Marsden-Weinstein reduction for symplectic connections

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Abstract

We propose a reduction procedure for symplectic connections with symmetry. This is applied to coadjoint orbits whose isotropy is reductive.

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The aim of this paper is to show that under very mild conditions, Marsden-Weinstein reduction is “compatible” with a symplectic connection. This means that if a symplectic manifold (M, ω) is endowed with a strongly Hamiltonian action of a connected Lie group G and with a G-invariant symplectic connection ∇, there is a natural way to construct a symplectic connection ∇^r on a reduced manifold (M^r, ω^r). The construction always works when G is compact, and in many non-compact cases as well.

The interest of the construction if two-fold. First it leads to interesting examples of symplectic connections when (M, ω) is a very simple symplectic manifold and G is, for example, one-dimensional or multidimensional but abelian ([2]). Secondly, it may be a useful tool in dealing with the general problem of commutation of quantization and reduction in the framework of deformation quantization.

The paper is organized as follows. We first recall some classical results about strongly Hamiltonian actions. In the second paragraph we show how to construct a reduced connection with a technical assumption and we prove that this is always

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possible in the compact case. The third paragraph collects several examples where this construction gives interesting results. We finally indicate some possible further developments.

1

Let (M,ω) be a symplectic manifold and let $\sigma\colon G\times M\to M$ be a strongly Hamiltonian action of a connected Lie group G, $(g,x)\mapsto g\cdot x$, which we will assume to be effective. If $\mathfrak{g}$ is the Lie algebra of G, we denote by $J\colon M\to\mathfrak{g}^*$ the corresponding G-equivariant momentum map:

$$i(X^*)\omega = d(J^*X),\ \forall X\in\mathfrak{g} \tag{1}$$

where X^* is the infinitesimal generator of the action corresponding to X:

$$X^*_x = \frac{d}{dt}\exp(-tX)\cdot x\Big|_{t=0} \tag{2}$$

and $J^*\colon \mathfrak{g}\subset C^\infty(\mathfrak{g}^*)\to C^\infty(M)$ the map defined by

$$(J^*X)(x) = \langle J(x), X\rangle,\ \forall x\in M. \tag{3}$$

Let $\mu\in\mathfrak{g}^*$ be a regular value of J and let $\Sigma_\mu = J^{-1}(\mu)$ be the constraint manifold; it is a closed embedded submanifold of M.

The following two lemmas are classical [1] and presented here for the sake of completeness.

Lemma 1.1. *In the neighborhood of Σ_μ, the action of G is locally free, i.e. for any $x\in\Sigma_\mu$, there exists a neighborhood Ω_x of the identity element e of G and a neighborhood U_x of x in M such that for any $g\in\Omega_x$, $y\in U_x$, the equation $g\cdot y = y$ implies $g = e$.*

Proof. Let $x\in\Sigma_\mu$. The map $J_{*x}\colon T_xM\to T_\mu\mathfrak{g}^*\cong\mathfrak{g}^*$ is surjective; hence the map $(J_{*x})^*\colon(\mathfrak{g}^*)^*\cong\mathfrak{g}\to T^*_xM$ is injective, i.e. $\forall X\in\mathfrak{g}$, $X\neq 0$, one has:

$$(J_{*x})^*(X) = (dJ^*X)_x = i(X^*_x)\omega_x \neq 0;$$

hence $X^*_x\neq 0$. This means that the stabilizer G_x of x is discrete. Let $\chi\colon G\times M\to M\times M$ be the map $(g,y)\mapsto(g\cdot y, y)$. By the above $\chi_{*(e,x)}$ is injective; hence there exist neighborhouds Ω_x of e in G and U_x of x in M such that $\chi|_{\Omega_x\times U_x}$ is injective. ∎

Let G_μ be the stabilizer of μ under the coadjoint action.

Lemma 1.2. *(i) Let $x\in\Sigma_\mu$ and denote by O_x the orbit of x under the action of G. Then $(T_x\Sigma_\mu)^\perp = T_xO_x$ (where $\perp$ means orthogonal with respect to ω_x).*

(ii) Let $\Delta_x = (T_x\Sigma_\mu)^\perp\cap T_x\Sigma_\mu$; then Δ_x has constant dimension (independent of x) and the orbit of x under the action of G_μ is an integral manifold of Δ.

Proof. (i) For $Z \in T_xM$, we have:

$$Z \in T_x\Sigma_\mu \Leftrightarrow J_{*x}Z = 0 \Leftrightarrow \langle J_{*x}Z, X\rangle = \omega_x(X^*_x, Z) = 0,\ \forall X \in \mathfrak{g}.$$

Consequently $T_x\Sigma_\mu \subset (T_xO_x)^\perp$. But $\dim \Sigma_\mu = \dim M - \dim G = \operatorname{codim} O_x$ (by Lemma 1.1). Hence $T_x\Sigma_\mu = (T_xO_x)^\perp$.

(ii) $Z \in T_x\Sigma_\mu \Leftrightarrow J_{*x}Z = 0$; $Z \in (T_x\Sigma_\mu)^\perp = T_xO_x \Leftrightarrow$ there exists $Y \in \mathfrak{g}$ such that $Z = Y^*$; so, by equivariance of J, $Z \in \Delta_x \Leftrightarrow Z = Y^*$ with $Y \in \mathfrak{g}_\mu$, where $\mathfrak{g}_\mu$ is the Lie algebra of G_μ. Hence, $\dim \Delta_x = \dim \mathfrak{g}_\mu$ and Δ_x is both the radical of $\omega|_{T_x\Sigma_\mu \times T_x\Sigma_\mu}$ and the tangent space to the orbit of G_μ passing through x. ■

Assumption 1. *The constraint manifold Σ_μ is a G_μ-principal bundle over the reduced manifold $M^r = G_\mu\backslash\Sigma_\mu$.*

Remark 1. If the action of G on M is free and proper, Assumption 1 is satisfied; in particular this is true if the action is free and the group G is compact.

The restriction to the constraint submanifold Σ_μ of the tangent bundle TM, denoted $TM|_{\Sigma_\mu}$ is a vector bundle over Σ_μ; the group G_μ acts by automorphisms on this bundle. It contains four G_μ-stable vector subbundles, $T\Sigma_\mu$, $(T\Sigma_\mu)^\perp$, $T\Sigma_\mu + (T\Sigma_\mu)^\perp$ and $T\Sigma_\mu \cap (T\Sigma_\mu)^\perp$.

Assumption 2. *There exists a G_μ-stable vector subbundle $\tilde S$ of $TM|_{\Sigma_\mu}$ such that:*

$$TM|_{\Sigma_\mu} = (T\Sigma_\mu + (T\Sigma_\mu)^\perp) \oplus \tilde S.$$

Remark 2. If the group G is compact, such a vector subbundle always exists. Indeed, we can build a G_μ-invariant metric on $TM|_{\Sigma_\mu}$ and choose $\tilde S$ to be the orthogonal complement, relative to this metric, of $T\Sigma_\mu + (T\Sigma_\mu)^\perp$.

Lemma 1.3. *One may assume that $\tilde S$ is isotropic (relative to ω).*

Proof. By dimension argument, $\dim \tilde S = \dim(T\Sigma_\mu \cap (T\Sigma_\mu)^\perp)$ and ω induces a non-singular pairing between these two G_μ-invariant subbundles. Let $x \in \Sigma_\mu$ and let V_x be the symplectic subspace of T_xM defined by:

$$V_x = \tilde S_x \oplus \Delta_x.$$

For any $u \in \tilde S_x$, there is a unique element $L_xu \in \Delta_x$ so that $-\omega_x(u, v) = 2\omega_x(L_xu, v)\ \forall v \in \tilde S_x$; hence there is a unique linear map $L_x\colon \tilde S_x \to \Delta_x$ such that, $\forall u, v \in \tilde S_x$,

$$\omega_x(L_xu, v) = \omega_x(u, L_xv),$$

$$\omega_x(L_xu, v) + \omega_x(u, L_xv) = -\omega_x(u, v).$$

The graph of L_x in V_x, $\{u + L_xu \mid u \in \tilde S_x\}$, is an isotropic subspace S_x of V_x such that

$$V_x = S_x \oplus \Delta_x.$$

Let $g \in G$; then

$$\begin{aligned} 0 &= \omega_x(L_x u, v) - \omega_x(u, L_x v) = (g^*\omega)_x(L_x u, v) - (g^*\omega)_x(u, L_x v) \\ &= \omega_{g\cdot x}(g_* L_x u, g_* v) - \omega_{g\cdot x}(g_* u, g_* L_x v) \end{aligned}$$

$$\begin{aligned} -\omega_x(u, v) &= -(g^*\omega)_x(u, v) = -\omega_{g\cdot x}(g_* u, g_* v) \\ &= \omega_x(L_x u, v) + \omega_x(u, L_x v) = (g^*\omega)_x(L_x u, v) + (g^*\omega)_x(u, L_x v) \\ &= \omega_{g\cdot x}(g_* L_x u, g_* v) + \omega_{g\cdot x}(g_* u, g_* L_x v). \end{aligned}$$

By unicity, $L_{g\cdot x} = g_* \circ L_x \circ g_*^{-1}$ and hence the subbundle S is G_μ-stable. ■

Remark 3. By dimension argument:

$$\begin{aligned} (S \oplus \Delta)^\perp &= \left((S \oplus \Delta)^\perp \cap T\Sigma\right) \oplus \left((S \oplus \Delta)^\perp \cap T\Sigma^\perp\right) \\ &\overset{not}{=\!=} W_1 \oplus W_2 \end{aligned}$$

and the two subbundles W_1 and W_2 are G_μ-stable.

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We consider the situation where one has a symplectic manifold (M, ω), a Hamiltonian action $\sigma\colon G \times M \to M$ of a connected Lie group G and a symplectic connection $\overset{\circ}{\nabla}$ which is G-invariant.

Lemma 2.1. *If the group G is compact such a connection always exist.*

Proof. Let $\tilde{\nabla}$ be any symplectic connection and let X, Y be smooth vector fields on M. Define:

$$(\overset{\circ}{\nabla}_X Y)_x = \int_G \left[(g \cdot \tilde{\nabla})_X Y\right]_x dg = \int_G \left(g_* \tilde{\nabla}_{g_*^{-1} X} g_*^{-1} Y\right)(x) dg.$$

One checks that $\overset{\circ}{\nabla}$ is a torsion free linear connection. Furthermore:

$$\omega_x(\overset{\circ}{\nabla}_X Y, Z) + \omega_x(Y, \overset{\circ}{\nabla}_X Z) =$$

$$\begin{aligned} &= \int_G \left[\omega_x \left(g_* \tilde{\nabla}_{g_*^{-1} X} g_*^{-1} Y, Z\right) + \omega_x \left(Y, g_* \tilde{\nabla}_{g_*^{-1} X} g_*^{-1} Z\right)\right] dg \\ &= \int_G \left[\omega_{g^{-1}\cdot x} \left(\tilde{\nabla}_{g_*^{-1} X} g_*^{-1} Y, g_*^{-1} Z\right) + \omega_{g^{-1}\cdot x} \left(g_*^{-1} Y, \tilde{\nabla}_{g_*^{-1} X} g_*^{-1} Z\right)\right] dg \\ &= \int_G (g_*^{-1} X)_{g^{-1}\cdot x} \omega(g_*^{-1} Y, g_*^{-1} Z) dg = \int_G X_x \omega(Y, Z) dg \\ &= X_x \omega(Y, Z), \end{aligned}$$

if the Haar measure dg is properly normalized. ■

If Assumptions 1 and 2 are satisfied, Σ_μ (the constraint manifold) is a G_μ-principal bundle over the reduced manifold M^r:

$$\pi\colon \Sigma_\mu \to M^r.$$

Furthermore, at a point $x \in \Sigma_\mu$, the tangent space $T_x\Sigma_\mu$ is the direct sum of two G_μ-invariant distributions:

$$T_x\Sigma_\mu = \Delta_x \oplus (W_1)_x$$

where $\Delta_x = \ker \pi_{*x} = \operatorname{rad}^\omega(T_x\Sigma_\mu)$. The distribution W_1 will be called the **horizontal distribution**. To W_1 is canonically associated a **connection 1-form** α on Σ_μ (with values in $\mathfrak{g}_\mu$):

$$\alpha(U) = X,$$

if $U = \delta + w_1$ with $\delta_x = (d/dt)\exp(-tX)\cdot x\Big|_{t=0} = X^*_x$. Remark that

$$\alpha_{g\cdot x}(g_{*_x}U) = \operatorname{Ad} g(\alpha_x(U))\ \forall g \in G_\mu.$$

Observe that in this framework

$$T_xM = \Delta_x \oplus (W_1)_x \oplus (W_2)_x \oplus S_x.$$

Hence we have a projection operator $P_x\colon T_xM \to T_x\Sigma_\mu$.

Definition 1. If X, Y are smooth vector fields, along Σ_μ, tangent at each point to Σ_μ, we define a linear **connection ∇ along** Σ_μ, by:

$$\nabla_X Y = P(\overset{\circ}{\nabla}_X Y). \tag{4}$$

Lemma 2.2. *∇ is a torsion free linear connection on Σ_μ. Furthermore, G_μ is a group of affine transformations of ∇.*

Proof. One has for $f \in C^\infty(\Sigma_\mu)$:

$$[\nabla_X(fY)]_x = P(\overset{\circ}{\nabla}_X fY)_x = P((Xf)Y + f\overset{\circ}{\nabla}_X Y)_x = (X_x f)Y_x + f(x)(\nabla_X Y)_x$$

$$\nabla_X Y - \nabla_Y X - [X,Y] = P(\overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X,Y]) = 0.$$

Also, if $Z \in \mathfrak{g}_\mu$:

$$\begin{aligned}(L_{Z^*}\nabla)_X Y &= [Z^*, \nabla_X Y] - \nabla_{[Z^*,X]}Y - \nabla_X[Z^*,Y]\\ &= [Z^*, P\overset{\circ}{\nabla}_X Y] - P\overset{\circ}{\nabla}_{[Z^*,X]}Y - P\overset{\circ}{\nabla}_X[Z^*,Y]\\ &= P\left([Z^*, \overset{\circ}{\nabla}_X Y] - \overset{\circ}{\nabla}_{[Z^*,X]}Y - \overset{\circ}{\nabla}_X[Z^*,Y]\right)\end{aligned}$$

using the G_μ-invariance of P. Hence the conclusion since $\overset{\circ}{\nabla}$ is G_μ-invariant. ■

Lemma 2.3. *The orbits of G_μ in Σ_μ are totally geodesic with respect to ∇ if and only if for all $X, Y \in \mathfrak{g}_\mu$ and for all vector fields Z on M, one has:*

$$\omega(P\overset{\circ}{\nabla}_{X^*}Y^*, PZ) = 0.$$

Proof. The totally geodesic condition means that $(\nabla_{X^*}Y^*)(x)$ belongs to Δ_x which is the radical of $T_x\Sigma_\mu$. ■

Definition 2. If X is a smooth vector field on M^r, its horizontal lift $\bar{X}$ to Σ_μ is defined by $\bar{X}_x \in W_{1x}$ and $\pi_{*_x}\bar{X}_x = X_{\pi(x)}$. Remark that $\bar{X}_{g\cdot x} = g_{*_x}\bar{X}_x \ \forall g \in G_\mu$. The **reduced connection** ∇^r on M^r is defined as follows. Let X, Y be smooth vector fields on M^r; denote by $\bar{X}, \bar{Y}$ their horizontal lifts to Σ_μ. Then:

$$\overline{(\nabla^r_X Y)}(x) = (\nabla_{\bar{X}}\bar{Y})(x) - [\alpha_x(\nabla_{\bar{X}}\bar{Y})]^*. \tag{5}$$

Proposition 2.4. *Formula* 5 *defines a torsion free linear connection on M^r. Furthermore, if ω^r is the 2-form on M^r such that*

$$\omega^r_{\pi(x)}(X,Y) = \omega_x(\bar{X},\bar{Y}),$$

then ω^r is symplectic and parallel relative to ∇^r.

Proof. Formula 5 defines a linear connection on M^r. Indeed, one has, if $g \in G_\mu$:

$$\begin{aligned}
\nabla_{\bar{X}}\bar{Y}\Big|_{g\cdot x} - [\alpha_{g\cdot x}(\nabla_{\bar{X}}\bar{Y})]^* &= \nabla_{g_*\bar{X}}g_*\bar{Y}\Big|_{g\cdot x} - g_*\left(\mathrm{Ad}(g^{-1})\alpha_{g\cdot x}(\nabla_{\bar{X}}\bar{Y})\right)^*_x \\
&= g_{*_x}\left[(g^{-1}\cdot\nabla)_{\bar{X}}\bar{Y}\Big|_x - \mathrm{Ad}(g^{-1})\,\mathrm{Ad}(g)\alpha_x\left((g^{-1}\cdot\nabla)_{\bar{X}}\bar{Y}\right)^*\right] \\
&= g_*\left[\nabla_{\bar{X}}\bar{Y}\Big|_x - \alpha_x(\nabla_{\bar{X}}\bar{Y})^*\right].
\end{aligned}$$

Thus formula 5 is independent of the choice of x in the fibre over $\pi(x)$. Also:

$$\begin{aligned}
\overline{\nabla^r_X Y - \nabla^r_Y X - [X,Y]} &= \nabla_{\bar{X}}\bar{Y} - \alpha_x(\nabla_{\bar{X}}\bar{Y})^* - \nabla_{\bar{Y}}\bar{X} + \alpha_x(\nabla_{\bar{Y}}\bar{X})^* - \overline{[X,Y]} \\
&= [\bar{X},\bar{Y}] - \alpha_x([\bar{X},\bar{Y}])^* - \overline{[X,Y]} = 0
\end{aligned}$$

and ∇^r is torsion free.

The 2-form ω^r has maximal rank; furthermore, if $\circlearrowleft\!\!\!\!\!+$ denotes the cyclic sum, we have:

$$\begin{aligned}
(d\omega^r)_{\pi(x)}(X,Y,Z) &= \underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!+}\left[X_{\pi(x)}\omega^r(Y,Z) - \omega^r_{\pi(x)}([X,Y],Z)\right] \\
&= \underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!+}\left[\bar{X}_x\omega(\bar{Y},\bar{Z}) - \omega_x([\bar{X},\bar{Y}] - \alpha_x([\bar{X},\bar{Y}])^*,\bar{Z})\right] \\
&= (d\omega)_x(\bar{X},\bar{Y},\bar{Z}),
\end{aligned}$$

hence ω^r is closed. Finally:

$$\begin{aligned}
X_{\pi(x)}\omega^r(Y,Z) &= \bar{X}_x\omega(\bar{Y},\bar{Z}) = \omega_x(\overset{\circ}{\nabla}_{\bar{X}},\bar{Y},\bar{Z}) + \omega_x(\bar{Y},\overset{\circ}{\nabla}_{\bar{X}}\bar{Z}) \\
&= \omega_x(P\overset{\circ}{\nabla}_{\bar{X}}\bar{Y},\bar{Z}) + \omega_x(\bar{Y},P\overset{\circ}{\nabla}_{\bar{X}}\bar{Z}) = \omega_x(\nabla_{\bar{X}}\bar{Y},\bar{Z}) + \omega_x(\bar{Y},\nabla_{\bar{X}}\bar{Z}) \\
&= \omega_x(\overline{\nabla^r_X Y},\bar{Z}) + \omega_x(\bar{Y},\overline{\nabla^r_X Z}) \\
&= \omega^r_{\pi(x)}(\nabla^r_X Y, Z) + \omega^r(Y,\nabla^r_X Z),
\end{aligned}$$

which proves that ∇^r is symplectic. ∎

Formula for the curvature of the reduced connection. Let X, Y, Z be vector fields on M^r. Then:

$$\begin{aligned}
\overline{R^r(X,Y)Z} &= \overline{\left(\nabla^r_X\nabla^r_Y - \nabla^r_Y\nabla^r_X - \nabla^r_{[X,Y]}\right)Z} \\
&= \nabla_{\bar X}(\overline{\nabla^r_Y Z}) - \alpha\left(\nabla_{\bar X}(\overline{\nabla^r_Y Z})\right)^* - \nabla_{\bar Y}(\overline{\nabla^r_X Z}) + \alpha\left(\nabla_{\bar Y}(\overline{\nabla^r_X Z})\right)^* \\
&\quad -\nabla_{\overline{[X,Y]}}\bar Z + \alpha\left(\nabla_{\overline{[X,Y]}}\bar Z\right)^* \\
&= \nabla_{\bar X}\left(\nabla_{\bar Y}\bar Z - \alpha(\nabla_{\bar Y}\bar Z)^*\right) - \alpha\left(\nabla_{\bar X}\left(\nabla_{\bar Y}\bar Z - \alpha(\nabla_{\bar Y}\bar Z)^*\right)\right)^* \\
&\quad -\nabla_{\bar Y}\left(\nabla_{\bar X}\bar Z - \alpha(\nabla_{\bar X}\bar Z)^*\right) + \alpha\left(\nabla_{\bar Y}\left(\nabla_{\bar X}\bar Z - \alpha(\nabla_{\bar X}\bar Z)^*\right)\right)^* \\
&\quad -\nabla_{[\bar X,\bar Y]-\alpha([\bar X,\bar Y])^*}\bar Z + \alpha\left(\nabla_{[\bar X,\bar Y]-\alpha([\bar X,\bar Y])^*}\bar Z\right)^* \\
&= R(\bar X,\bar Y)\bar Z - \alpha(R(\bar X,\bar Y)\bar Z)^* - \nabla_{\bar X}(\alpha(\nabla_{\bar Y}\bar Z)^*) + \alpha\left(\nabla_{\bar X}(\alpha(\nabla_{\bar Y}\bar Z)^*)\right)^* \\
&\quad +\nabla_{\bar Y}(\alpha(\nabla_{\bar X}\bar Z)^*) - \alpha\left(\nabla_{\bar Y}(\alpha(\nabla_{\bar X}\bar Z)^*)\right)^* + \nabla_{\alpha([\bar X,\bar Y])^*}\bar Z \\
&\quad -\alpha\left(\nabla_{\alpha([\bar X,\bar Y])^*}\bar Z\right)^*
\end{aligned}$$

In the special case where Σ_μ is totally geodesic with respect to the connection $\overset{\circ}{\nabla}$ (i.e. autoparallel, i.e. $\overset{\circ}{\nabla}_X Y$ is tangent to Σ_μ at each point of Σ_μ for all smooth vector fields X, Y along Σ_μ tangent at each point to Σ_μ), we have $\nabla_X Y = \overset{\circ}{\nabla}_X Y$ for all vector fields X, Y tangent to Σ_μ and the vertical subbundle ($\ker \pi_*$) in $T\Sigma_\mu$ (which coincides with the radical of $\omega|_{\Sigma_\mu}$) is preserved by the connection ∇. Furthermore, the reduced connection ∇^r does not depend on the choice of S. Indeed, for another subbundle $\hat S$ with the same properties as S, we have another horizontal distribution $\hat W_1$; if X is a vector field on M^r, $\bar X$ and $\hat X$ its horizontal lifts with respect to W_1 and $\hat W_1$, and $\hat\alpha$ the connection 1-form defining $\hat W_1$, then $\hat X = \bar X + \alpha(\hat X)^* = \bar X - \hat\alpha(\bar X)^*$. If $\nabla^{\hat r}$ is the reduced connection defined by 5 for the connection $\hat\alpha$, then one easily sees that $\widehat{\nabla^{\hat r}_X Y} = \overline{\nabla^r_X Y} - \hat\alpha(\overline{\nabla^r_X Y}) = \widehat{\nabla^r_X Y}$, which simply means that ∇^r and $\nabla^{\hat r}$ coincide. The reduction of the symplectic connection when Σ_μ is autoparallel is natural and can be performed without the machinery we introduce here (see [4] for more details).

3

Coadjoint orbits are standard examples of reduced symplectic manifolds [1].
Let $p\colon T^*G \to G$ be the cotangent bundle to a connected Lie group G; it can be identified, as manifold, to the direct product $G \times \mathfrak{g}^*$ by:

$$\phi\colon T^*G \to G \times \mathfrak{g}^*,\ a \mapsto (g, L_g^* a),\ g = p(a),$$

where $\mathfrak{g}$ is the Lie algebra of G. The left translation by g_1 of G, lifts to T^*G and can be read by the above identification, as:

$$L(g_1)\colon G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*,\ (g, \xi) \mapsto (g_1 g, \xi).$$

Similarly, the right translation by g_1 reads:

$$R(g_1)\colon G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*,\ (g, \xi) \mapsto (g g_1, \mathrm{Coad}(g_1^{-1})\xi).$$

The Liouville 1-form θ on T^*G, reads on $G \times \mathfrak{g}^*$:

$$\left((\phi^{-1})^*\theta\right)_{(g,\xi)}(L_{g*}X + \eta) \overset{not}{=\!=} \bar{\theta}_{(g,\xi)}(L_{g*}X + \eta) = \xi(X),$$

for $X \in \mathfrak{g}$, $\eta \in \mathfrak{g}^*$. This gives the symplectic form

$$\omega_{(g,\xi)}(L_{g*}X + \eta, L_{g*}X' + \eta') = \langle \eta, X' \rangle - \langle \eta', X \rangle - \langle \xi, [X, X'] \rangle.$$

The fundamental vector field corresponding to the left action is

$$X^l(g, \xi) = -R_{g*}X.$$

Similarly, the fundamental vector field corresponding to the right action is

$$X^r(g, \xi) = L_{g*}X + \xi \circ \mathrm{ad}(X).$$

From this one deduces the expression of the left (resp. right) momentum maps:

$$J^l(g, \xi) = \mathrm{Coad}(g)\xi$$

$$J^r(g, \xi) = \xi.$$

If $\mu \in \mathfrak{g}^*$ one constructs a constraint submanifold Σ^l_μ (resp. Σ^r_μ) corresponding to the left (resp. right) action:

$$\Sigma^l_\mu = \left\{(g, \mathrm{Coad}(g^{-1})\mu) \mid g \in G\right\}$$

$$\Sigma^r_\mu = \{(g, \mu) \mid g \in G\}.$$

Let us consider the constraint manifold corresponding to the right action:

$$T_{(g,\mu)}\Sigma^r_\mu = \{L_{g_*}X \mid X \in \mathfrak{g}\};$$

$$\left(T_{(g,\mu)}\Sigma^r_\mu\right)^\perp = \{X^r(g, \mu) \mid X \in \mathfrak{g}\};$$

$$\left(T\Sigma^r_\mu \cap (T\Sigma^r_\mu)^\perp\right)_{(g,\mu)} = \{\tilde{Y} \mid Y \in \mathfrak{g},\ \mu \circ \mathrm{ad}(Y) = 0\} \cong \mathfrak{g}_\mu,$$

where $\mathfrak{g}_\mu$ is the Lie algebra of the stabilizer G_μ of μ in the coadjoint action and where $\tilde{Y}_{(g,\mu)} = L_{g_*}Y$ for $Y \in \mathfrak{g}$.

$$\left(T\Sigma^r_\mu + (T\Sigma^r_\mu)^\perp\right)_{(g,\mu)} = \{\tilde{X} + \mu \circ \mathrm{ad}(Y) \mid X, Y \in \mathfrak{g}\}.$$

Lemma 2.5. *On $(T^*G \cong G \times \mathfrak{g}^*, \omega)$ there exists a symplectic connection ∇ invariant by the right action of G.*

Proof. Let ∇^0 be the linear connection on $G \times \mathfrak{g}^*$ defined by:

$$\nabla^0{}_{\tilde{X}+\eta}(\tilde{X}' + \eta') = \frac{1}{2}\widetilde{[X, X']}.$$

This connection is right and left invariant but not symplectic; indeed, one has:

$$\begin{aligned}
(\nabla^0{}_{\tilde{X}+\eta}\omega)_{(g,\xi)}(\tilde{Y}+\zeta,\tilde{Y}'+\zeta') &= (\tilde{X}+\eta)[\langle\zeta,Y'\rangle - \langle\zeta',Y\rangle - \langle\zeta,[Y,Y']\rangle] \\
&\quad -\frac{1}{2}\Big(-\langle\zeta',[X,Y]\rangle - \langle\xi,[[X,Y],Y']\rangle\Big) \\
&\quad -\frac{1}{2}\Big(\langle\zeta,[X,Y']\rangle - \langle\xi,[Y,[X,Y']]\rangle\Big) \\
&= -\langle\eta,[Y,Y']\rangle + \frac{1}{2}\langle\zeta',[X,Y]\rangle - \frac{1}{2}\langle\zeta,[X,Y']\rangle \\
&\quad +\frac{1}{2}\langle\xi,[X,[Y,Y']]\rangle.
\end{aligned}$$

This can be projected on the space of symplectic connections as follows. Write

$$\nabla_U V = \nabla^0{}_U V + A(U)V$$

where $A(U)$ is an endomorphism such that

$$A(U)V = A(V)U \quad \text{(torsion free condition)}.$$

Then choose:

$$\omega(A(U)V,W) = \frac{1}{3}[(\nabla^0{}_U\omega)(V,W) + (\nabla^0{}_V\omega)(U,W)].$$

This gives a symplectic connection which is G-invariant. ■

Proposition 2.6. *If the group G_μ is reductive, there exists on the reduced symplectic manifold a symplectic connection.*

Proof. The action of G on T^*G is free; hence Assumption 1 is satisfied. The reductiveness hypothesis ensures Assumption 2. ■

Curvature properties of these reduced connections are worth investigating. We recall in particular the examples given in [2]. It seems also worthwhile to read the nice Gotay-Tuynman paper [3] thinking of connections.

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