Invariant locally φ-symmetric contact structures on Lie groups

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Abstract

We are interested in the question whether every strongly locally φ-symmetric contact metric space is a (κ, μ)-space. In this paper, we show that the answer is positive for left-invariant contact metric structures on Lie groups.

1 Introduction

In K-contact and Sasakian geometry, local symmetry is a very strong condition: a locally K-contact manifold is necessarily a space of constant curvature equal to 1 ([12], [15]). The appropriate notion to consider in the Sasakian context was introduced by T. Takahashi in [14]: he calls a Sasakian space *(locally) φ-symmetric* if its Riemann curvature tensor R satisfies the condition

$$g((\nabla_X R)(Y, Z)V, W) = 0 \tag{1}$$

for all vector fields X, Y, Z, V and W orthogonal to the characteristic vector field ξ, where ∇ denotes the Levi Civita connection. He shows that this corresponds geometrically to the fact that the characteristic reflections (i.e., reflections with respect to the integral curves of ξ) are local automorphisms of the Sasakian structure. In fact, it is already sufficient that the reflections are local isometries ([4]). In the broader context of K-contact geometry, it was proved in [9] that a K-contact manifold admitting locally isometric characteristic reflections is necessarily a (locally) φ-symmetric Sasakian space.

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At least two generalizations of the notion of local φ-symmetry to the class of contact metric spaces have appeared in the literature. The first one, in [3], defines a locally φ-symmetric contact metric space to be one for which the curvature property (1) holds. It is as yet unclear whether there is a geometric reality corresponding to (1) in the contact metric setting. A second generalization was proposed by the author and L. Vanhecke in [8]: they define a contact metric space to be locally φ-symmetric if its characteristic reflections are local isometries. This gives rise to an infinite number of curvature restrictions (see further), including (1). Hence, this second generalization is a priori more restrictive than the first. To distinguish between the two, we speak about *weak local φ-symmetry* (for the first one) and *strong local φ-symmetry* (for the second). That the two classes do not agree was shown explicitly in [7]: there, left-invariant contact metric structures on three-dimensional non-unimodular Lie groups were constructed which are weakly, but not strongly locally φ-symmetric. Quite recently, D. Perrone has presented another three-dimensional contact metric space with this property, but which is moreover not locally homogeneous ([13]).

The first examples of strongly locally φ-symmetric contact metric spaces (which are not Sasakian) were found in [8]: these are the unit tangent sphere bundles of spaces of constant curvature c, $c \neq 1$, equipped with their natural contact metric structure. Later, this family of examples was extended further to include all (non)-Sasakian contact metric (κ, μ)*-spaces.* These are contact metric manifolds for which the Riemann curvature tensor R satisfies

$$R(X,Y)\xi = \kappa\Big(\eta(Y)X - \eta(X)Y\Big) + \mu\Big(\eta(Y)hX - \eta(X)hY\Big) \qquad (2)$$

for some real numbers κ and μ and for all vector fields X and Y. Here h denotes, up to a scaling factor, the Lie derivative of the structure tensor φ in the direction of ξ. For convenience, we will call such contact metric spaces (κ, μ)*-spaces.* Note that Sasakian spaces also satisfy (2) ($\kappa = 1$ and $h = 0$). The class of (κ, μ)-spaces was introduced in [2], and there it was shown that the only unit tangent sphere bundles with this curvature property are precisely those of spaces of constant curvature c (with $\kappa = c(2-c)$ and $\mu = -2c$). That the (non-Sasakian) (κ, μ)-spaces are strongly locally φ-symmetric was shown by the present author in [5], and also that they are all locally homogeneous. Finally, a full local classification of (κ, μ)-spaces was realized in [6]. In this classification, a prominent role is played by, on the one hand, the unit tangent sphere bundles of spaces of constant curvature and, on the other hand, special Lie groups equipped with specific left-invariant contact metric spaces.

Apart from the (non-Sasakian) contact metric (κ, μ)-spaces, not a single example is known of a non-Sasakian strongly locally φ-symmetric space. This raises the question whether there actually exist any. In dimension three, the answer is known: G. Calvaruso, D. Perrone and L. Vanhecke have shown in [10] that the two classes agree. In higher dimensions, the question is wide open.

In this paper, we make the first contribution to this problem. Inspired by the role played by Lie groups in the classification of (κ, μ)-spaces, we investigate whether there exist left-invariant contact metric structures on Lie groups which are strongly locally φ-symmetric without being (κ, μ). The invariance allows to reduce the study to one on the Lie algebra. Still, the calculations to be made are enormous. Only after detailed and explicit computations for the five- and seven-dimensional cases did we acquire enough insight to treat the most general situation. We prove

Main Theorem. *Every left-invariant contact metric structure on a Lie group which is strongly locally φ-symmetric is a (κ, μ)-contact metric structure.*

2 Strongly locally φ-symmetric contact metric spaces

In this section we collect the formulas and results we need on contact metric manifolds. We refer to [1] for a more detailed treatment. All manifolds in this note are assumed to be connected and smooth.

An odd-dimensional differentiable manifold M^{2n+1} has an *almost contact structure* if it admits a vector field ξ, a one-form η and a $(1,1)$-tensor field φ satisfying

$$\eta(\xi) = 1 \quad \text{and} \quad \varphi^2 = -\mathrm{id} + \eta \otimes \xi.$$

In that case, one can always find a compatible Riemannian metric g, i.e., such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y on M. $(M, \xi, \eta, \varphi, g)$ is an *almost contact metric manifold.* If the additional property $d\eta(X, Y) = g(X, \varphi Y)$ holds, then $(M, \xi, \eta, \varphi, g)$ is called a *contact metric manifold.* As a consequence, the characteristic curves (i.e., the integral curves of the characteristic vector field ξ) are geodesics.

On a contact metric manifold M, we define the $(1,1)$-tensor h by

$$hX = \frac{1}{2}\,(\mathcal{L}_\xi \varphi)(X)$$

where $\mathcal{L}_\xi$ denotes Lie differentiation in the direction of ξ. The tensor h is self-adjoint, $h\xi = 0$, $\operatorname{tr} h = 0$ and $h\varphi = -\varphi h$. The covariant derivative of ξ is given explicitly by

$$\nabla_X \xi = -\varphi X - \varphi h X. \tag{3}$$

If the vector field ξ on a contact metric manifold $(M, \xi, \eta, \varphi, g)$ is a Killing vector field, then the manifold is called a *K-contact manifold.* This is the case if and only if $h = 0$. Finally, if the Riemann curvature tensor satisfies

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi = \eta(Y)X - \eta(X)Y \tag{4}$$

for all vector fields X and Y on M, then the contact metric manifold is *Sasakian.* In that case, ξ is a Killing vector field, hence every Sasakian manifold is K-contact.

Recall that a contact metric space $(M, \xi, \eta, \varphi, g)$ is called a (strongly) locally φ-symmetric space if the local reflections with respect to the integral curves of ξ are local isometries. This geometric property is reflected in an infinite list of curvature conditions (see also [11]):

Proposition. *Let $(M, \xi, \eta, \varphi, g)$ be a contact metric manifold. If it is a (strongly) locally φ-symmetric space, then the following infinite list of curvature conditions hold:*

$$g((\nabla^{2k}_{X\cdots X} R)(X, Y)X, \xi) = 0, \tag{5}$$

$$g((\nabla^{2k+1}_{X\cdots X} R)(X, Y)X, Z) = 0, \tag{6}$$

$$g((\nabla^{2k+1}_{X\cdots X} R)(X, \xi)X, \xi) = 0, \tag{7}$$

for all vectors X, Y and Z orthogonal to ξ and $k = 0, 1, 2, \dots$. Moreover, if (M, g) is analytic, these conditions are also sufficient for the contact metric manifold to be a locally φ-symmetric space.

Note that (6) for $k = 0$ is precisely the condition (1), implying that any strongly locally φ-symmetric space is also weakly locally φ-symmetric.

3 Left-invariant contact structures on Lie groups

Let G^{2n+1} be a Lie group equipped with a left-invariant contact metric structure (ξ, η, φ, g). By invariance, this structure is completely determined if we know it at the identity element e of G. Conversely, if we take an appropriate structure on the associated Lie algebra $\mathfrak{g} \simeq T_e G$, then we can transport it by left translations to a left-invariant structure on G.

Consider the operator $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ as acting on left-invariant vector fields. Since it is symmetric with respect to the metric g and since $h\varphi = -\varphi h$, we can find an orthonormal basis of $\mathfrak{g}$ of the form $\{\xi, X_1, \dots, X_n, Y_1, \dots, Y_n\}$ satisfying

$$h(X_i) = \lambda_i X_i, \quad h(Y_i) = -\lambda_i Y_i, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i$$

for $i = 1, \dots, n$ and such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$.

In the rest of this section, we express the Lie bracket on $\mathfrak{g}$ in terms of this specific basis. We start from the general situation

$$\begin{aligned}
[\xi, X_i] &= \sum_{k=1}^{n} (\alpha_{ik} X_k + \beta_{ik} Y_k) + x_i \xi, \\
[\xi, Y_i] &= \sum_{k=1}^{n} (\gamma_{ik} X_k + \epsilon_{ik} Y_k) + y_i \xi, \\
[X_i, X_j] &= \sum_{k=1}^{n} (A_{ij}^k X_k + B_{ij}^k Y_k) + x_{ij} \xi, \\
[Y_i, Y_j] &= \sum_{k=1}^{n} (C_{ij}^k X_k + D_{ij}^k Y_k) + y_{ij} \xi, \\
[X_i, Y_j] &= \sum_{k=1}^{n} (a_{ij}^k X_k + b_{ij}^k Y_k) + z_{ij} \xi
\end{aligned} \tag{8}$$

where A_{ij}^k,B_{ij}^k, C_{ij}^k, D_{ij}^k, x_{ij} and y_{ij} are skew-symmetric in i and j. We now express some of the necessary conditions on the coefficients of this bracket in order to obtain a contact metric structure.

The conditions for an almost contact metric structure pose no restrictions on the coefficients, but the additional one for a contact metric structure does. Indeed, the equality

$$\frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])) = d\eta(X, Y) = g(X, \varphi Y)$$

reduces for the case of left-invariant vector fields X and Y to the condition

$$g([X, Y], \xi) = -2\, g(X, \varphi Y).$$

Using (8), this implies $x_i = y_i = x_{ij} = y_{ij} = 0$ and $z_{ij} = 2\,\delta_{ij}$.

Next, we express the fact that $h(X_i) = \lambda_i X_i$ using (8):

$$\begin{aligned}\lambda_i X_i &= h(X_i) = \frac{1}{2}(\mathcal{L}_\xi\varphi)X_i = \frac{1}{2}([\xi,\varphi X_i] - \varphi[\xi, X_i]) \\ &= \frac{1}{2}\sum_{k=1}^{n}\Big((\gamma_{ik}+\beta_{ik})X_k + (\epsilon_{ik}-\alpha_{ik})Y_k\Big).\end{aligned}$$

Hence,

$$\gamma_{ik}+\beta_{ik} = 2\lambda_i\,\delta_{ik}, \quad \epsilon_{ik}-\alpha_{ik} = 0. \tag{9}$$

With (9), the corresponding conditions $h(Y_i) = -\lambda_i Y_i$ are already satisfied.

Finally, in the present situation, the equality (3) gives

$$\nabla_{X_i}\xi = -(1+\lambda_i)Y_i, \quad \nabla_{Y_i}\xi = (1-\lambda_i)X_i. \tag{10}$$

Using the Koszul formula for the covariant derivative in its simplified form for left-invariant vector fields

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)\Big), \tag{11}$$

this yields

$$\begin{aligned} 0 &= 2g(\nabla_{X_i}\xi, X_j) = -(\alpha_{ij}+\alpha_{ji}), \\ -2\delta_{ij}(1+\lambda_i) &= 2g(\nabla_{X_i}\xi, Y_j) = -(\beta_{ij}+\gamma_{ji}+2\delta_{ij}), \\ 2\delta_{ij}(1-\lambda_i) &= 2g(\nabla_{Y_i}\xi, X_j) = -(\gamma_{ij}+\beta_{ji}-2\delta_{ij}), \\ 0 &= 2g(\nabla_{Y_i}\xi, Y_j) = -(\epsilon_{ij}+\epsilon_{ji}). \end{aligned} \tag{12}$$

Solving (9) and (12) for α_{ij}, β_{ij}, γ_{ij} and ϵ_{ij} and substituting in (8), we obtain

$$\begin{aligned} [\xi, X_i] &= \sum_{k=1}^{n}(\alpha_{ik}X_k + (\lambda_i\delta_{ik}+\nu_{ik})Y_k), \\ [\xi, Y_i] &= \sum_{k=1}^{n}((\lambda_i\delta_{ik}-\nu_{ik})X_k + \alpha_{ik}Y_k), \\ [X_i, X_j] &= \sum_{k=1}^{n}(A_{ij}^k X_k + B_{ij}^k Y_k), \\ [Y_i, Y_j] &= \sum_{k=1}^{n}(C_{ij}^k X_k + D_{ij}^k Y_k), \\ [X_i, Y_j] &= \sum_{k=1}^{n}(a_{ij}^k X_k + b_{ij}^k Y_k) + 2\delta_{ij}\xi \end{aligned} \tag{13}$$

where α_{ij}, A_{ij}^k, B_{ij}^k, C_{ij}^k en D_{ij}^k are skew-symmetric in i and j and ν_{ij} is symmetric in these indices.

Of course, one should still add conditions on these coefficients following from the Jacobi identity for the Lie bracket. Since most of these are *quadratic* conditions in the coefficients, we do not consider these now. We come back to them in Section 5.

Note. It may seem odd that we use the property (3) in order to arrive at the form (13) for the Lie bracket. After all, this property is automatic as soon as we have a contact metric structure. Indeed, the conditions (12) also follow from the Jacobi identity for the bracket.

4 The first curvature condition

Recall that we are looking for left-invariant strongly locally φ-symmetric contact metric structures on Lie groups. The necessary condition (5) for $k = 0$ implies that $R(X,Y)\xi = 0$ for all vector fields X and Y orthogonal to ξ. Expressing this requirement in our present situation, starting from the Lie algebra defined by (13), gives rise to a system of *linear* equations in the coefficients A^k_{ij}, B^k_{ij}, C^k_{ij}, D^k_{ij}, a^k_{ij} and b^k_{ij} which can be solved explicitly. We describe this process in some more detail now, giving the explicit solutions.

4.1 One index

Using again the Koszul formula (11) and (10), we calculate

$$\begin{aligned}
0 &= g(R(X_i,Y_i)\xi, X_i) = (\nabla_{X_i}\nabla_{Y_i}\xi - \nabla_{Y_i}\nabla_{X_i}\xi - \nabla_{[X_i,Y_i]}\xi, X_i)\\
&= g((1-\lambda_i)\nabla_{X_i}X_i + (1+\lambda_i)\nabla_{Y_i}Y_i - g([X_i,Y_i],Y_i)\,\nabla_{Y_i}\xi, X_i)\\
&= (1+\lambda_i)b^i_{ii} - (1-\lambda_i)b^i_{ii} = 2\lambda_i b^i_{ii},\\
0 &= g(R(X_i,Y_i)\xi, Y_i) = 2\lambda_i a^i_{ii}.
\end{aligned}$$

Hence,

- if $\lambda_i > 0$: $a^i_{ii} = b^i_{ii} = 0$;
- if $\lambda_i = 0$: a^i_{ii} and b^i_{ii} are arbitrary.

4.2 Two different indices

This time we look at the conditions of the form $g(R(X,Y)\xi, Z) = 0$ where $X, Y, Z \in \{X_i, Y_i, X_j, Y_j\}$, $i < j$, with both indices occurring. This gives rise to a list of linear equations in the coefficients with two different indices. However, this system subdivides into systems of five equations for only five coefficients. Since all the subsystems are similar, we treat only the one involving A^i_{ij}, C^i_{ij}, b^j_{ii}, b^i_{ij} and b^i_{ji}. This subsystem corresponds to the conditions

$$\begin{aligned}
0 &= g(R(X_i,Y_i)\xi, X_j),\\
0 &= g(R(X_i,X_j)\xi, Y_i),\\
0 &= g(R(Y_i,Y_j)\xi, Y_i),\\
0 &= g(R(X_i,Y_j)\xi, X_i),\\
0 &= g(R(X_j,Y_i)\xi, X_i),
\end{aligned}$$

or, after some calculations, using (10) and (11), to

$$M \begin{pmatrix} A^i_{ij} \\ C^i_{ij} \\ b^j_{ii} \\ b^i_{ij} \\ b^i_{ji} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

where

$$M = \begin{pmatrix} 1-\lambda_i & 0 & 1-\lambda_j & 0 & -(1+\lambda_i) \\ 2(1+\lambda_i) & -(1+\lambda_j) & 1+\lambda_j & -(1+\lambda_j) & 0 \\ 0 & 3\lambda_i+1 & 1-\lambda_i & 1-\lambda_i & -2(1-\lambda_j) \\ 2(1-\lambda_j) & -(1+\lambda_i) & 1+\lambda_i & 3\lambda_i-1 & 0 \\ 0 & 1+\lambda_j & 1+\lambda_j & 1+\lambda_j & -2(1-\lambda_i) \end{pmatrix}.$$

The determinant of the matrix M is given by $64{\lambda_i}^2(\lambda_i+\lambda_j)(\lambda_i-\lambda_j)$. Keeping in mind that $\lambda_i \geq \lambda_j \geq 0$, we have the following solutions for the system (14):

- if $\lambda_i > \lambda_j \geq 0$:
$$A^i_{ij} = C^i_{ij} = b^j_{ii} = b^i_{ij} = b^i_{ji} = 0;$$
- if $\lambda_i = \lambda_j > 0$:
$$C^i_{ij} = b^i_{ji} = 0, \quad A^i_{ij} = b^i_{ij} = -b^j_{ii}, \quad b^j_{ii} \text{ is arbitrary;}$$
- if $\lambda_i = \lambda_j = 0$:
$$A^i_{ij} = b^i_{ji} - b^j_{ii}, \quad C^i_{ij} = 2b^i_{ji} - b^i_{ij} - b^j_{ii}, \quad b^j_{ii},\, b^i_{ij} \text{ and } b^i_{ji} \text{ are arbitrary.}$$

Working similarly for the subsystems $\{A^j_{ij}, C^j_{ij}, b^i_{jj}, b^j_{ji}, b^j_{ij}\}$, $\{D^i_{ij}, B^i_{ij}, a^j_{ii}, a^i_{ji}, a^i_{ij}\}$ and $\{D^j_{ij}, B^j_{ij}, a^i_{jj}, a^j_{ij}, a^j_{ji}\}$, we find

- if $\lambda_i > \lambda_j > 0$:
$$\begin{aligned} A^j_{ij} &= C^j_{ij} = b^i_{jj} = b^j_{ji} = b^j_{ij} = 0, \\ D^i_{ij} &= B^i_{ij} = a^j_{ii} = a^i_{ji} = a^i_{ij} = 0, \\ D^j_{ij} &= B^j_{ij} = a^i_{jj} = a^j_{ij} = a^j_{ji} = 0; \end{aligned}$$
- if $\lambda_i > \lambda_j = 0$:
$$\begin{aligned} &A^j_{ij} = b^j_{ij} = b^i_{jj} = 0, \quad C^j_{ij} = b^j_{ji}, \quad b^j_{ji} \text{ is arbitrary,} \\ &D^i_{ij} = B^i_{ij} = a^j_{ii} = a^i_{ji} = a^i_{ij} = 0, \\ &D^j_{ij} = a^i_{jj} = a^j_{ji} = 0, \quad B^j_{ij} = -a^j_{ij}, \quad a^j_{ij} \text{ is arbitrary;} \end{aligned}$$
- if $\lambda_i = \lambda_j > 0$:
$$\begin{aligned} &C^j_{ij} = b^j_{ij} = 0, \quad A^j_{ij} = -b^j_{ji} = b^i_{jj}, \quad b^i_{jj} \text{ is arbitrary,} \\ &B^i_{ij} = a^i_{ij} = 0, \quad D^i_{ij} = -a^i_{ji} = a^j_{ii}, \quad a^j_{ii} \text{ is arbitrary,} \\ &B^j_{ij} = a^j_{ji} = 0, \quad D^j_{ij} = a^j_{ij} = -a^i_{jj}, \quad a^i_{jj} \text{ is arbitrary;} \end{aligned}$$
- if $\lambda_i = \lambda_j = 0$:
$$\begin{aligned} &A^j_{ij} = b^i_{jj} - b^j_{ij}, \quad C^j_{ij} = b^i_{jj} + b^j_{ji} - 2b^j_{ij}, \\ &b^i_{jj},\, b^j_{ji} \text{ and } b^j_{ij} \text{ are arbitrary,} \\ &D^i_{ij} = a^j_{ii} - a^i_{ij}, \quad B^i_{ij} = a^j_{ii} + a^i_{ji} - 2a^i_{ij}, \\ &a^j_{ii},\, a^i_{ij} \text{ and } a^i_{ji} \text{ are arbitrary,} \\ &D^j_{ij} = a^j_{ji} - a^i_{jj}, \quad B^j_{ij} = 2a^j_{ji} - a^j_{ij} - a^i_{jj}, \\ &a^i_{jj},\, a^j_{ji} \text{ and } a^j_{ij} \text{ are arbitrary.} \end{aligned}$$

4.3 Three different indices

As in the previous case, the conditions $g(R(X,Y)\xi,Z)=0$ where $X,Y,Z\in\{X_i,Y_i,X_j,Y_j,X_k,Y_k\}$, $i<j<k$, and with three different indices occurring, lead to subsystems of twelve equations in twelve coefficients: $\{A^k_{ij},A^j_{ik},A^i_{jk},C^k_{ij},C^j_{ik},C^i_{jk},b^k_{ij},b^k_{ji},b^j_{ik},b^j_{ki},b^i_{jk},b^i_{kj}\}$ and $\{D^k_{ij},D^j_{ik},D^i_{jk},B^k_{ij},B^j_{ik},B^i_{jk},a^k_{ij},a^k_{ji},a^j_{ik},a^j_{ki},a^i_{jk},a^i_{kj}\}$. Again, both cases are quite similar. We do not give the equations explicitly here. It suffices to say that the rank of the system is equal to

- 6 if $\lambda_i=\lambda_j=\lambda_k=0$;
- 9 if $\lambda_i=\lambda_j=\lambda_k>0$;
- 10 if $\lambda_i>\lambda_j=\lambda_k=0$;
- 11 in all other cases.

The corresponding solutions are given by

- if $\lambda_i=\lambda_j=\lambda_k=0$:

$$\begin{aligned}
&A^k_{ij}=b^i_{jk}-b^j_{ik},\quad A^j_{ik}=b^i_{kj}-b^k_{ij},\quad A^i_{jk}=b^j_{ki}-b^k_{ji},\\
&C^k_{ij}=b^k_{ji}+b^i_{jk}-b^j_{ik}-b^k_{ij},\\
&C^j_{ik}=b^i_{kj}+b^j_{ki}-b^j_{ik}-b^k_{ij},\\
&C^i_{jk}=b^i_{kj}+b^j_{ki}-b^k_{ji}-b^i_{jk},\\
&b^k_{ij},b^k_{ji},b^j_{ik},b^j_{ki},b^i_{jk}\text{ and }b^i_{kj}\text{ are arbitrary,}\\
&D^k_{ij}=a^j_{ki}-a^i_{kj},\quad D^j_{ik}=a^k_{ji}-a^i_{jk},\quad D^i_{jk}=a^k_{ij}-a^j_{ik},\\
&B^k_{ij}=a^j_{ki}+a^k_{ji}-a^k_{ij}-a^i_{kj},\\
&B^j_{ik}=a^j_{ki}+a^k_{ji}-a^i_{jk}-a^j_{ik},\\
&B^i_{jk}=a^k_{ij}+a^i_{kj}-a^i_{jk}-a^j_{ik},\\
&a^k_{ij},a^k_{ji},a^j_{ik},a^j_{ki},a^i_{jk}\text{ and }a^i_{kj}\text{ are arbitrary;}
\end{aligned}$$

- if $\lambda_i=\lambda_j=\lambda_k>0$:

$$\begin{aligned}
&A^k_{ij}=b^i_{jk}+b^k_{ij},\quad A^j_{ik}=-b^j_{ki}-b^k_{ij},\quad A^i_{jk}=b^j_{ki}+b^i_{jk},\\
&C^k_{ij}=C^j_{ik}=C^i_{jk}=0,\\
&b^k_{ji}=-b^i_{jk},\quad b^j_{ik}=-b^k_{ij},\quad b^i_{kj}=-b^j_{ki},\\
&b^k_{ij},\ b^j_{ki}\text{ and }b^i_{jk}\text{ are arbitrary,}\\
&D^k_{ij}=a^j_{ki}+a^k_{ij},\quad D^j_{ik}=-a^j_{ki}-a^i_{jk},\quad D^i_{jk}=a^k_{ij}+a^i_{jk},\\
&B^k_{ij}=B^j_{ik}=B^i_{jk}=0,\\
&a^k_{ji}=-a^j_{ki},\quad a^j_{ik}=-a^i_{jk},\quad a^i_{kj}=-a^k_{ij},\\
&a^k_{ij},\ a^j_{ki}\text{ and }a^i_{jk}\text{ are arbitrary;}
\end{aligned}$$

- if $\lambda_i > \lambda_j = \lambda_k = 0$:

$$\begin{aligned}
&A^i_{jk} = C^i_{jk} = b^i_{jk} = b^i_{kj} = 0,\\
&A^k_{ij} = -A^j_{ik} = -b^j_{ik} = b^k_{ij}, \quad C^k_{ij} = C^j_{ik} = b^j_{ki} = b^k_{ji},\\
&b^k_{ij} \text{ and } b^k_{ji} \text{ are arbitrary},\\
&D^i_{jk} = B^i_{jk} = a^i_{jk} = a^i_{kj} = 0,\\
&D^k_{ij} = -D^j_{ik} = a^j_{ki} = -a^k_{ji}, \quad B^k_{ij} = B^j_{ik} = -a^j_{ik} = -a^k_{ij},\\
&a^k_{ij} \text{ and } a^k_{ji} \text{ are arbitrary;}
\end{aligned}$$

- otherwise:

$$\begin{aligned}
&A^k_{ij} = (\lambda_i+\lambda_j)(\lambda_i+\lambda_k)(\lambda_j+\lambda_k)(\lambda_i-\lambda_j)^2 A_{ijk},\\
&A^j_{ik} = -(\lambda_i+\lambda_j)(\lambda_i+\lambda_k)(\lambda_j+\lambda_k)(\lambda_i-\lambda_k)^2 A_{ijk},\\
&A^i_{jk} = (\lambda_i+\lambda_j)(\lambda_i+\lambda_k)(\lambda_j+\lambda_k)(\lambda_j-\lambda_k)^2 A_{ijk},\\
&C^k_{ij} = (\lambda_i+\lambda_j)(\lambda_i-\lambda_j)^2(\lambda_i-\lambda_k)(\lambda_j-\lambda_k) A_{ijk},\\
&C^j_{ik} = (\lambda_i+\lambda_k)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)^2(\lambda_j-\lambda_k) A_{ijk},\\
&C^i_{jk} = (\lambda_j+\lambda_k)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)(\lambda_j-\lambda_k)^2 A_{ijk},\\
&b^k_{ij} = (\lambda_i+\lambda_j)^2(\lambda_j+\lambda_k)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k) A_{ijk},\\
&b^k_{ji} = (\lambda_i+\lambda_j)^2(\lambda_i+\lambda_k)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k) A_{ijk},\\
&b^j_{ik} = -(\lambda_i+\lambda_k)^2(\lambda_j+\lambda_k)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k) A_{ijk},\\
&b^j_{ki} = (\lambda_i+\lambda_j)(\lambda_i+\lambda_k)^2(\lambda_i-\lambda_k)(\lambda_j-\lambda_k) A_{ijk},\\
&b^i_{jk} = -(\lambda_i+\lambda_k)(\lambda_j+\lambda_k)^2(\lambda_i-\lambda_j)(\lambda_j-\lambda_k) A_{ijk},\\
&b^i_{kj} = -(\lambda_i+\lambda_j)(\lambda_j+\lambda_k)^2(\lambda_i-\lambda_k)(\lambda_j-\lambda_k) A_{ijk},\\
&D^k_{ij} = (\lambda_i+\lambda_j)(\lambda_i+\lambda_k)(\lambda_j+\lambda_k)(\lambda_i-\lambda_j)^2 D_{ijk},\\
&D^j_{ik} = -(\lambda_i+\lambda_j)(\lambda_i+\lambda_k)(\lambda_j+\lambda_k)(\lambda_i-\lambda_k)^2 D_{ijk},\\
&D^i_{jk} = (\lambda_i+\lambda_j)(\lambda_i+\lambda_k)(\lambda_j+\lambda_k)(\lambda_j-\lambda_k)^2 D_{ijk},\\
&B^k_{ij} = (\lambda_i+\lambda_j)(\lambda_i-\lambda_j)^2(\lambda_i-\lambda_k)(\lambda_j-\lambda_k) D_{ijk},\\
&B^j_{ik} = (\lambda_i+\lambda_k)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)^2(\lambda_j-\lambda_k) D_{ijk},\\
&B^i_{jk} = (\lambda_j+\lambda_k)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)(\lambda_j-\lambda_k)^2 D_{ijk},\\
&a^k_{ij} = -(\lambda_i+\lambda_j)^2(\lambda_i+\lambda_k)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k) D_{ijk},\\
&a^k_{ji} = -(\lambda_i+\lambda_j)^2(\lambda_j+\lambda_k)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k) D_{ijk},\\
&a^j_{ik} = -(\lambda_i+\lambda_j)(\lambda_i+\lambda_k)^2(\lambda_i-\lambda_k)(\lambda_j-\lambda_k) D_{ijk},\\
&a^j_{ki} = (\lambda_i+\lambda_k)^2(\lambda_j+\lambda_k)(\lambda_i-\lambda_j)(\lambda_i-\lambda_k) D_{ijk},\\
&a^i_{jk} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)^2(\lambda_i-\lambda_k)(\lambda_j-\lambda_k) D_{ijk},\\
&a^i_{kj} = (\lambda_i+\lambda_k)(\lambda_j+\lambda_k)^2(\lambda_i-\lambda_j)(\lambda_j-\lambda_k) D_{ijk},\\
&A_{ijk} \text{ and } D_{ijk} \text{ are arbitrary.}
\end{aligned}$$

5 Jacobi identities

The results in the previous section have already reduced dramatically the number of independent coefficients for the Lie bracket of a Lie group with a strongly locally φ-symmetric left-invariant contact metric structure. We draw further conclusions by expressing the Jacobi identity. Since our aim is to show that all the contact metric structures are (κ, μ)-contact metric structures, we may suppose that $\lambda_1 > 0$. Indeed, otherwise we have $h = 0$ and the structure would be K-contact. But a K-contact metric space with isometric characteristic reflections is necessarily Sasakian ([9]), hence also a (κ, μ)-contact metric space.

For the rest of this section, we slightly change the notation in the following way: we work with an orthonormal basis $\{\xi, X_{11}, \dots, X_{1k_1}, X_{21}, \dots, X_{2k_2}, \dots, X_{t1}, \dots, X_{tk_t}, Y_{11}, \dots, Y_{1k_1}, \dots, Y_{t1}, \dots, Y_{tk_t}\}$ such that

$$h(X_{si}) = \lambda_s X_{si}, \quad h(Y_{si}) = -\lambda_s Y_{si}, \quad \varphi(X_{si}) = Y_{si}, \quad \varphi(Y_{si}) = -X_{si}$$

for $s = 1, \dots, t$, $i = 1, \dots, k_s$ and such that $\lambda_1 > \lambda_2 > \cdots > \lambda_t = 0$. The indexing of the coefficients in (13) will be changed accordingly.

5.1 The zero eigenvalue

Suppose first that h has a zero eigenvalue on $\xi^\perp$. With the notation above, this means $k_t > 0$. We compute the coefficient of Y_{11} in

$$0 = [[X_{tk_t}, Y_{tk_t}], X_{11}] + [[Y_{tk_t}, X_{11}], X_{tk_t}] + [[X_{11}, X_{tk_t}], Y_{tk_t}].$$

We do this term by term. First, we have

$$\begin{aligned} [X_{tk_t}, Y_{tk_t}] &= \sum_{s=1}^{t}\sum_{i=1}^{k_s}(a^{si}_{tk_t\,tk_t} X_{si} + b^{si}_{tk_t\,tk_t} Y_{si}) + 2\xi \\ &= \sum_{i=1}^{k_t}(a^{ti}_{tk_t\,tk_t} X_{ti} + b^{ti}_{tk_t\,tk_t} Y_{ti}) + 2\xi, \end{aligned}$$

$$\begin{aligned} g([[X_{tk_t}, Y_{tk_t}], X_{11}], Y_{11}) &= -\sum_{i=1}^{k_t}(a^{ti}_{tk_t\,tk_t} B^{11}_{11\,ti} + b^{ti}_{tk_t\,tk_t} b^{11}_{11\,ti}) + 2(\lambda_1 + \nu_{11\,11}) \\ &= 2(\lambda_1 + \nu_{11\,11}), \end{aligned}$$

$$
\begin{aligned}
[Y_{tk_t}, X_{11}] &= -\sum_{s=1}^{t}\sum_{i=1}^{k_s}(a^{si}_{11\,tk_t}X_{si} + b^{si}_{11\,tk_t}Y_{si}) \\
&= -\sum_{i=1}^{k_1} a^{1i}_{11\,tk_t}X_{1i} - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} a^{si}_{11\,tk_t}X_{si} - \sum_{i=1}^{k_t} a^{ti}_{11\,tk_t}X_{ti} \\
&\quad -\sum_{i=1}^{k_1} b^{1i}_{11\,tk_t}Y_{1i} - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} b^{si}_{11\,tk_t}Y_{si} - \sum_{i=1}^{k_t} b^{ti}_{11\,tk_t}Y_{ti} \\
&= \sum_{i=2}^{k_1} 2{\lambda_1}^5 D_{11\,1i\,tk_t}X_{1i} \\
&\quad + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^3\lambda_s(\lambda_1+\lambda_s)D_{11\,si\,tk_t}X_{si} - \sum_{i=1}^{k_t} a^{tk_t}_{11\,ti}X_{ti} \\
&\quad + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^3\lambda_s(\lambda_1-\lambda_s)A_{11\,si\,tk_t}Y_{si} + \sum_{i=1}^{k_t-1} b^{tk_t}_{11\,ti}Y_{ti},
\end{aligned}
$$

$$
\begin{aligned}
g([[Y_{tk_t}, X_{11}], X_{tk_t}], Y_{11}) &= \sum_{i=2}^{k_1} 2{\lambda_1}^5 D_{11\,1i\,tk_t}B^{11}_{1i\,tk_t} \\
&\quad + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^3\lambda_s(\lambda_1+\lambda_s)D_{11\,si\,tk_t}B^{11}_{si\,tk_t} - \sum_{i=1}^{k_t-1} a^{tk_t}_{11\,ti}B^{11}_{ti\,tk_t} \\
&\quad - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^3\lambda_s(\lambda_1-\lambda_s)A_{11\,si\,tk_t}b^{11}_{tk_t\,si} - \sum_{i=1}^{k_t-1} b^{tk_t}_{11\,ti}b^{11}_{tk_t\,ti} \\
&= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^3\lambda_s(\lambda_1+\lambda_s)D_{11\,si\,tk_t}\lambda_1{\lambda_s}^3(\lambda_1-\lambda_s)D_{11\,si\,tk_t} \\
&\quad + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^3\lambda_s(\lambda_1-\lambda_s)A_{11\,si\,tk_t}\lambda_1{\lambda_s}^3(\lambda_1+\lambda_s)A_{11\,si\,tk_t} \\
&= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^4{\lambda_s}^4({\lambda_1}^2-{\lambda_s}^2)({D_{11\,si\,tk_t}}^2 + {A_{11\,si\,tk_t}}^2)
\end{aligned}
$$

and similarly

$$
g([[X_{11}, X_{tk_t}], Y_{tk_t}], Y_{11}) = \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^4{\lambda_s}^4({\lambda_1}^2-{\lambda_s}^2)({D_{11\,si\,tk_t}}^2 + {A_{11\,si\,tk_t}}^2).
$$

Hence, we obtain

$$
\lambda_1 + \nu_{11\,11} + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^4{\lambda_s}^4({\lambda_1}^2-{\lambda_s}^2)({D_{11\,si\,tk_t}}^2 + {A_{11\,si\,tk_t}}^2) = 0. \tag{15}
$$

Computing in an analogous way the X_{11}-component of

$$
0 = [[X_{tk_t}, Y_{tk_t}], Y_{11}] + [[Y_{tk_t}, Y_{11}], X_{tk_t}] + [[Y_{11}, X_{tk_t}], Y_{tk_t}],
$$

we obtain

$$
\lambda_1 - \nu_{11\,11} - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} {\lambda_1}^4{\lambda_s}^4({\lambda_1}^2-{\lambda_s}^2)({D_{11\,si\,tk_t}}^2 + {A_{11\,si\,tk_t}}^2) = 0. \tag{16}
$$

From (15) and (16), if follows that $\lambda_1 = 0$, which is contrary to our assumption. Hence, $k_t = 0$ and zero is not an eigenvalue of h on $\xi^{\perp}$.

5.2 The number of eigenvalues

Suppose next that h has at least two different positive eigenvalues on $\xi^\perp$, i.e., $k_2 > 0$. Calculating the Y_{11}-component of

$$0 = [[X_{21}, Y_{21}], X_{11}] + [[Y_{21}, X_{11}], X_{21}] + [[X_{11}, X_{21}], Y_{21}]$$

as before, we find

$$\begin{aligned}\lambda_1 + \nu_{11\,11} &= \sum_{s=3}^{t-1}\sum_{i=1}^{k_s}({\lambda_1}^2 - {\lambda_2}^2)^2({\lambda_1}^2 - {\lambda_s}^2)({\lambda_2}^2 - {\lambda_s}^2)\cdot \\ &\qquad \Big((\lambda_2+\lambda_s)^2 {A_{11\,21\,si}}^2 + (\lambda_2-\lambda_s)^2 {D_{11\,21\,si}}^2\Big),\end{aligned} \tag{17}$$

while the X_{11}-component of

$$0 = [[X_{21}, Y_{21}], Y_{11}] + [[Y_{21}, Y_{11}], X_{21}] + [[Y_{11}, X_{21}], Y_{21}]$$

yields

$$\begin{aligned}\lambda_1 - \nu_{11\,11} &= -\sum_{s=3}^{t-1}\sum_{i=1}^{k_s}({\lambda_1}^2 - {\lambda_2}^2)^2({\lambda_1}^2 - {\lambda_s}^2)({\lambda_2}^2 - {\lambda_s}^2)\cdot \\ &\qquad \Big((\lambda_2-\lambda_s)^2 {A_{11\,21\,si}}^2 + (\lambda_2+\lambda_s)^2 {D_{11\,21\,si}}^2\Big).\end{aligned} \tag{18}$$

Adding (17) and (18), we find the relation

$$\lambda_1 = 2\sum_{s=3}^{t-1}\sum_{i=1}^{k_s}({\lambda_1}^2 - {\lambda_2}^2)^2({\lambda_1}^2 - {\lambda_s}^2)({\lambda_2}^2 - {\lambda_s}^2)\lambda_2\lambda_s({A_{11\,21\,si}}^2 - {D_{11\,21\,si}}^2).$$

On the other hand, we can play the same game with the roles of $\{X_{11}, Y_{11}\}$ and $\{X_{21}, Y_{21}\}$ reversed. This leads, after some calculation, to

$$\lambda_2 = 2\sum_{s=3}^{t-1}\sum_{i=1}^{k_s}({\lambda_1}^2 - {\lambda_2}^2)^2({\lambda_1}^2 - {\lambda_s}^2)({\lambda_2}^2 - {\lambda_s}^2)\lambda_1\lambda_s({A_{11\,21\,si}}^2 - {D_{11\,21\,si}}^2).$$

From the last two expressions, it follows that $\lambda_1/\lambda_2 = \lambda_2/\lambda_1$ or ${\lambda_1}^2 = {\lambda_2}^2$. But since $\lambda_1 > \lambda_2 > 0$, this is impossible. Hence, $k_2 = 0$ and h has only one positive eigenvalue on $\xi^\perp$.

5.3 (κ, μ)-spaces

So, we are left with the situation where we have an orthonormal basis $\{\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ for the Lie algebra $\mathfrak{g}$ for which

$$h(X_i) = \lambda X_i, \quad h(Y_i) = -\lambda Y_i, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i$$

for $i = 1, \dots, n$ and $\lambda > 0$. Using the results from Sections 3 and 4, the Lie bracket can be written in the form

$$\begin{aligned}
[\xi, X_i] &= \sum_{k=1}^{n} (\alpha_{ik} X_k + (\lambda\delta_{ik} + \nu_{ik}) Y_k), \\
[\xi, Y_i] &= \sum_{k=1}^{n} ((\lambda\delta_{ik} - \nu_{ik}) X_k + \alpha_{ik} Y_k), \\
[X_i, X_j] &= \sum_{k=1}^{n} A_{ij}^k X_k, \\
[Y_i, Y_j] &= \sum_{k=1}^{n} D_{ij}^k Y_k, \\
[X_i, Y_j] &= \sum_{k=1}^{n} (a_{ij}^k X_k + b_{ij}^k Y_k) + 2\delta_{ij}\xi
\end{aligned} \tag{19}$$

where α_{ij}, A_{ij}^k en D_{ij}^k are skew-symmetric in i and j and ν_{ij} is symmetric in these indices. Moreover, for $1 \le i < j < k \le n$, it holds

$$\begin{aligned}
& a_{ii}^i = b_{ii}^i = 0, \\
& a_{ij}^i = a_{ji}^j = b_{ji}^i = b_{ij}^j = 0, \\
& A_{ij}^i = b_{ij}^i = -b_{ii}^j, \quad A_{ij}^j = -b_{ji}^j = b_{jj}^i, \\
& D_{ij}^i = -a_{ji}^i = a_{ii}^j, \quad D_{ij}^j = a_{ij}^j = -a_{jj}^i, \\
& b_{ii}^j,\ b_{jj}^i,\ a_{ii}^j \text{ and } a_{jj}^i \text{ are arbitrary}, \\
& A_{ij}^k = b_{jk}^i + b_{ij}^k, \quad A_{ik}^j = -b_{ki}^j - b_{ij}^k, \quad A_{jk}^i = b_{ki}^j + b_{jk}^i, \\
& b_{ji}^k = -b_{jk}^i, \quad b_{ik}^j = -b_{ij}^k, \quad b_{kj}^i = -b_{ki}^j, \\
& D_{ij}^k = a_{ki}^j + a_{ij}^k, \quad D_{ik}^j = -a_{ki}^j - a_{jk}^i, \quad D_{jk}^i = a_{ij}^k + a_{jk}^i, \\
& a_{ji}^k = -a_{ki}^j, \quad a_{ik}^j = -a_{jk}^i, \quad a_{kj}^i = -a_{ij}^k, \\
& b_{ij}^k,\ b_{ki}^j,\ b_{jk}^i,\ a_{ij}^k,\ a_{ki}^j \text{ and } a_{jk}^i \text{ are arbitrary.}
\end{aligned}$$

Additionally, the coefficients have to satisfy the conditions arising from the Jacobi identity for the Lie bracket.

First, we investigate under which conditions on the coefficients we obtain a (κ, μ)-space, i.e., a space for which the curvature tensor satisfies

$$R(X, Y)\xi = \kappa \Big(\eta(Y)X - \eta(X)Y\Big) + \mu \Big(\eta(Y)hX - \eta(X)hY\Big).$$

In Section 4, we have already looked at this condition for both X and Y orthogonal to ξ. This leaves only the case $R(X, \xi)\xi$ for X orthogonal to ξ. We compute:

$$\begin{aligned}
R(X_i, \xi)\xi &= -\nabla_\xi \nabla_{X_i} \xi - \nabla_{[X_i, \xi]} \xi \\
&= (1 + \lambda)\nabla_\xi Y_i + \sum_{k=1}^{n} \Big(\alpha_{ik} \nabla_{X_k} \xi + (\lambda\delta_{ik} + \nu_{ik}) \nabla_{Y_k} \xi\Big) \\
&= (1 + \lambda)([\xi, Y_i] + \nabla_{Y_i} \xi) + \sum_{k=1}^{n} \Big(\alpha_{ik} \nabla_{X_k} \xi + (\lambda\delta_{ik} + \nu_{ik}) \nabla_{Y_k} \xi\Big) \\
&= \left((1 - \lambda^2) + 2\lambda(1 - \nu_{ii})\right) X_i - 2\lambda \sum_{k=1,\, k \ne i}^{n} \nu_{ik} X_k
\end{aligned}$$

and similarly

$$R(Y_i,\xi)\xi = \left((1-\lambda^2) - 2\lambda(1-\nu_{ii})\right) Y_i + 2\lambda \sum_{k=1,\, k\neq i}^{n} \nu_{ik} Y_k.$$

On the other hand, for a (κ,μ)-contact metric structure, we need

$$R(X_i,\xi)\xi = (\kappa+\lambda\mu)X_i, \quad R(Y_i,\xi)\xi = (\kappa-\lambda\mu)Y_i.$$

The necessary and sufficient conditions for such a structure on the Lie group are therefore given by

$$\nu_{11} = \cdots = \nu_{nn}, \qquad \nu_{ij} = 0, \quad i \neq j. \tag{20}$$

If these hold, $\kappa = 1-\lambda^2$ and $\mu = 2(1-\nu_{11})$. We now show that both of these conditions are consequences of the Jacobi identity, thereby proving the main result.

First, for $i<j$, we compute as previously the Y_i-component of

$$0 = [[X_i,Y_i],X_j] + [[Y_i,X_j],X_i] + [[X_j,X_i],Y_i] \tag{21}$$

using the formulas (19) for the bracket:

$$\begin{aligned}
[X_i,Y_i] &= \sum_{k<i} a_{ii}^k X_k + \sum_{i<k<j} a_{ii}^k X_k + a_{ii}^j X_j + \sum_{j<k} a_{ii}^k X_k \\
&\quad + \sum_{k<i} b_{ii}^k Y_k + \sum_{i<k<j} b_{ii}^k Y_k + b_{ii}^j Y_j + \sum_{j<k} b_{ii}^k Y_k + 2\xi,
\end{aligned}$$

$$\begin{aligned}
g([[X_i,Y_i],X_j],Y_i) &= -\sum_{k<i} b_{ii}^k b_{jk}^i - \sum_{i<k<j} b_{ii}^k b_{jk}^i - b_{ii}^j b_{jj}^i - \sum_{j<k} b_{ii}^k b_{jk}^i + 2\nu_{ji} \\
&= -\sum_{k<i} b_{ii}^k b_{jk}^i + \sum_{i<k<j} b_{ii}^k b_{ji}^k - b_{ii}^j b_{jj}^i - \sum_{j<k} b_{ii}^k b_{jk}^i + 2\nu_{ji},
\end{aligned}$$

$$\begin{aligned}
[Y_i,X_j] &= -\sum_{k<i} a_{ji}^k X_k - a_{ji}^i X_i - \sum_{i<k<j} a_{ji}^k X_k - a_{ji}^j X_j - \sum_{j<k} a_{ji}^k X_k \\
&\quad - \sum_{k<i} b_{ji}^k Y_k - b_{ji}^i Y_i - \sum_{i<k<j} b_{ji}^k Y_k - b_{ji}^j Y_j - \sum_{j<k} b_{ji}^k Y_k \\
&= \sum_{k<i} a_{ki}^j X_k + a_{ii}^j X_i - \sum_{i<k<j} a_{ji}^k X_k + \sum_{j<k} a_{ki}^j X_k \\
&\quad + \sum_{k<i} b_{jk}^i Y_k - \sum_{i<k<j} b_{ji}^k Y_k + b_{jj}^i Y_j + \sum_{j<k} b_{jk}^i Y_k,
\end{aligned}$$

$$\begin{aligned}
g([[Y_i,X_j],X_i],Y_i) &= -\sum_{k<i} b_{jk}^i b_{ik}^i + \sum_{i<k<j} b_{ji}^k b_{ik}^i - b_{jj}^i b_{ij}^i - \sum_{j<k} b_{jk}^i b_{ik}^i \\
&= \sum_{k<i} b_{jk}^i b_{ii}^k - \sum_{i<k<j} b_{ji}^k b_{ii}^k + b_{jj}^i b_{ii}^j + \sum_{j<k} b_{jk}^i b_{ii}^k,
\end{aligned}$$

$$\begin{aligned}
[X_j,X_i] &= -\sum_{k<i} A_{ij}^k X_k - A_{ij}^i X_i - \sum_{i<k<j} A_{ij}^k X_k - A_{ij}^j X_j - \sum_{j<k} A_{ij}^k X_k \\
&= -\sum_{k<i} (b_{jk}^i + b_{ij}^k) X_k + b_{ii}^j X_i + \sum_{i<k<j} (b_{ji}^k + b_{ik}^j) X_k \\
&\quad - b_{jj}^i X_j - \sum_{j<k} (b_{jk}^i + b_{ij}^k) X_k,
\end{aligned}$$

$$\begin{aligned} g([[X_j, X_i], Y_i], Y_i) &= -\sum_{k<i}(b^i_{jk} + b^k_{ij})b^i_{ki} + \sum_{i<k<j}(b^k_{ji} + b^j_{ik})b^i_{ki} \\ &\quad - b^i_{jj}b^i_{ji} - \sum_{j<k}(b^i_{jk} + b^k_{ij})b^i_{ki} \\ &= 0. \end{aligned}$$

Hence, we obtain $\nu_{ji} = 0$ for $i < j$. So, the symmetric matrix (ν_{ij}) is diagonal. To show it is actually a multiple of the identity, we compute the Y_j-component of (21). It is given by

$$\begin{aligned} 0 &= \sum_{k<i} b^k_{ii}b^k_{jj} + \sum_{i<k<j} b^k_{ii}b^k_{jj} + \sum_{j<k} b^k_{ii}b^k_{jj} + 2(\lambda + \nu_{jj}) \\ &\quad + \sum_{k<i} b^i_{jk}b^k_{ij} + \sum_{i<k<j} b^k_{ji}b^j_{ik} + \sum_{j<k} b^i_{jk}b^k_{ij} \\ &\quad - \sum_{k<i}(b^i_{jk} + b^k_{ij})b^j_{ki} + b^j_{ii}b^j_{ii} - \sum_{i<k<j}(b^k_{ji} + b^j_{ik})b^i_{kj} \\ &\quad + b^i_{jj}b^i_{jj} - \sum_{j<k}(b^j_{ki} + b^i_{jk})b^j_{ki}. \end{aligned}$$

On the other hand, the Y_i-component of

$$0 = [[X_j, Y_j], X_i] + [[Y_j, X_i], X_j] + [[X_i, X_j], Y_j]$$

is given by

$$\begin{aligned} 0 &= \sum_{k<i} b^k_{jj}b^k_{ii} + \sum_{i<k<j} b^k_{jj}b^k_{ii} + \sum_{j<k} b^k_{jj}b^k_{ii} + 2(\lambda + \nu_{ii}) \\ &\quad + \sum_{k<i} b^k_{ij}b^i_{jk} + \sum_{i<k<j} b^j_{ik}b^k_{ji} + \sum_{j<k} b^k_{ij}b^i_{jk} \\ &\quad - \sum_{k<i}(b^i_{jk} + b^k_{ij})b^j_{ki} + b^j_{ii}b^j_{ii} - \sum_{i<k<j}(b^k_{ji} + b^j_{ik})b^i_{kj} \\ &\quad + b^i_{jj}b^i_{jj} - \sum_{j<k}(b^j_{ki} + b^i_{jk})b^j_{ki}. \end{aligned}$$

Comparing the last two equalities, it follows $\nu_{ii} = \nu_{jj}$.

Note that we have only used the first curvature condition to prove the main result. We could therefore rephraze it as follows:

Proposition. *Every left-invariant contact metric structure on a Lie group which satisfies $R(X,Y)\xi = 0$ for all vector fields X and Y orthogonal to ξ is a (κ, μ)-contact metric structure.*

Proof. Since we used $\lambda_1 > 0$ to prove the main theorem, we still have to look at the case when the contact structure is K-contact. Then $h = 0$ or, equivalently, $\lambda_i = 0$ for $i = 1, \ldots, n$. By (13), we have

$$\begin{aligned} [\xi, X_i] &= \sum_{k=1}^{n}(\alpha_{ik}X_k + \nu_{ik}Y_k), \\ [\xi, Y_i] &= \sum_{k=1}^{n}(-\nu_{ik}X_k + \alpha_{ik}Y_k) \end{aligned}$$

where α_{ij} is skew-symmetric and ν_{ij} is symmetric in the indices i and j. Using this, a simple computation as above gives

$$R(X_i,\xi)\xi = X_i, \quad R(Y_i,\xi)\xi = Y_i.$$

Together with the condition $R(X,Y)\xi = 0$ for all vector fields X and Y orthogonal to ξ, it follows

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

for arbitrary vector fields X and Y and the structure is Sasakian, hence (κ,μ). ∎

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