p'(t) > dt - \int_0^T V^*(t, -p'(t))dt\right\} \\
&\quad + < x(T), k(T)x'(T) > -\frac{1}{2}\int_0^T k(t) |x'(t)|^2 \, dt = -\int_0^T (-\breve{V}(t, x(t)) \\
&\quad + \frac{k(t)}{2} |x'(t)|^2)dt + k(T) < x(T), x'(T) >= -J(x).
\end{aligned} \tag{2.6}$$

which is our claim.

Now we can formulate the following duality principle

Theorem 2.1 *For functionals J and J_D we have the duality relation*

$$\inf_{x \in X} J(x) \leq \inf_{p \in X^d} J_D(p). \tag{2.7}$$

Proof From (2.5) and (2.4) we obtain the below chain of equalities:

$$\begin{aligned}\inf_{x\in X} J(x) &= -\sup_{x\in X}(-J(x)) \le -\sup_{x\in X}\sup_{p\in X^d}\left(-J_x^{\#}(-p)\right) = -\sup_{p\in X^d}\sup_{x\in X}\left(-J_x^{\#}(-p)\right)\\ &= -\sup_{p\in X^d}(-J_D(p)) = \inf_{p\in X^d} J_D(p).\end{aligned}$$

Denote by $\partial J_x(y)$ the subdifferential of J_x. In particular,

$$\partial J_x(0) = \{p' \in L^2 : \int_0^T V^*(t, p'(t))dt + \int_0^T \check{V}(t, x(t))dt = \int_0^T < p'(t), x(t) > dt\} \tag{2.8}$$

The next result formulates a variational principle for "min" arguments.

Theorem 2.2 *Let $\bar{x} \in X$ be such that $J(\bar{x}) = \inf_{x\in X} J(x)$. Then there exists $\bar{p} \in X^d$ with $\bar{p}(t) = \bar{p}(T) - \int_t^T \bar{p}'(s)ds$, where $\bar{p}(T) = k(T)\bar{x}'(T)$* and $-\bar{p}' \in \partial J_{\bar{x}}(0)$, *such that $\bar{p}$ satisfies*

$$J_D(\bar{p}) = \inf_{p\in X^d} J_D(p).$$

Furthermore

$$J_{\overline{x}}(0) + J_{\bar{x}}^{\#}(-\bar{p}) = 0 \tag{2.9}$$

$$J_D(\bar{p}) - J_{\bar{x}}^{\#}(-\bar{p}) = 0. \tag{2.10}$$

Proof By Theorem 2.1 to prove the first assertion it suffices to show that $J(\bar{x}) \ge J_D(\bar{p})$. Using Remark 1.1 we can observe that for $\bar{x}$ there exists $\bar{p} \in X^d$ such that $\bar{p}'(t) = -V_x(t, \overline{x}(t))$ a.e. on $[0,T]$, which implies

$$\int_0^T < -\bar{p}'(t), \overline{x}(t) > dt - \int_0^T V^*(t, -\bar{p}'(t))dt = \int_0^T \check{V}(t, \overline{x}(t))dt. \tag{2.11}$$

Combining (2.11) and (2.8) we get the inclusion $-\bar{p}' \in \partial J_{\bar{x}}(0)$. Thus $J_{\bar{x}}^*(-\bar{p}') = J_{\bar{x}}^{\#}(-\bar{p})$ (where $J_{\bar{x}}^*(-\bar{p}')$ denotes the Fenchel transform of $J_{\bar{x}}$ at $-\bar{p}'$) gives (2.9). It follows that

$$-J(\bar{x}) = -J_{\bar{x}}^{\#}(-\bar{p}) \le \sup_{x\in X}(-J_x^{\#}(-\bar{p})) = -J_D(\bar{p}),$$

where the last equality is due to (2.4). Hence $J(\bar{x}) \ge J_D(\bar{p})$ and so $J_D(\bar{p}) = J(\bar{x}) = \inf_{x\in X} J(x) = \inf_{p\in X^d} J_D(p)$. (2.10) is a simply consequence of (2.9) and the chain of equalities $J_{\bar{x}}(0) = -J(\bar{x}) = -J_D(\overline{p})$.

From equalities (2.9), (2.10) we are able to derive a dual to (1.1) Euler - Lagrange equations.

Corollary 2.1 *Let $\bar{x} \in X$ be such that $J(\bar{x}) = \inf_{x\in X} J(x)$.* Then there exists $\bar{p} \in X^d$ such that the pair $(\bar{x}, \bar{p})$ satisfies the relations

$$-\bar{p}'(t) = V_x(t, \bar{x}(t)), \tag{2.12}$$

$$\bar{p}(t) = k(t)\bar{x}'(t), \tag{2.13}$$

$$J_D(\bar{p}) = \inf_{p\in X^d} J_D(p) = \inf_{x\in X} J(x) = J(\bar{x}). \tag{2.14}$$

Proof. Equalities (2.9) and (2.10) imply

$$\int_0^T V(t,\bar{x}(t))dt + \int_0^T V^*(t,-\bar{p}'(t))dt - \int_0^T < \bar{x}(t), -\bar{p}'(t) > dt = 0,$$

$$\int_0^T \frac{1}{2k(t)} |\bar{p}(t)|^2 \, dt + \int_0^T \frac{k(t)}{2} |\bar{x}'(t)|^2 \, dt - \int_0^T < \bar{x}'(t), \bar{p}(t) > dt = 0,$$

and then (2.12), (2.13). Relations (2.14) are a direct consequence of Theorem 2.1 and Theorem 2.2.

As a direct consequence of the above corollary and definition of X^d we have

Corollary 2.2 *By the same assumptions as in Corollary 2.1 there exists a pair* $(\bar{x}, \bar{p}) \in X \times X^d$ *satisfying relations (2.14), and the pair* $(\bar{x},\ \bar{p})$ *is a solution to (1.1). Conversely, each pair* $(\bar{x}, \bar{p})$ *satisfying relations (2.14) satisfies also equations (2.12), (2.13).*

3 Variational principles and a duality gap for minimizing sequences

In this section we show that a statement similar to Theorem 2.2 is true for a minimizing sequence of J.

Theorem 3.1 *Let* $\{x_j\}$, $x_j \in X$, $j = 1, 2, ...,$ *be a minimizing sequence for* J *and let*

$$+\infty > \inf_{j \in N} J(x_j) = a > -\infty.$$

Then there exist $p_j \in X^d$ *with* $-p_j' \in \partial J_{x_j}(0)$ *such that* $\{p_j\}$ *is a minimizing sequence for* J_D *i.e.*

$$\inf_{x \in X} J(x) = \inf_{j \in N} J(x_j) = \inf_{j \in N} J_D(p_j) = \inf_{p \in X^d} J_D(p). \tag{3.1}$$

Furthermore

$$J_{x_j}(0) + J^{\#}_{x_j}(-p_j) = 0,$$

$$J_D(p_j) - J^{\#}_{x_j}(-p_j) \le \varepsilon, \tag{3.2}$$

$$0 \le J(x_j) - J_D(p_j) \le \varepsilon \tag{3.3}$$

for a given $\varepsilon > 0$ *and sufficiently large* j.

Proof. We have that $\infty > \inf_{j \in N} J(x_j) = a > -\infty$, and therefore for a given $\varepsilon > 0$ there exists j_0 such that $J(x_j) - a < \varepsilon$, for all $j \ge j_0$. Further, the proof is similar to that of Theorem 2.2, so we only sketch it. First we observe that there exists $p_j \in X^d$ such that $p_j'(t) = -V_x(t, x_j(t))$ a.e. on $[0, T]$, which implies $-p_j' \in \partial J_{x_j}(0)$. Therefore, we also have for all $j \in N$

$$\int_0^T \breve{V}(t, x_j(t))dt = \int_0^T -V^*(t, -p_j'(t))dt - \int_0^T < x_j(t), p_j'(t) > dt.$$

Hence, we get inclusion $-p_j' \in \partial J_{x_j}(0)$ and the inequality

$$-J(x_j) \le -J_D(p_j)$$

which together with Theorem 2.1 implies (3.1). Again by Theorem 2.1

$$J_D(p_j) + \varepsilon \geq J(x_j) \quad \text{for} \quad j \geq j_0,$$

which implies (3.3).

Since $-p_j' \in \partial J_{x_j}(0)$ we infer $J_{x_j}(0) + J_{x_j}^{\#}(-p_j) = 0$ for all $j \in N$. (3.2) follows from two facts: $J_{x_j}(0) = -J(x_j) = -J_{x_j}^{\#}(-p_j)$ and $\inf_{x \in X} J(x) = \inf_{j \in N} J_D(p_j) = a$.

A direct consequence of this theorem is the following corollary.

Corollary 3.1. *Let* $\{x_j\}$, $x_j \in X$, $j = 1, 2, ...$, *be a minimizing sequence for* J *and let*

$$+\infty > \inf_{j \in N} J(x_j) = a > -\infty.$$

If

$$-p_j'(t) = V_x(t, x_j(t))$$

then $p_j(t) = p_j(T) - \int_t^T p_j'(s)ds$, $p_j(T) = k(T)x_j'(T)$, *belongs to* X^d, *and* $\{p_j\}$ *is a minimizing sequence for* J_D *i.e.*

$$\inf_{x \in X} J(x) = \inf_{j \in N} J(x_j) = \inf_{j \in N} J_D(p_j) = \inf_{p \in X^d} J_D(p).$$

Furthermore

$$\begin{aligned} &J_D(p_j) - J_{x_j}^{\#}(-p_j) \leq \varepsilon, \\ &0 \leq J(x_j) - J_D(p_j) \leq \varepsilon \end{aligned} \tag{3.4}$$

for a given $\varepsilon > 0$ *and sufficiently large* j.

4 The existence of a minimum of *J*

The last problem which we have to solve is to prove the existence of $\bar{x} \in X$ such that

$$J(\bar{x}) = \min_{x \in X} J(x).$$

To obtain this it is enough to use hypotheses **(H)** and **(H1)-(H4),** the results of the former section and known compactness theorems.

Theorem 4.1 *Under hypotheses* (H) and $(H1) - (H4)$ *there exists* $\bar{x} \in X$ *such that* $J(\bar{x}) = \min_{x \in X} J(x)$.

Proof. Let us observe that by **(H4),** $J(x)$ is bounded below on X:

$$J(x) \geq \int_0^T \frac{k(t)}{2} |x'(t)|^2 dt - \int_0^T l(t)dt - k(T)|v||\beta||\alpha| \left| \int_\eta^T \frac{1}{k(t)} \int_t^T l_1(s)dsdg(t) \right|. \tag{4.1}$$

From (4.1) we infer the boundedness below of J on X as well as that the sets $S_b = \{x \in X, \ J(x) \leq b\}$, $b \in \mathbf{R}$ are nonempty for sufficiently large b and bounded with respect to the norm $\| \ x' \ \|_{L^2}$. The last means that S_b, $b \in \mathbf{R}$ are relatively weakly compact in A_{0b}. It is a well known fact that the functional J is weakly lower semicontinuous in A_{0b} and thus also in X. Therefore there exists a sequence $\{x_n\}$, $x_n \in X$, such that $x_n \rightharpoonup \bar{x}$ weakly in A_{0b} with $\bar{x} \in A_{0b}$ (we use the fact that $\{x_n\}$ is uniformly convergent to $\bar{x}$ and formula (1.6)) and $\liminf_{n\to\infty} J(x_n) \geq J(\bar{x})$. Moreover, the uniform convergence of $\{x_n\}$ to $\bar{x}$, implies that $\overline{x} \leq \beta v$. In order to finish the proof we must only show that $\bar{x} \in X$.

To proof that we apply the duality results of Section 3. To this effect let us recall from Corollary 3.1 that for

$$p_n'(t) = -V_x(t, x_n(t)),\ t \in [0, T] \tag{4.2}$$

$p_n(t) = p_n(T) - \int_t^T p_n'(s)ds$, where $p_n(T) = k(T)x_n'(T)$, belongs to X^d. Then $\{p_n\}$ is a minimizing sequence for J_D. We easily check that $\{p_n(T)\}$ is a bounded sequence and therefore we may assume (up to a subsequence) that it is convergent. From (4.2) we infer that $\{p_n'\}$ is a bounded sequence in L^2 norm and that it is pointwise convergent to

$$\bar{p}'(t) = -V_x(t, \bar{x}(t))$$

and so $\{p_n\}$ is uniformly convergent to $\bar{p}$ where $\bar{p}(t) = \bar{p}(T) - \int_t^T \bar{p}'(s)ds$.

By Corollary 3.1 (see (3.4)) we also have (taking into account (4.2)) that for $\varepsilon_n \to 0$ $(n \to \infty)$

$$0 \leq \int_0^T (\frac{1}{2k(t)} |p_n(t)|^2 + \frac{k(t)}{2} |x_n'(t)|^2)dt - \int_0^T < x_n'(t), p_n(t) > dt \quad \leq \varepsilon_n$$

and so, taking a limit

$$0 = \int_0^T \frac{1}{2k(t)} |\bar{p}(t)|^2 dt + \lim_{n\to\infty} \int_0^T \frac{k(t)}{2} |x_n'(t)|^2 dt - \int_0^T < \bar{x}'(t), \bar{p}(t) > dt$$

and next, in view of the property of Fenchel inequality,

$$0 = \int_0^T \frac{1}{2k(t)} |\bar{p}(t)|^2 dt + \int_0^T \frac{k(t)}{2} |\bar{x}'(t)|^2 dt - \int_0^T < \bar{x}'(t), \bar{p}(t) > dt$$

and further

$$\bar{p}(t) = k(t)\bar{x}'(t).$$

Thus, as $\bar{x} \in A_{0b}$ and $\overline{x} \leq \beta v$, $\bar{x} \in X$ and so the proof is completed.

A direct consequence of Theorem 4.1 and Corollary 2.2 is the following main theorem.

Theorem 4.2 *Under hypothesis* (H) *and* $(H1) - (H4)$ *there exists a pair* $(\bar{x}, \bar{p})$ *being a solution to (1.1), (1.1a) and such that*

$$J(\bar{x}) = \min_{x\in X} J(x) = \min_{p\in X^d} J_D(p) = J_D(\bar{p}).$$

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