CUBIC COMMUTATIVE IDEALS OF BCK-ALGEBRAS

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Abstract. In this paper, we apply the concept of cubic set to commutative ideals of BCK-algebras, and then characterize their basic properties. We discuss relations among cubic commutative ideals, cubic subalgebras, and cubic ideals of BCK-algebras. We provide a condition for a cubic ideal to be a cubic commutative ideal. We define inverse images of cubic commutative ideals and establish how the inverse images of a cubic commutative ideal becomes a cubic commutative ideal. We introduce products of cubic BCK-algebras. Finally, we discuss the relationships between (cubic) commutative ideals, implicative ideals, and positive implicative ideals in BCK/BCI-algebras.

1. Introduction

Combining the idea of fuzzy set [19] and interval-valued fuzzy set [20], Jun et al. [3] introduced the concept of cubic sets, and applied it to subalgebras, ideals and q-ideals in BCK/BCI-algebras [4, 5]. Jun et al. [6, 7] applied cubic soft sets and double-framed soft sets in BCK/BCI-algebras. Muhiuddin et al. [10, 11, 12] applied cubic soft sets and (α, β)-type fuzzy sets in BCK/BCI-algebras. Senapati together with his collaborators [2, 16, 17, 18] applied the notion of cubic sets in G-algebras, B-algebras, BF-algebras, and BG-algebras. Recently, Senapati et al. [13] introduced cubic implicative ideals of BCK-algebras.

The objective of this paper is to introduce the concept of cubic set to commutative ideals of BCK-algebras. We prove that every cubic commutative ideal must be a cubic ideal and a cubic subalgebra. In addition to this we observe that in a commutative BCK-algebra, every cubic ideal is a cubic commutative ideal. By using the notion of level sets, we hence give some theorems of characterizations of cubic commutative ideals of BCK-algebras.

The remainder of this paper is organized as follows: in Section 2, we recall important preliminary definitions and properties. Section 3 contains definition and related results of cubic subalgebras and ideals of BCK-algebras. In Section 4, we propose concepts and operations of cubic commutative ideals and discuss their properties in details. In Section 5, we give cubic extension property of cubic commutative ideals. In Section 6, we investigate properties of cubic commutative ideals under homomorphisms. In Section 7, we study products of cubic commutative ideals. Finally, in Section 8, we discuss the relationship between (cubic) commutative ideals, implicative ideals and positive implicative ideals in BCK/BCI-algebras.

2. Preliminaries

To make this work self-contained, we briefly mention some of the definitions and results employed in the rest of the work.

An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BCI-algebra [12] if it satisfies the following axioms for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$
(ii) $(x * (x * y)) * y = 0$
(iii) $x * x = 0$
(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra. Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:

(1) $(x * y) * z = (x * z) * y$
(2) $((x * z) * (y * z)) * (x * y) = 0$
(3) $x * 0 = x$
(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0.$

Throughout this paper, X always means a BCK-algebra without any specification.

A BCK-algebra X is said to be commutative [8] if it satisfies the identity $x * (x * y) = y * (y * x)$ for all $x, y \in X$.

A mapping $f : X \to Y$ of BCK-algebras is called a homomorphism if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. A non-empty subset S of X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$. A nonempty subset I of X is called an ideal of X if it satisfies

(I_1) $0 \in I$ and
(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

A non-empty subset I of X is said to be an *commutative ideal* of X (see [8]) if it satisfies (I_1) and (I_3) $(x*y)*z \in I$ and $z \in I$ imply $x*(y*(y*x)) \in I$, for all $x, y, z \in X$.

Our aim of this paper is to study properties of commutative ideals of cubic sets. By a cubic set, we mean a particular type of fuzzy set. A fuzzy set A in X is of the form $A = \{< x, \mu_A(x) >: x \in X\}$, where $\mu_A(x)$ is called the membership value of x in A and $0 \le \mu_A(x) \le 1$.

An interval-valued fuzzy set A over X is an object having the form $A = \{\langle x, \tilde{\mu}_A(x)\rangle : x \in X\}$, where $\tilde{\mu}_A(x) : X \to D[0, 1]$, where $D[0, 1]$ is the set of all subintervals of $[0, 1]$. The intervals $\tilde{\mu}_A(x)$ denote the intervals of the degree of membership of the element x to the set A, where $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ for all $x \in X$.

The determination of maximum and minimum between two real numbers is very simple but it is not simple for two intervals. Biswas [1] described a method to find max/sup and min/inf between two intervals or a set of intervals.

Definition 2.1. [1] *Consider two elements $D_1, D_2 \in D[0,1]$. If $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$, then $\operatorname{rmin}(D_1, D_2) = [\min(a_1^-, a_2^-), \min(a_1^+, a_2^+)]$ which is denoted by $D_1 \wedge^r D_2$. Thus, if $D_i = [a_i^-, a_i^+] \in D[0,1]$ for $i = 1,2,3,4,\ldots$, then we define $\operatorname{rsup}_i(D_i) = [\sup_i(a_i^-), \sup_i(a_i^+)]$, i.e, $\vee_i^r D_i = [\vee_i a_i^-, \vee_i a_i^+]$. Now we call $D_1 \gg D_2$ if and only if $a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$. Similarly, the relations $D_1 \ll D_2$ and $D_1 = D_2$ are defined.*

Based on the (interval valued) fuzzy sets, Jun et al. [3] introduced the notion of (internal, external) cubic sets, and investigated several properties.

Definition 2.2. [3] *Let X be a nonempty set. A cubic set A in X is a structure $A = \{\langle x, \tilde{\mu}_A(x), \nu_A\ (x)\rangle : x \in X\}$ which is briefly denoted by $A = (\tilde{\mu}_A, \nu_A)$ where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set in X and ν_A is a fuzzy set in X.*

3. Cubic Subalgebras and Ideals of BCK-Algebras

Throughout this section, unless otherwise stated, we denote the BCK-algabra by X. In [4, 5], Jun et al. defined the cubic subalgebras and ideals of X. The definitions are given in below.

Definition 3.1. [4] *Let $A = (\tilde{\mu}_A, \nu_A)$ be cubic set in X. Then the set A is cubic subalgebra over the binary operator $*$ if it satisfies the following conditions for all $x, y \in X$:*

$(F1)$ $\tilde{\mu}_A(x * y) \gg \operatorname{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$
$(F2)$ $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Definition 3.2. [4] *A cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X is called a cubic ideal of X if it satisfies:*

$(T1)$ $\tilde{\mu}_A(0) \gg \tilde{\mu}_A(x)$
$(T2)$ $\nu_A(0) \leq \nu_A(x)$
$(T3)$ $\tilde{\mu}_A(x) \gg \operatorname{rmin}\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$
$(T4)$ $\nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\}$

for all $x, y \in X$.

Lemma 3.3. [4] *Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic ideal of X. If the inequality $x \leq y$ holds in X, then $\tilde{\mu}_A(x) \gg \tilde{\mu}_A(y)$ and $\nu_A(x) \leq \nu_A(y)$.*

Theorem 3.4. [4] *Let X be a BCK-algebra. Then every cubic ideal of X is a cubic subalgebra of X.*

Proposition 3.5. [4] *Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic ideal of X. If the inequality $x * y \leq z$ holds in X, then $\tilde{\mu}_A(x) \gg \mathrm{rmin}\{\tilde{\mu}_A(y), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \max\{\nu_A(y), \nu_A(z)\}$.*

4. Cubic Commutative Ideals of BCK-Algebras

In this section, cubic commutative ideals of BCK-algebras are defined and proved some related results.

Definition 4.1. *A cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X is called a cubic commutative ideal of X if it satisfies $(T1)$, $(T2)$, and*

$(T5)$ $\tilde{\mu}_A(x * (y * (y * x))) \gg \mathrm{rmin}\{\tilde{\mu}_A((x * y) * z), \tilde{\mu}_A(z)\}$

$(T6)$ $\nu_A(x * (y * (y * x))) \leq \max\{\nu_A((x * y) * z), \nu_A(z)\}$

for all $x, y, z \in X$.

Let us illustrate Definition 4.1 using the following example.

Example 4.2. *Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following Cayley table*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic set of X defined as $\tilde{\mu}_A(0) = [0.7, 0.8]$, $\tilde{\mu}_A(a) = [0.4, 0.5]$, $\tilde{\mu}_A(b) = \tilde{\mu}_A(c) = [0.2, 0.4]$, $\nu_A(0) = 0.2$, $\nu_A(a) = 0.3$, and $\nu_A(b) = \nu_A(c) = 0.5$. Routine calculation gives that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic commutative ideal of X.

Now we give a relation between a cubic commutative ideal and a cubic ideal.

Theorem 4.3. *Any cubic commutative ideal of X must be a cubic ideal of X.*

Proof. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic commutative ideal of X. Substituting 0 for y in $(T5)$ and $(T6)$, we get $\tilde{\mu}_A(x * (0 * (0 * x))) \gg \mathrm{rmin}\{\tilde{\mu}_A((x * 0) * z), \tilde{\mu}_A(z)\} = \mathrm{rmin}\{\tilde{\mu}_A(x * z), \tilde{\mu}_A(z)\}$ and $\nu_A(x * (0 * (0 * x))) \leq \max\{\nu_A((x * 0) * z), \nu_A(z)\} = \max\{\nu_A(x * z), \nu_A(z)\}$. Using (3) and $0 * x = 0$, we get

$$\tilde{\mu}_A(x) = \tilde{\mu}_A(x * (0 * (0 * x))) \gg \mathrm{rmin}\{\tilde{\mu}_A(x * z), \tilde{\mu}_A(z)\},$$
$$\nu_A(x) = \nu_A(x * (0 * (0 * x))) \leq \max\{\nu_A(x * z), \nu_A(z)\}.$$

This shows that $A = (\tilde{\mu}_A, \nu_A)$ satisfies $(T3)$ and $(T4)$. Combining $(T1)$ and $(T2)$, A is cubic ideal of X, proving the theorem. □

By applying Theorem 3.4 and 4.3, we get the following corollary.

Corollary 4.4. *Every cubic commutative ideal of X must be a cubic subalgebra of X.*

The converse of Theorem 4.3 may not be true as shown in the following example.

Example 4.5. *Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Define a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X by $\tilde{\mu}_A(0) = [0.6, 0.7]$, $\tilde{\mu}_A(a) = [0.5, 0.6]$, $\tilde{\mu}_A(b) = \tilde{\mu}_A(c) = \tilde{\mu}_A(d) = [0.3, 0.4]$, and $\nu_A(0) = 0.2$, $\nu_A(a) = 0.3$, $\nu_A(b) = \nu_A(c) = \nu_A(d) = 0.6$. It is easy to check that A is a cubic ideal of X, but it is not a cubic commutative ideal of X because $\tilde{\mu}_A(b * (c * (c * b))) \gg \mathrm{rmin}\{\tilde{\mu}_A((b * c) * 0), \tilde{\mu}_A(0)\}$ does not hold, and $\nu_A(b * (c * (c * b))) \not\leq \max\{\nu_A((b * c) * 0), \nu_A(0)\}$.

We provide a condition for a cubic ideal to be a cubic commutative ideal.

Theorem 4.6. *Let A be a cubic ideal of X. Then A is a cubic commutative ideal of X if and only if it satisfies the conditions $\tilde{\mu}_A(x * (y * (y * x))) \gg \tilde{\mu}_A(x * y)$ and $\nu_A(x * (y * (y * x))) \leq \nu_A(x * y)$ for all $x, y \in X$.*

Proof. Assume that A is a cubic commutative ideal of X. Taking $z = 0$ in $(T5)$ and $(T6)$, and using $(T1)$, $(T2)$, and (3), we get the conditions.

Conversely, suppose A satisfies the above two conditions. As A is a cubic ideal, hence,

$$\tilde{\mu}_A(x * y) \gg \mathrm{rmin}\{\tilde{\mu}_A((x * y) * z), \tilde{\mu}_A(z)\},$$
$$\nu_A(x * y) \leq \max\{\nu_A((x * y) * z), \nu_A(z)\},$$

for all $x, y, z \in X$. Therefore, combining with the given two conditions, we obtain

$$\tilde{\mu}_A(x * (y * (y * x))) \gg \tilde{\mu}_A(x * y) \gg \mathrm{rmin}\{\tilde{\mu}_A((x * y) * z), \tilde{\mu}_A(z)\},$$
$$\nu_A(x * (y * (y * x))) \leq \nu_A(x * y) \leq \max\{\nu_A((x * y) * z), \nu_A(z)\}.$$

The proof is complete. □

Observing $x * y \leq x * (y * (y * x))$ and using Lemma 3.3, we have $\tilde{\mu}_A(x * (y * (y * x))) \ll \tilde{\mu}_A(x * y)$ and $\nu_A(x * (y * (y * x))) \geq \nu_A(x * y)$ for all $x, y \in X$.

Hence, Theorem 4.6 can be improved as follows.

Theorem 4.7. *A cubic ideal A of X is a cubic commutative ideal of X if and only if it satisfies the conditions $\tilde{\mu}_A(x*(y*(y*x))) = \tilde{\mu}_A(x*y)$ and $\nu_A(x*(y*(y*x))) = \nu_A(x*y)$ for all $x, y \in X$.*

In the following theorem, we can see that the converse of Theorem 4.3 also holds in a commutative BCK-algebra.

Theorem 4.8. *In a commutative BCK-algebra X, every cubic ideal is a cubic commutative ideal.*

Proof. Let A be a cubic ideal of a commutative BCK-algebra X. It is sufficient to show that A satisfies conditions $(T5)$ and $(T6)$. Now

$$\begin{aligned}
&((x*(y*(y*x)))*((x*y)*z))*z \\
&= ((x*(y*(y*x)))*z)*((x*y)*z) \\
&\leq (x*(y*(y*x)))*(x*y) \\
&= (x*(x*y))*(y*(y*x)) \\
&= 0,
\end{aligned}$$

for all $x, y, z \in X$. Thus, $(x*(y*(y*x)))*((x*y)*z) \leq z$. It follows from Proposition 3.5 that

$$\begin{aligned}
\tilde{\mu}_A(x*(y*(y*x))) &\gg \mathrm{rmin}\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\}, \\
\nu_A(x*(y*(y*x))) &\leq \max\{\nu_A((x*y)*z), \nu_A(z)\}.
\end{aligned}$$

Hence, A is a cubic commutative ideal of X. □

Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic set in X. For any $r \in [0,1]$ and $[s,t] \in D[0,1]$, we define $U(A;[s,t],r)$ as follows

$$U(A;[s,t],r) = \{x \in X \mid \tilde{\mu}_A(x) \gg [s,t], \nu_A(x) \leq r\}$$

and say it is a cubic level set of $A = (\tilde{\mu}_A, \nu_A)$.

Theorem 4.9. *For a cubic set A in X, the following are equivalent.*
  (i) A is a cubic commutative ideal of X.
  (ii) Every nonempty cubic level set of A is a commutative ideal of X.

Proof. Assume that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic commutative ideal of X. Let $x, y \in X$, $r \in [0,1]$ and $[s,t] \in D[0,1]$. If $x \in U(A;[s,t],r)$, then $\tilde{\mu}_A(0) \gg \tilde{\mu}_A(x) \gg [s,t]$ and $\nu_A(0) \leq \nu_A(x) \leq r$. Thus, $0 \in U(A;[s,t],r)$. Let $x, y, z \in X$ be such that $(x*y)*z \in U(A;[s,t],r)$ and $z \in U(A;[s,t],r)$. Then $\tilde{\mu}_A((x*y)*z) \gg [s,t]$, $\nu_A((x*y)*z) \leq r$, $\tilde{\mu}_A(z) \gg [s,t]$ and $\nu_A(z) \leq r$. It follows from $(T5)$ and $(T6)$ that $\tilde{\mu}_A(x*(y*(y*x))) \gg \mathrm{rmin}\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\} \gg \mathrm{rmin}\{[s,t],[s,t]\} = [s,t]$ and $\nu_A(x*(y*(y*x))) \leq \max\{\mu_A((x*y)*z), \mu_A(z)\} \leq \{r, r\} = r$. Therefore, $x*(y*(y*x)) \in U(A;[s,t],r)$. Hence, $U(A;[s,t],r)$ is a commutative ideal of X.

Conversely, suppose that (ii) is valid, that is, $U(A;[s,t],r)$ is non-empty and is a commutative ideal of X for all $r \in [0,1]$ and $[s,t] \in D[0,1]$. Let $\tilde{\mu}_A(x) = [s,t]$ and $\nu_A(y) = r$ for any $x, y \in X$. Since $0 \in U(A;[s,t],r)$, we have $\tilde{\mu}_A(0) \gg [s,t] = \tilde{\mu}_A(x)$ and $\nu_A(0) \leq r = \nu_A(x)$ for all $x \in X$. Hence, $(T1)$ and $(T2)$ hold.

For any $x, y, z \in X$, let

$$\begin{aligned}&\operatorname{rmin}\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\} := [s,t],\\ &\max\{\nu_A((x*y)*z), \nu_A(z)\} := r.\end{aligned}$$

Then $\tilde{\mu}_A((x*y)*z) \gg [s,t]$, $\tilde{\mu}_A(z) \gg [s,t]$, $\nu_A((x*y)*z) \leq r$, $\nu_A(z) \leq r$, that is, $(x*y)*z \in U(A;[s,t],r)$ and $z \in U(A;[s,t],r)$. It follows from hypothesis that $x*(y*(y*x)) \in U(A;[s,t],r)$. Thus,

$$\begin{aligned}&\tilde{\mu}_A(x*(y*(y*x))) \gg [s,t] = \operatorname{rmin}\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\},\\ &\nu_A(x*(y*(y*x))) \leq r = \max\{\nu_A((x*y)*z), \nu_A(z)\}.\end{aligned}$$

Therefore, A is a cubic commutative ideal of X. $\square$

Theorem 4.10. *If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic commutative ideal of X, then the set*

$$I_A = \{x \in X | \tilde{\mu}_A(x) = \tilde{\mu}_A(0), \nu_A(x) = \nu_A(0)\}$$

is a commutative ideal of X.

Proof. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic commutative ideal of X. Then it is obvious that $0 \in I_A$. Let $x, y, z \in X$ such that $(x*y)*z \in I_A$ and $z \in I_A$. Then $\tilde{\mu}_A((x*y)*z) = \tilde{\mu}_A(0) = \tilde{\mu}_A(z)$ and $\nu_A((x*y)*z) = \nu_A(0) = \nu_A(z)$, and so,

$$\begin{aligned}&\tilde{\mu}_A(x*(y*(y*x))) \gg \operatorname{rmin}\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\} = \tilde{\mu}_A(0),\\ &\nu_A(x*(y*(y*x))) \leq \max\{\nu_A((x*y)*z), \nu_A(z)\} = \nu_A(0).\end{aligned}$$

It follows from $(T1)$ and $(T2)$ that $\tilde{\mu}_A(x*(y*(y*x))) = \tilde{\mu}_A(0)$ and $\nu_A(x*(y*(y*x))) = \nu_A(0)$ so that $x*(y*(y*x)) \in I$. Therefore, I_A is a commutative ideal of X. $\square$

Theorem 4.11. *If P is a commutative ideal of X, then there is a cubic commutative ideal $A = (\tilde{\mu}_A, \nu_A)$ of X such that $U(A;[s,t],r) = P$ for any $r \in [0,1]$ and $[s,t] \in D[0,1]$.*

Proof. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic set in X defined by

$$\tilde{\mu}_A(x) = \begin{cases}[s,t], & \text{if } x \in P;\\ [0,0], & \text{otherwise;}\end{cases} \quad \text{and } \nu_A(x) = \begin{cases}0, & \text{if } x \in P;\\ r, & \text{otherwise.}\end{cases}$$

Now we aim to verify that A is a cubic commutative ideal of X.

We will divide the following cases to verify that A is a cubic commutative ideal of X.

Case I. If $(x*y)*z \in P$ and $z \in P$, then $x*(y*(y*x)) \in P$ by (I_3); hence,

$$\tilde{\mu}_A((x*y)*z) = \tilde{\mu}_A(z) = \tilde{\mu}_A(x*(y*(y*x))) = [s,t],$$
$$\nu_A((x*y)*z) = \nu_A(z) = \nu_A(x*(y*(y*x))) = r,$$

and so

$$\tilde{\mu}_A(x*(y*(y*x))) = \mathrm{rmin}\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\},$$
$$\nu_A(x*(y*(y*x))) = \max\{\nu_A((x*y)*z), \nu_A(z)\}.$$

Case II. If $(x*y)*z \notin P$ and $z \notin P$, then

$$\tilde{\mu}_A((x*y)*z) = \tilde{\mu}_A(z) = 0,$$
$$\nu_A((x*y)*z) = \nu_A(z) = 0.$$

Hence,

$$\tilde{\mu}_A(x*(y*(y*x))) \gg \mathrm{rmin}\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(z)\},$$
$$\nu_A(x*(y*(y*x))) \leq \max\{\nu_A((x*y)*z), \nu_A(z)\}.$$

Case III. If exactly one of $(x*y)*z$ and z is not in P, then

exactly one of $\tilde{\mu}_A((x*y)*z)$ and $\tilde{\mu}_A(z)$ is equal to 0,
exactly one of $\nu_A((x*y)*z)$ and $\nu_A(z)$ is equal to 0.

Hence, condition $(T5)$ and $(T6)$ are satisfied.

Summarizing the above three cases, we know that $(T5)$ and $(T6)$ hold for all $x, y, z \in X$. Since $0 \in I$, it is clear that $\tilde{\mu}_A(0) = [s,t] \gg \tilde{\mu}_A(x)$, $\nu_A(0) = r \leq \nu_A(x)$, for all $x \in X$. Thus, condition $(T1)$ and $(T2)$ holds. Therefore, A is a cubic commutative ideal of X. Obviously, $U(A;[s,t],r) = P$. The proof is complete. □

5. Cubic Extension Property

Theorem 5.1. [8] *Let I and A be ideals of X with $I \subseteq A$. If I is a commutative ideal, then so is A.*

Definition 5.2. *Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be two cubic sets of X. Then $B = (\tilde{\mu}_B, \nu_B)$ is called cubic extension of $A = (\tilde{\mu}_A, \nu_A)$, denoted by $A \lesssim B$, if $\tilde{\mu}_A(x) \ll \tilde{\mu}_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$.*

We next give a cubic extension of cubic commutative ideals.

Theorem 5.3. *Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be cubic ideals of X such that $A \lesssim B$, $\tilde{\mu}_A(0) = \tilde{\mu}_B(0)$, and $\nu_A(0) = \nu_B(0)$. If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic commutative ideal of X, then so is $B = (\tilde{\mu}_B, \nu_B)$.*

Proof. To prove that $B = (\tilde{\mu}_B, \nu_B)$ is a cubic commutative ideal of X it sufficient to show that for any $[s,t] \in D[0,1]$ and $r \in [0,1]$, $U(B;[s,t],r) = \{x \in X \mid \tilde{\mu}_B(x) \geq [s,t], \nu_B(x) \leq r\}$ is either empty or a commutative ideal of X. Suppose $U(B;[s,t],r)$ is non-empty and $A \lesssim B$. In fact, if $x \in U(A;[s,t],r)$ then $\tilde{\mu}_A(x) \gg [s,t]$ and $\nu_A(x) \leq r$. Hence, $\tilde{\mu}_B(x) \gg [s,t]$ and $\nu_B(x) \leq r$, that is, $x \in U(A;[s,t],r)$. So, $U(A;[s,t],r) \subseteq U(B;[s,t],r)$. By the hypothesis, $A = (\tilde{\mu}_A, \nu_A)$ is a cubic commutative ideal of X. It follows from Theorem 4.9 that the set $U(A;[s,t],r)$ is a commutative ideal of X. By Theorem 5.1, $U(B;[s,t],r)$ is also a commutative ideal of X. Hence, by using Theorem 4.9, we get that $B = (\tilde{\mu}_B, \nu_B)$ is a cubic commutative ideal of X. The proof is complete. □

6. Images and Preimages of Cubic Commutative Ideals

Throughout this section, we always use X and Y to denote the BCK-algebras.

Definition 6.1. *Let f be a mapping from a set X into a set Y. Let $B = (\tilde{\mu}_B, \nu_B)$ be a cubic set in Y. Then the inverse image of B is defined as $f^{-1}(B) = (f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B))$ of B, where $f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x))$ and $f^{-1}(\nu_B)(x) = \nu_B(f(x))$.*

Theorem 6.2. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If $B = (\tilde{\mu}_B, \nu_B)$ is a cubic commutative ideal of Y, then the preimage $f^{-1}(B)$ of B under f is a cubic commutative ideal of X.*

Proof. Assume that $B = (\tilde{\mu}_B, \nu_B)$ is a cubic commutative ideal of Y. For all $x \in X$, $f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x)) \leq \tilde{\mu}_B(0) = \tilde{\mu}_B(f(0)) = f^{-1}(\tilde{\mu}_B)(0)$ and $f^{-1}(\nu_B)(x) = \nu_B(f(x)) \gg \nu_B(0) = \nu_B(f(0)) = f^{-1}(\nu_B)(0)$.

Let $x, y, z \in X$. Then $f^{-1}(\tilde{\mu}_B)(x*(y*(y*x))) = \tilde{\mu}_B(f(x*(y*(y*x)))) = \tilde{\mu}_B(f(x)*(f(y)*(f(y)*f(x)))) \gg \mathrm{rmin}\{\tilde{\mu}_B((f(x)*f(y))*f(z)), \tilde{\mu}_B(f(z))\} = \mathrm{rmin}\{\tilde{\mu}_B(f((x*y)*z), \tilde{\mu}_B(f(z))\} = \mathrm{rmin}\{f^{-1}(\tilde{\mu}_B)((x*y)*z), f^{-1}(\tilde{\mu}_B)(z)\}$ and $f^{-1}(\nu_B)(x*(y*(y*x))) = \nu_B(f(x*(y*(y*x)))) = \nu_B(f(x)*(f(y)*(f(y)*f(x)))) \leq \max\{\nu_B((f(x)*f(y))*f(z)), \nu_B(f(z))\} = \max\{\nu_B(f((x*y)*z), \nu_B(f(z))\} = \max\{f^{-1}(\nu_B)((x*y)*z), f^{-1}(\nu_B)(z)\}$. Hence, $f^{-1}(B)$ is a cubic commutative ideal of X. □

Definition 6.3. *A cubic set $A = (\tilde{\mu}_A, \nu_A)$ of X has rsup-property and infimum property if for any T of X there exist $t_0 \in T$ such that $\tilde{\mu}_A(t_0) = \mathrm{rsup}_{t_0 \in T} \tilde{\mu}_A(t)$ and $\nu_A(t_0) = \inf\limits_{t_0 \in T} \nu_A(t)$, respectively.*

Definition 6.4. *Let f be a mapping from the set X to the set Y. If $A = (\tilde{\mu}_A, \nu_A)$ is cubic set in X, then the cubic subset $B = (\tilde{\mu}_B, \nu_B)$ of Y is defined as*

$$f(\tilde{\mu}_A)(y) = \tilde{\mu}_B(y) = \begin{cases} \mathrm{rsup}_{x\in f^{-1}(y)}\tilde{\mu}_A(x); & \text{if } f^{-1}(y) \neq \emptyset; \\ [0,0]; & \text{otherwise;} \end{cases}$$

and

$$f(\nu_A)(y) = \nu_B(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \nu_A(x); & \text{if } f^{-1}(y) \neq \emptyset; \\ 1; & \text{otherwise;} \end{cases}$$

are said to be the images of $A = (\tilde{\mu}_A, \nu_A)$ under f.

Theorem 6.5. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic commutative ideal of X, then the image $B = (\tilde{\mu}_B, \nu_B)$ of A under f is a cubic commutative ideal of Y.*

Proof. Let A be a cubic commutative ideal of X with rsup-property and infimum property and B be the images of A under f. Since A is a cubic commutative ideal it must be a cubic ideal by Theorem 4.3. Therefore, we have $\tilde{\mu}_A(0) \gg \tilde{\mu}_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$.

Note that $0 \in f^{-l}(0')$, where 0 and $0'$ are the zero elements of X and Y, respectively. Thus, $\tilde{\mu}_B(0') = \mathrm{rsup}_{t\in f^{-1}(0')}\tilde{\mu}_A(t) = \tilde{\mu}_A(0) \gg \tilde{\mu}_A(x)$ and $\nu_B(0') = \inf\limits_{t\in f^{-1}(0')} \nu_A(t) = \nu_A(0) \leq \nu_A(x)$ for all $x \in X$, which implies that $\tilde{\mu}_B(0') \gg \mathrm{rsup}_{t\in f^{-1}(x')}\tilde{\mu}_A(t) = \tilde{\mu}_B(x')$ and $\nu_B(0') \leq \inf\limits_{t\in f^{-1}(x')} \nu_A(t) = \nu_B(x')$ for any $x' \in Y$.

For any $x', y', z' \in Y$, let $x_0 \in f^{-1}(x')$, $y_0 \in f^{-1}(y')$ and $z_0 \in f^{-1}(z')$ be such that $\tilde{\mu}_A(x_0) = \mathrm{rsup}_{t\in f^{-1}(x')}\tilde{\mu}_A(t)$, $\nu_A(x_0) = \inf\limits_{t\in f^{-1}(x')} \nu_A(t)$, $\tilde{\mu}_A(z_0) = \mathrm{rsup}_{t\in f^{-1}(z')}\tilde{\mu}_A(t)$, $\nu_A(z_0) = \inf\limits_{t\in f^{-1}(z')} \nu_A(t)$, $\tilde{\mu}_A((x_0 * y_0) * z_0) = \tilde{\mu}_B[f((x_0 * y_0) * z_0)] = \tilde{\mu}_B((x' * y') * z') = \mathrm{rsup}_{((x_0*y_0)*z_0)\in f^{-1}((x'*y')*z')}\tilde{\mu}_A((x_0 * y_0) * z_0) = \mathrm{rsup}_{t\in f^{-1}((x'*y')*z')}\tilde{\mu}_A(t)$ and $\nu_A((x_0 * y_0) * z_0) = \nu_B[f((x_0 * y_0) * z_0)] = \nu_B((x' * y') * z') = \inf\limits_{((x_0*y_0)*z_0)\in f^{-1}((x'*y')*z')} \nu_A((x_0 * y_0) * z_0) = \inf\limits_{t\in f^{-1}((x'*y')*z')} \nu_A(t)$. Then

$$\begin{aligned} \tilde{\mu}_B(x' * (y' * (y' * x'))) &= \mathrm{rsup}_{t\in f^{-1}(x'*(y'*(y'*x')))}\tilde{\mu}_A(t) \\ &= \tilde{\mu}_A(x_0 * (y_0 * (y_0 * x_0))) \\ &\gg \mathrm{rmin}\{\tilde{\mu}_A((x_0 * y_0) * z_0), \tilde{\mu}_A(z_0)\} \\ &= \mathrm{rmin}\{\mathrm{rsup}_{t\in f^{-1}((x'*y')*z')}\tilde{\mu}_A(t), \\ &\qquad \mathrm{rsup}_{t\in f^{-1}(z')}\tilde{\mu}_A(t)\} \\ &= \mathrm{rmin}\{\tilde{\mu}_B((x' * y') * z'), \tilde{\mu}_B(z')\} \end{aligned}$$

and

$$\begin{aligned}\nu_B(x' * (y' * (y' * x'))) &= \inf_{t \in f^{-1}(x' * (y' * (y' * x')))} \nu_A(t) \\ &= \nu_A(x_0 * (y_0 * (y_0 * x_0))) \\ &\leq \max\{\nu_A((x_0 * y_0) * z_0), \nu_A(z_0)\} \\ &= \max\Big\{ \inf_{t \in f^{-1}((x' * y') * z')} \nu_A(t), \\ &\qquad \inf_{t \in f^{-1}(z')} \nu_A(t) \Big\} \\ &= \max\{\nu_B((x' * y') * z'), \nu_B(z')\}.\end{aligned}$$

Hence, $B = (\tilde{\mu}_B, \nu_B)$ is a cubic commutative ideal of Y. □

7. Product of Cubic Commutative Ideals of BCK-Algebras

In this section, the products of cubic BCK-algebras are defined and considered. We obtain some new results for this topic.

Definition 7.1. *Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be two cubic sets of X and Y, respectively. The cartesian product $A \times B = (X \times Y, \tilde{\mu}_A \times \tilde{\mu}_B, \nu_A \times \nu_B)$ is defined by $(\tilde{\mu}_A \times \tilde{\mu}_B)(x, y) = \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\}$ and $(\nu_A \times \nu_B)(x, y) = \max\{\nu_A(x), \nu_B(y)\}$, where $\tilde{\mu}_A \times \tilde{\mu}_B : X \times Y \to D[0, 1]$ and $\nu_A \times \nu_B : X \times Y \to [0, 1]$ for all $(x, y) \in X \times Y$.*

Remark 7.2. *Let X and Y be BCK-algebras. We define $*$ on $X \times Y$ by $(x, y) * (z, p) = (x * z, y * p)$ for every (x, y) and $(z, p) \in X \times Y$. Then it is clear that $X \times Y$ is a BCK-algebra.*

Definition 7.3. *A cubic subset $A \times B = (X \times Y, \tilde{\mu}_A \times \tilde{\mu}_B, \nu_A \times \nu_B)$ is called a cubic commutative ideal if*
($T7$) $(\tilde{\mu}_A \times \tilde{\mu}_B)(0, 0) \gg (\tilde{\mu}_A \times \tilde{\mu}_B)(x, y)$, $(\nu_A \times \nu_B)(0, 0) \leq (\nu_A \times \nu_B)(x, y)$ for all $(x, y) \in X \times Y$;
($T8$) $(\tilde{\mu}_A \times \tilde{\mu}_B)((x_1, y_1) * ((x_2, y_2) * ((x_2, y_2) * (x_1, y_1)))) \gg \mathrm{rmin}\{(\tilde{\mu}_A \times \tilde{\mu}_B)(((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)), (\tilde{\mu}_A \times \tilde{\mu}_B)(x_3, y_3)\}$ and
($T9$) $(\nu_A \times \nu_B)((x_1, y_1) * ((x_2, y_2) * ((x_2, y_2) * (x_1, y_1)))) \leq \max\{(\nu_A \times \nu_B)(((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)), (\nu_A \times \nu_B)(x_3, y_3)\}$,
for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.

Theorem 7.4. *Let A and B be cubic commutative ideals of X and Y, respectively. Then $A \times B$ is a cubic commutative ideal of $X \times Y$.*

Proof. For any $(x, y) \in X \times Y$, we have

$$\begin{aligned}(\tilde{\mu}_A \times \tilde{\mu}_B)(0, 0) &= \mathrm{rmin}\{\tilde{\mu}_A(0), \tilde{\mu}_B(0)\} \\ &\gg \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\} = (\tilde{\mu}_A \times \tilde{\mu}_B)(x, y),\end{aligned}$$

and

$$\begin{aligned}(\nu_A \times \nu_B)(0,0) &= \max\{\nu_A(0), \nu_B(0)\} \leq \max\{\nu_A(x), \nu_B(y)\}\\ &= (\nu_A \times \nu_B)(x,y).\end{aligned}$$

Let (x_1, y_1), (x_2, y_2) and $(x_3, y_3) \in X \times Y$. Then
$(\tilde{\mu}_A \times \tilde{\mu}_B)((x_1, y_1) * ((x_2, y_2) * ((x_2, y_2) * (x_1, y_1))))$
$= (\tilde{\mu}_A \times \tilde{\mu}_B)((x_1 * (x_2 * (x_2 * x_1))), (y_1 * (y_2 * (y_2 * y_1))))$
$= \mathrm{rmin}\{\tilde{\mu}_A(x_1 * (x_2 * (x_2 * x_1))), \tilde{\mu}_B(y_1 * (y_2 * (y_2 * y_1)))\}$
$\gg \mathrm{rmin}\{\mathrm{rmin}\{\tilde{\mu}_A((x_1 * x_2) * x_3), \tilde{\mu}_A(x_3)\}, \mathrm{rmin}\{\tilde{\mu}_B((y_1 * y_2) * y_3), \tilde{\mu}_B(y_3)\}\}$
$= \mathrm{rmin}\{\mathrm{rmin}\{\tilde{\mu}_A((x_1 * x_2) * x_3), \tilde{\mu}_B((y_1 * y_2) * y_3)\}, \mathrm{rmin}\{\tilde{\mu}_A(x_3), \tilde{\mu}_B(y_3)\}\}$
$= \mathrm{rmin}\{(\tilde{\mu}_A \times \tilde{\mu}_B)(((x_1 * x_2) * x_3), ((y_1 * y_2) * y_3)), (\tilde{\mu}_A \times \tilde{\mu}_B)(x_3, y_3)\}$
$= \mathrm{rmin}\{(\tilde{\mu}_A \times \tilde{\mu}_B)(((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)), (\tilde{\mu}_A \times \tilde{\mu}_B)(x_3, y_3)\}$ and
$(\nu_A \times \nu_B)((x_1, y_1) * ((x_2, y_2) * ((x_2, y_2) * (x_1, y_1))))$
$= (\nu_A \times \nu_B)((x_1 * (x_2 * (x_2 * x_1))), (y_1 * (y_2 * (y_2 * y_1))))$
$= \max\{\nu_A(x_1 * (x_2 * (x_2 * x_1))), \nu_B(y_1 * (y_2 * (y_2 * y_1)))\}$
$\leq \max\{\max\{\nu_A((x_1 * x_2) * x_3), \nu_A(x_3)\}, \max\{\nu_B((y_1 * y_2) * y_3), \nu_B(y_3)\}\}$
$= \max\{\max\{\nu_A((x_1 * x_2) * x_3), \nu_B((y_1 * y_2) * y_3)\}, \max\{\nu_A(x_3), \nu_B(y_3)\}\}$
$= \max\{(\nu_A \times \nu_B)(((x_1 * x_2) * x_3), ((y_1 * y_2) * y_3)), (\nu_A \times \nu_B)(x_3, y_3)\}$
$= \max\{(\nu_A \times \nu_B)(((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)), (\nu_A \times \nu_B)(x_3, y_3)\}$.
Hence, $A \times B$ is a cubic commutative ideal of $X \times Y$. □

Definition 7.5. *Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be cubic subset of X and Y, respectively. For $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, the set $U(\tilde{\mu}_A \times \tilde{\mu}_B : [s_1, s_2]) = \{(x,y) \in X \times Y | (\tilde{\mu}_A \times \tilde{\mu}_B)(x,y) \gg [s_1, s_2]\}$ is called an upper $[s_1, s_2]$-level of $A \times B$ and $L(\nu_A \times \nu_B : t) = \{(x,y) \in X \times Y | (\nu_A \times \nu_B)(x,y) \leq t\}$ is called a lower t-level of $A \times B$.*

Theorem 7.6. *For any two cubic sets A and B, $A \times B$ is a cubic commutative ideals of $X \times Y$ if and only if the non-empty upper $[s_1, s_2]$-level cut $U(\tilde{\mu}_A \times \tilde{\mu}_B : [s_1, s_2])$ and the non-empty lower t-level cut $L(\nu_A \times \nu_B : t)$ are commutative ideals of $X \times Y$ for any $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$.*

Proof. The proof is straightforward. □

8. Relationship with (Cubic) Implictive Ideals and Positive Implicative Ideals

Proposition 8.1. *In a BCK-algebra X the following all hold for all $x, y, z \in X$.*

(i) $((x * z) * z) * (y * z) \leq (x * y) * z$.
(ii) $(x * z) * (x * (x * z)) = (x * z) * z$.
(iii) $(x * (y * (y * x))) * (y * (x * (y * (y * x)))) \leq x * y$.

Proof. See the proof of [9, Theorems 9 and 16]. □

Definition 8.2. [13] *A cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X is called a cubic implicative ideal of X if it satisfies $(T1)$, $(T2)$ and*

$$(T10)\quad \tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$$

$$(T11)\quad \nu_A(x) \leq \max\{\nu_A((x*(y*x))*z), \nu_A(z)\}$$

for all $x, y, z \in X$.

Theorem 8.3. [13] *Suppose that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic ideal of X. Then the following are equivalent.*

(i) A is a cubic implicative ideal of X.

(ii) $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(x*(y*x))$ and $\nu_A(x) \leq \nu_A(x*(y*x))$ for all $x, y \in X$.

(iii) $\tilde{\mu}_A(x) = \tilde{\mu}_A(x*(y*x))$ and $\nu_A(x) = \nu_A(x*(y*x))$ for all $x, y \in X$.

Definition 8.4. [14] *A cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X is called a cubic positive implicative ideal of X if it satisfies $(T1)$, $(T2)$, and*

$$(T12)\quad \tilde{\mu}_A(x*z) \geq rmin\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(y*z)\}$$

$$(T13)\quad \nu_A(x*z) \leq \max\{\nu_A((x*y)*z), \nu_A(y*z)\}$$

for all $x, y, z \in X$.

Theorem 8.5. [14] *A cubic ideal of X is called a cubic positive implicative ideal of X if and only if it satisfies the conditions $\tilde{\mu}_A(x*y) \gg \tilde{\mu}_A((x*y)*y)$ and $\nu_A(x*y) \geq \nu_A((x*y)*y)$ for all $x, y \in X$.*

Observing $(x*y)*y \leq x*y$ and using Lemma 3.3, we have $\tilde{\mu}_A(x*y) \ll \tilde{\mu}_A((x*y)*y)$ and $\nu_A(x*y) \leq \nu_A((x*y)*y)$ for all $x, y \in X$. Hence, Theorem 8.5 can be improved as follows.

Theorem 8.6. *A cubic ideal $A = (\tilde{\mu}_A, \nu_A)$ of X is a cubic positive implicative ideal of X if and only if it satisfies the identity $\tilde{\mu}_A(x*y) = \tilde{\mu}_A((x*y)*y)$ and $\nu_A(x*y) = \nu_A((x*y)*y)$ for all $x, y \in X$.*

We now describe relationship between cubic commutative ideals, cubic implicative ideals, and cubic positive implicative ideals.

Theorem 8.7. *A cubic ideal $A = (\tilde{\mu}_A, \nu_A)$ of X is cubic implicative ideal if and only if $A = (\tilde{\mu}_A, \nu_A)$ is both cubic commutative ideal and cubic positive implicative ideal.*

Proof. Assume $A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X. For all $x, y, z \in X$, we have

$$\begin{aligned}\tilde{\mu}_A(x*z) &= \tilde{\mu}_A((x*z)*(x*(x*z))) \text{ by Theorem 8.3 (iii)}\\ &= \tilde{\mu}_A((x*z)*z) \text{ by Proposition 8.1 (ii)}\\ &\gg \mathrm{rmin}\{\tilde{\mu}_A((x*y)*z), \tilde{\mu}_A(y*z)\} \text{ by Proposition 3.5, 8.1(i).}\end{aligned}$$

Similarly $\nu_A(x*z) \geq \min\{\nu_A((x*y)*z), \nu_A(y*z)\}$. Hence, $A = (\tilde{\mu}_A, \nu_A)$ is a cubic positive implicative ideal of X.

By using Lemma 3.3, Theorem 8.3 (iii), and Proposition 8.1 (iii), we get $\tilde{\mu}_A(x*y) \ll \tilde{\mu}_A((x*(y*(y*x)))*(y*(x*(y*(y*x))))) = \tilde{\mu}_A(x*(y*(y*x)))$, $\nu_A(x*y) \geq \nu_A((x*(y*(y*x)))*(y*(x*(y*(y*x))))) = \nu_A(x*(y*(y*x)))$, for all $x, y \in X$. It follows from Theorem 4.6 that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic commutative ideal of X.

Conversely, suppose that $A = (\tilde{\mu}_A, \nu_A)$ is both a cubic positive implicative ideal and a cubic commutative ideal. Since $(y*(y*x))*(y*x) \leq x*(y*x)$, it follows from Lemma 3.3 that $\tilde{\mu}_A(x*(y*x)) \ll \tilde{\mu}_A((y*(y*x))*(y*x))$ and $\nu_A(x*(y*x)) \leq \nu_A((y*(y*x))*(y*x))$. Using Theorem 8.6, we have $\tilde{\mu}_A((y*(y*x))*(y*x)) = \tilde{\mu}_A(y*(y*x))$, $\nu_A((y*(y*x))*(y*x)) = \nu_A(y*(y*x))$, and so

$$\left.\begin{aligned} &\tilde{\mu}_A(x*(y*x)) \ll \tilde{\mu}_A(y*(y*x)), \\ &\nu_A(x*(y*x)) \geq \nu_A(y*(y*x)). \end{aligned}\right\} \tag{8.1}$$

On the other hand, since $x*y \leq x*(y*x)$, we have $\tilde{\mu}_A(x*(yx)) \ll \tilde{\mu}_A(x*y)$ and $\nu_A(x*(yx)) \leq \nu_A(x*y)$, by Lemma 3.3. Since $A = (\tilde{\mu}_A, \nu_A)$ is a cubic commutative ideal of X, by Theorem 4.7 we have $\tilde{\mu}_A(x*y) = \tilde{\mu}_A(x*(y*(y*x)))$ and $\nu_A(x*y) = \nu_A(x*(y*(y*x)))$; hence, $\tilde{\mu}_A(x*(y*x)) \ll \tilde{\mu}_A(x*(y*(y*x)))$ and $\nu_A(x*(y*x)) \leq \nu_A(x*(y*(y*x)))$. Combining (8.1) we obtain

$$\begin{aligned} &\tilde{\mu}_A(x) \gg \mathrm{rmin}\{\tilde{\mu}_A(x*(y*(y*x))), \tilde{\mu}_A(y*(y*x))\} \gg \tilde{\mu}_A(x*(y*x)), \\ &\nu_A(x) \geq \min\{\nu_A(x*(y*(y*x))), \nu_A(y*(y*x))\} \geq \nu_A(x*(y*x)). \end{aligned}$$

Hence, $A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X by Theorem 8.3 (ii). The proof is complete. □

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