SOCCER BALLS, GOLF BALLS, AND THE EULER IDENTITY

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Abstract. We show, with simple combinatorics, that if the dimples on a golf ball are all 5-sided and 6-sided polygons, with three dimples at each "vertex", then no matter how many dimples there are and no matter the sizes and distribution of the dimples, there will always be exactly twelve 5-sided dimples. Of course, the same is true of a soccer ball and its faces.

What do a soccer ball and golf ball have in common? They are both spherical, but there is much more than that. When the second author played soccer sixty years ago, the ball consisted of a number of oblong brown panels sewn together. Today the surface of many soccer balls is covered with 12 black pentagons and 20 white hexagons (see Figure 1). For convenience, we also show a "flat" or "plane" image of a soccer ball in Figure 2. This is also known as the truncated icosahedral graph.

Figure 1. A common soccer ball.

The points at which three polygons, or *faces*, of the soccer ball meet are called *vertices*; the total number of vertices is denoted by V. By counting we find that $V = 60$. (Or we could note that there is exactly one pentagon at each vertex, so that $V = 12 \times 5$.) The line segments at which two faces meet are called *edges*, and the total number of edges is denoted by E. By laborious counting we find that $E = 90$. (We will find two easier ways to determine E below.) Since exactly three edges meet at any vertex, each

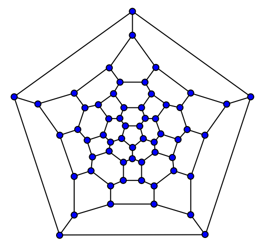

FIGURE 2. The plane image of a soccer ball.

vertex is said to have *degree* three. Finally, the total number of faces is $F = 12 + 20 = 32$. It follows that for such a soccer ball

$$V - E + F = 60 - 90 - 32 = 2.$$

In fact, this is guaranteed by Euler's identity, which states that

$$V - E + F = 2, \tag{1}$$

for all connected plane graphs with V vertices, E edges and F faces (see [1, Theorem 10.1], for example).

Since $V = 60$ and $E = 90$ in most soccer balls today,

$$2E = 3V \tag{2}$$

for such a ball. In fact, this is a special case of a more general theorem sometimes referred to as the "First Theorem of Graph Theory" (see [1, Theorem 1.4]); in summing the vertex degrees, each edge is counted twice — once at each end).

Finally, if F_5 gives the number of 5-sided faces on the soccer ball and F_6 the number of 6-sided faces, then $F_5 = 12$, $F_6 = 20$, and we have both

$$F = F_5 + F_6, \tag{3}$$

and

$$2E = 5F_5 + 6F_6. \tag{4}$$

Equation (4) holds because, in summing the face lengths, each edge is again counted twice — this time once on each side. For a more detailed discussion of the construction of soccer balls, see [2].

Perhaps surprisingly, conditions (1)–(4) hold also for many golf balls (and all those we consider). More specifically, dimples (faces) first appeared on golf balls more than a century ago when golfers discovered that golf balls that were beat up traveled farther than smooth balls. The nicks and scratches were acting as "turbulators", that is, were inducing turbulence in the layer of air next to the ball. Sometimes, a turbulent boundary layer reduces what golfers refer to as "drag", and so golf balls with dimples travel farther.

Although there is a great deal of science involved in creating the ideal distribution of dimples in a golf ball, we will concentrate on some of the counting and, more generally, the mathematics that arises.

Depending on the source, we are told that the total number of dimples on a golf ball ranges from 300 to 500, with 336 being a common number. The dimples are concave and might be round or polygonal (like the faces of many soccer balls). The most common golf balls (as observed by the first author's golfer spouse and our other golfer friends) have dimples that are a combination of 5-sided and 6-sided polygons, with three dimples at each vertex. In such golf balls, just as for a soccer ball, the number of 5-sided dimples is always exactly 12. Thus, all hexagons is impossible (in contrast to the infinite Euclidean plane) and all pentagons could hold only for a golf ball with exactly 12 dimples! Another such example, with only pentagons, is the dodecahedron, the Platonic solid with exactly 12 faces.

In golf balls for which hexagonal dimples dominate, we get something like the hexagonal tessellation of the plane — only now we are tessellating a sphere, and finitely. Thus it is natural to expect exactly three dimples at every vertex. (A graph for which all vertices have degree three is called a *cubic graph.*)

Now we prove that if a connected plane cubic graph has all faces either pentagons or hexagons, then there must be exactly twelve pentagons. We note that this surprising result follows as a special case of the dual of Corollary 6.10 of [1], but we prefer the following instructive proof for the present context. A good proof should make us wiser: Why exactly 12 pentagons???.

So, let G be a connected plane cubic graph having all faces either pentagons or hexagons. Our discussion so far has established that

$$V - E + F = 2, \tag{1}$$

$$2E = 3V, \tag{2}$$

$$F = F_5 + F_6, \tag{3}$$

and

$$2E = 5F_5 + 6F_6. \tag{4}$$

Consequently,

by (1) $6V - 6E + 6F = 12$, and so
by (2) $4E - 6E + 6F = 12$. Then
by (3) $-2E + 6(F_5 + F_6) = 12$, and finally
by (4) $-5F_5 - 6F_6 + 6F_5 + 6F_6 = 12$.

We conclude that $F_5 = 12$.

In other words, there will always be exactly 12 pentagons, no matter what the total number of 5-sided and 6-sided faces. This result applies to three common objects of our discussion: the dodecahedron, soccer balls, and golf balls. The dodecahedron has no hexagons and 12 pentagons, and the soccer ball has 20 hexagons and 12 pentagons. A golf ball with 336 dimples (all 5-sided or 6-sided) will have 324 hexagons and 12 pentagons, whereas 500 dimples would yield 488 hexagons and 12 pentagons. As long as the dimples on a golf ball are all 5-sided and 6-sided (with three at each vertex), no matter how many dimples there are, and no matter what the sizes of the dimples, the 12 pentagons are constant!

If the dimples were uniformly distributed around the spherical ball, they would appear as six antipodal pairs, in accordance with the vertices of an icosahedron — the dual of a dodecahedron — inscribed in our golf ball. It appears that manufacturers have not achieved this uniformity. Moreover, a fairly lengthy internet search, a few phone calls, and informal discussions seem to indicate few sales reps, golf ball manufacturers, or golfers are aware that all golf balls have $F_5 = 12$!

We conclude by commenting that retired Brown University and Hall of Fame soccer coach Cliff Stevenson introduced the now classic black and white soccer ball in the United States. Before leaving Oberlin for Brown in 1960, Cliff was the soccer coach of the second author for three years.

References

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