$(\in, \in \vee q)$-BIPOLAR FUZZY BCK/BCI-ALGEBRAS

CHIRANJIBE JANA, MADHUMANGAL PAL, AND
ARSHAM BORUMAND SAIED

ABSTRACT. In this paper, the concept of quasi-coincidence of a bipolar fuzzy point within a bipolar fuzzy set is introduced. The notion of $(\in, \in \vee q)$-bipolar fuzzy subalgebras and ideals of BCK/BCI-algebras are introduced and their related properties are investigated by some examples. We study bipolar fuzzy BCK/BCI-subalgebras and bipolar fuzzy BCK/BCI-ideals by their level subalgebras and level ideals. We also provide the relationship between $(\in, \in \vee q)$-bipolar fuzzy BCK/BCI-subalgebras and bipolar fuzzy BCK/BCI-subalgebras, and $(\in, \in \vee q)$-bipolar fuzzy BCK/BCI-ideals and bipolar fuzzy BCK/BCI-ideals by counter examples.

1. INTRODUCTION

In 1965, the concept of fuzzy sets, a remarkable idea in mathematics, was proposed by Zadeh [45]. In this traditional concept of fuzzy set, the membership degree expresses belongingness of an element to a fuzzy set. The membership degree of an element ranges over the interval $[0, 1]$. When the membership degree of an element is 1, then the element completely belongs to its corresponding fuzzy set, and the membership degree of an element is 0 means an element does not belong to the fuzzy set. Based on this tool, different fuzzy algebraic structures have been developed by many researchers, fuzzy BCK/BCI-algebras is one of them. The BCK/BCI-algebras are two classes of algebras of logic which was initiated by Imai and Iseki [8] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. The fuzzy structures of BCK/BCI-algebras worked out by many researchers such as Jun [18, 19, 23, 35], Liu [28], Lee [27], Bej and Pal [2], Jana et al. and others [8-16, 31] have done much investigations on $BCK/BCI/G/B$-algebras related to these algebras.

In 1994, the notion of bipolar fuzzy sets was proposed by Zhang [49, 50] as a generalization of fuzzy sets [45]. Bipolar-valued fuzzy sets [24, 25] are seen as an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets are similar, they are different sets as introduced by Lee [25]. Bipolar fuzzy sets have various applications in fuzzy algebras. For example, bipolar fuzzy ideals [1] in LA-semigroups, bipolar fuzzy subalgebras and ideals [26] of BCK/BCI-algebras, bipolar fuzzy a-ideals in BCK/BCI-algebras [27] and bipolar valued fuzzy BCK/BCI-algebras [42] are some of them.

In 2004, Murali [38] introduced the definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy subset. The quasi-coincidence of a fuzzy point to a fuzzy subset, as mentioned by Pu [39] (1980), played a vital role to derive some types of fuzzy subsystems. Bhakat and Das [3, 4] utilized the concept of (α, β)-fuzzy subgroups by using the 'belongs to' relation $(\in)$ and 'quasi-coincident with' relation (q) between a fuzzy point and a fuzzy subset. It is seen that $(\in, \in \vee q)$-fuzzy subgroups are an important generalization of Rosenfeld's [41] fuzzy subgroup. Similar types of generalizations have been made to the other algebraic structures by Zhan [46, 47, 48]. Jun et al. [14–16] introduced the concept of (α, β)-fuzzy subalgebras and ideals and investigated their related properties. Ma et al. [29, 30, 31, 32] introduced some kinds of $(\in, \in \vee q)$-interval-valued fuzzy ideals of BCI-algebras. In 2015, Muhiuddin et al. [36, 37] studied subalgebras of BCK/BCI-algebras based on (α, β)-type fuzzy sets. These works are enough to motivate us and, to the best of our knowledge, no other works are available on $(\in, \in \vee q)$-bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras and other fuzzy algebraic structures. For this reason we have developed the theoretical study of $(\in, \in \vee q)$-bipolar fuzzy BCK/BCI-subalgebras and $(\in, \in \vee q)$-bipolar fuzzy ideals of BCK/BCI-algebras.

The remainder of this article is structured as follows: Section 2 proceeds with a recapitulation of all required definitions and properties. In Section 3, concepts and operations of $(\in, \in \vee q)$-bipolar fuzzy BCK/BCI-subalgebras are introduced and properties are investigated. In Section 4, $(\in, \in \vee q)$-bipolar fuzzy ideals of BCK/BCI-algebras are proposed and their properties are discussed in detail. Finally, in Section 5, conclusions and the scope of future research is given.

2. Preliminaries

In this section, some elementary aspects necessary for this paper are included.

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$
(ii) $(x * (x * y)) * y = 0$
(iii) $x * x = 0$
(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We define a partial ordering "$\leq$ " by $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say X is a BCK-algebra. Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:
(1) $(x * y) * z = (x * z) * y$
(2) $((x * z) * (y * z)) * (x * y) = 0$
(3) $x * 0 = x$
(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0.$

Throughout this paper X always means a BCK/BCI-algebra without any specification.

A non-empty subset S of X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$. A nonempty subset I of X is called an ideal of X if it satisfies
(I_1) $0 \in I$ and,
(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

We refer the reader to the books Huang [7] and Meng [34] for further information regarding BCK/BCI-algebras. A fuzzy set A in a set X is of the form

$$\mu_A(y) = \begin{cases} t \in (0,1], & \text{if } y = x; \\ 0, & \text{if } y \neq x. \end{cases}$$

We denote a fuzzy point with support x and value t as x_t. For a fuzzy point x_t and a fuzzy set A of a set X, Pu and Liu [39] gave meaning to the symbol $x_t \Phi A$, where $\Phi \in \{\in, q, \in \vee q, \wedge q\}$. To say that $x_t \in A$ (respectively, $x_t q \mu$) means that $\mu_A(x) \geq t$ (respectively, $\mu_A(x) + t > 1$), and in this case, x_t is said to belong to (respectively, be quasi-coincident with) a fuzzy set A. To say that $x_t \in \vee q A$ (respectively, $x_t \in \wedge q A$) means that $x_t \in A$ or $x_t q A$ (respectively, $x_t \in A$ and $x_t q A$). To say that $x_t \overline{\phi} A$ means that $x_t \Phi A$ does not hold, where $\Phi \in \{\in, q, \in \vee q, \in \wedge q\}$.

A fuzzy set A in a BCK/BCI-algebra X is said to be a fuzzy subalgebra of X if it satisfies $\mu_A(x * y) \geq min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.

A fuzzy set A of X is said to be a fuzzy ideal of X if it satisfies (i) $\mu_A(0) \geq \mu_A(x))$ and (ii) $\mu_A(x) \geq \{\mu_A(x * y), \mu_A(y)\}$, for all $x, y \in X$.

Proposition 2.1. [18] *A fuzzy set A of X is called a fuzzy subalgebra of X if and only if it satisfies $x_t \in A, y_s \in A \Rightarrow (x * y)_{min(t,s)} \in A$ for all $x, y \in X$ and $t, s \in (0, 1]$.*

Proposition 2.2. [20] *A fuzzy set A of X is called a fuzzy ideal of X if and only if it satisfies (i) $x_t \in A \Rightarrow 0_t \in A$, (ii) $(x * y)_t \in A, y_s \in A \Rightarrow x_{min(t,s)} \in A$, for all $x, y \in X$ and $t, s \in (0, 1]$.*

Definition 2.3. [24] *A bipolar fuzzy set A of X is defined as*

$$A = \{(x, \mu_A^P(x), \mu_A^N(x)) : x \in X\}$$

where $\mu_A^P : X \to [0,1]$ and $\mu_A^N : X \to [-1,0]$ are mappings. The positive membership degree $\mu_A^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set $A = \{(x, \mu_A^P(x), \mu_A^N(x) : x \in X\}$ and the negative membership degree $\mu_A^N(x)$ denotes the satisfaction degree of an element x to some implicit counter property of $A = \{(x, \mu_A^P(x), \mu_A^N(x) : x \in X\}$. If $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) = 0$, this case is regarded as having only a positive satisfaction degree for $A = \{(x, \mu_A^P(x), \mu_A^N(x) : x \in X\}$. If $\mu_A^P(x) = 0$ and $\mu_A^N(x) \neq 0$, x does not satisfy the property of $A = \{(x, \mu_A^P(x), \mu_A^N(x) : x \in X\}$, but somewhat satisfies the counter-property of $A = \{(x, \mu_A^P(x), \mu_A^N(x) : x \in X\}$. In some cases it is possible for an element x to be $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property of its portion of domain (Lee [25]*). We shall simply use the symbol $A = (\mu_A^P, \mu_A^N)$ for the bipolar fuzzy set $A = \{(x, \mu_A^P(x), \mu_A^N(x)) | x \in X\}$.*

Definition 2.4. [49] *For every two bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ in X, we define*

$$(A \cup B)(x) = \{\max\{\mu_A^P(x), \mu_B^P(x)\}, \min\{\mu_A^N(x), \mu_B^N(x)\}\}$$
$$(A \cap B)(x) = \{\min\{\mu_A^P(x), \mu_B^P(x)\}, \max\{\mu_A^N(x), \mu_B^N(x)\}\}.$$

Proposition 2.5. [26] *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of X is called a bipolar fuzzy subalgebra of X if it satisfies $\mu_A^P(x * y) \geq min\{\mu_A^P(x), \mu_A^P(y)\}$ and $\mu_A^N(x * y) \leq max\{\mu_A^N(x), \mu_A^N(y)\}$ for all $x, y \in X$.*

Definition 2.6. [26] *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of X is called a bipolar fuzzy ideal of X if it satisfies the following assertions*
(i) $\mu_A^P(0) \geq \mu_A^P(x)$ *and* $\mu_A^N(0) \leq \mu_A^N(x)$
(ii) $\mu_A^P(x) \geq min\{\mu_A^P(x * y), \mu_A^P(y)\}$
(iii) $\mu_A^N(x) \leq max\{\mu_A^N(x * y), \mu_A^N(y)\}$ *for all $x, y \in X$.*

3. $(\in, \in \vee q)$-Bipolar Fuzzy BCK/BCI-Subalgebras

In this section, $(\in, \in \vee q)$-bipolar fuzzy subalgebras of BCK/BCI-algebras are defined and some important properties are presented.

Definition 3.1. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set in a set X of the form*

$$\mu_A^P(y) = \begin{cases} t \in (0,1], & \text{if } y = x; \\ 0, & \text{if } y \neq x. \end{cases}$$
$$\mu_A^N(y) = \begin{cases} m \in [-1,0), & \text{if } y = x; \\ 0, & \text{if } y \neq x. \end{cases}$$

A bipolar fuzzy point with support x and values t and m is denoted by $\langle x, t, m\rangle$. For a bipolar fuzzy point $\langle x, t, m\rangle$ and a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in a set X, we give meaning to the symbol $(x_t\Phi\mu_A^P,\ x_m\Phi\mu_A^N)$, where $\Phi \in \{\in, q, \in \vee q, \in \wedge q\}$. To say that $x_t \in \mu_A^P$ (respectively, $x_t q\mu_A^P$) and $x_m \in \mu_A^N$ (respectively, $x_m q\mu_A^N$) means that $\mu_A^P(x) \geq t$ (respectively, $\mu_A^P(x) + t > 1$) and $\mu_A^N(x) \leq m$ (respectively, $\mu_A^N(x) + m < -1$), and in this case we say that x_t is said to belong to (respectively, be quasi-coincident with) and x_m is said to belong to (respectively, be quasi-coincident with) a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$. To say that $x_t \in \vee q$ (respectively, $x_t \in \wedge q$) and $x_m \in \vee q$ (respectively, $x_m \in \wedge q$) means that $x_t \in \mu_A^P$ or $x_t q\mu_A^P$ (respectively, $x_t \in \mu_A^P$ and $x_t q\mu_A^P$) and $x_m \in \mu_A^N$ or $x_m q\mu_A^N$ (respectively, $x_m \in \mu_A^N$ and $x_m q\mu_A^N$). To say that $(x_t\overline{\Phi}\mu_A^P, x_m\overline{\Phi}\mu_A^N)$ means that $x_t\Phi\mu_A^P$ does not hold and $x_m\Phi\mu_A^N$ does not hold, where $\Phi \in \{\in, q, \in \vee q, \in \wedge q\}$.

Definition 3.2. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of X and $(m, t) \in [-1, 0] \times [0, 1]$, we define $U(\mu_A^P, \mu_A^N; t, m) = \{x \in X | \mu_A^P(x) \geq t$ and $\mu_A^N(x) \leq m\}$ is called a t-level cut of μ_A^P and m-level cut of μ_A^N of the bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$.*

Theorem 3.3. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of X is called a bipolar fuzzy subalgebra of X if and only if the following assertion is valid $x_t \in \mu_A^P, y_s \in \mu_A^P \Rightarrow (x * y)_{\min(t,s)} \in \mu_A^P$ and $x_m \in \mu_A^N, y_n \in \mu_A^N \Rightarrow (x * y)_{\max(m,n)} \in \mu_A^N$, for all $x, y \in X$, $t, s \in (0, 1]$ and $m, n \in [-1, 0)$.*

Proof. Assume that Proposition 2.5 is valid. Let $x, y \in X$ and $t, s \in (0, 1]$ and $m, n \in [-1, 0)$ be such that $x_t, y_s \in \mu_A^P$ and $x_m, y_n \in \mu_A^N$. Then $\mu_A^P(x) \geq t, \mu_A^P(y) \geq s$ and $\mu_A^N(x) \leq m, \mu_A^N(y) \leq n$ which imply, from Proposition 2.5, that $\mu_A^P(x * y) \geq \min\{\mu_A^P(x), \mu_A^P(y)\} \geq \min\{t, s\}$ and $\mu_A^N(x * y) \leq \max\{\mu_A^N(x), \mu_A^N(y)\} \leq \max\{m, n\}$. Hence, $(x * y) \in \mu_A^P$ and $(x * y) \in \mu_A^N$.

Assume that $x_{\mu_A^P(x)} \in \mu_A^P$ and $y_{\mu_A^P(y)} \in \mu_A^P$, and $x_{\mu_A^N(x)} \in \mu_A^N$ and $y_{\mu_A^N(y)} \in \mu_A^N$ hold for all $x, y \in X$. Then $(x * y)_{\min\{\mu_A^P(x), \mu_A^P(y)\}} \in \mu_A^P$ and $(x * y)_{\max\{\mu_A^N(x), \mu_A^N(y)\}} \in \mu_A^N$ by Theorem 3.3. Thus, $\mu_A^P(x * y) \geq \min\{\mu_A^P(x), \mu_A^P(y)\}$ and $\mu_A^N(x * y) \leq \max\{\mu_A^N(x), \mu_A^N(y)\}$ hold for all $x, y \in X$. Hence, the proof is completed. □

Definition 3.4. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of X is called an $(\in, \in \vee q)$-bipolar fuzzy subalgebras of X if it satisfies the following conditions*
(i) $x_t \in \mu_A^P, y_s \in \mu_A^P \Rightarrow (x * y)_{\min(t,s)} \in \vee q\mu_A^P$, for all $x, y \in X$ and $t, s \in (0, 1]$
(ii) $x_m \in \mu_A^N, y_n \in \mu_A^N \Rightarrow (x * y)_{\max(m,n)} \in \vee q\mu_A^N$, for all $x, y \in X$ and $m, n \in [-1, 0)$.

Example 3.5. *Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Caley Table 1 as follows.*

Table 1

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let A be a bipolar fuzzy set of X defined by $\mu_A^P(0) = 0.6$, $\mu_A^P(a) = 0.7$, $\mu_A^P(b) = \mu_A^P(c) = 0.3$ and $\mu_A^N(0) = -0.8$, $\mu_A^N(a) = \mu_A^N(c) = -0.3$, $\mu_A^N(b) = -0.7$. Then A is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X but not a bipolar fuzzy subalgebra of X as $\mu_A^P(a * a) = 0.6 \not\geq 0.7 = min\{\mu_A^P(a), \mu_A^P(a)\}$.

Corollary 3.6. *Theorem 3.3 shows that every $(\in, \in)$-bipolar fuzzy subalgebra is precisely a bipolar fuzzy subalgebra and vice versa. Obviously, every $(\in, \in)$-bipolar fuzzy subalgebra is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra.*

In general, the converse of the corollary is not true, justified by the following example.

Example 3.7. *Consider $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X which is given in example 3.5. It is seen that $A = (\mu_A^P, \mu^N)$ is not an $(\in, \in)$-bipolar fuzzy subalgebra of X because $(a, 0.63) \in \mu_A^P$ and $(a, 0.68) \in \mu^P$, but $(a * a, 0.63 \wedge 0.68) = (0, 0.63)\overline{\in}\mu_A^P$.*

Theorem 3.8. *Every $(\in, q)$-bipolar fuzzy subalgebra of X is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X.*

Proof. The proof of the theorem is straightforward. □

In general, the converse of the theorem 3.8 is not true by the following example.

Example 3.9. *Consider an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X given in Example 3.5 such that $(a, 0.6) \in \mu_A^P$ and $(c, 0.22) \in \mu^P$, but $(a * b, 0.6 \wedge 0.22) = (b, 0.22)\overline{q}\mu_A^P$ as $\mu_A^P(b) + 0.22 < 1$. Again, $(b, -0.25) \in \mu_A^N$ and $(c, -0.45) \in \mu_A^N$, since $(b * c, -0.25 \vee -0.45) = (a, -0.25)\overline{q}\mu_A^N$ as $\mu_A^N(a) - 0.25 > -1$. Therefore, $A = (\mu_A^P, \mu_A^N)$ is not an $(q, \in)$-bipolar fuzzy subalgebra of X.*

Theorem 3.10. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of X is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X if and only if it satisfies*

$$\mu_A^P(x * y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\}$$

and

$$\mu_A^N(x * y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\}$$

for all $x, y \in X$.

Proof. Let A be an $(\in,\in\vee q)$-bipolar fuzzy subalgebras of X and $x,y\in X$. If $\min\{\mu_A^P(x),\mu_A^P(y)\}<0.5$ and $\max\{\mu_A^N(x),\mu_A^N(y)\}>-0.5$, then $\mu_A^P(x*y)\geq\min\{\mu_A^P(x),\mu_A^N(y)\}$ and $\mu_A^N(x*y)\leq\max\{\mu_A^N(x),\mu_A^N(y)\}$. Assume that $\mu_A^P(x*y)<\min\{\mu_A^P(x),\mu_A^P(x)\}$ and $\mu_A^N(x*y)>\max\{\mu_A^N(x),\mu_A^N(y)\}$. Let us choose $t\in A$ and $m\in\neg A$ such that $\mu_A^P(x*y)<t<\min\{\mu_A^P(x),\mu_A^P(y)\}$ and $\mu_A^N(x*y)>m>\max\{\mu_A^N(x),\mu_A^N(y)\}$. Then $x_t\in\mu_A^P$, $y_t\in\mu_A^P$ and $x_m\in\mu_A^N$, $y_m\in\mu_A^N$ but $(x*y)_{\min(t,t)}=(x*y)_t\overline{\in,\in\vee q}\mu_A^P$ and $(x*y)_{\max(m,m)}=(x*y)_m\overline{\in,\in\vee q}\mu^N$, a contradiction. Hence, $\mu_A^P(x*y)\geq\{\mu_A^P(x),\mu_A^P(y)\}$ whenever $\min\{\mu_A^P(x),\mu_A^P(y)\}<0.5$ and $\mu_A^N(x*y)\leq\max\{\mu_A^N(x),\mu_A^N(y)\}$ whenever $\max\{\mu_A^N(x),\mu_A^N(y)\}>-0.5$. Suppose that $\min\{\mu_A^P(x),\mu_A^P(y)\}\geq 0.5$ and $\max\{\mu_A^N(x),\mu_A^N(y)\}\leq -0.5$. Then, $x_{0.5}\in\mu_A^P$, $y_{0.5}\in\mu_A^P$ and $x_{-0.5}\in\mu_A^N$, $y_{-0.5}\in\mu_A^N$, which imply that

$$(x*y)_{\min(0.5,0.5)}=(x*y)_{0.5}\in\vee q\mu_A^P$$

$$(x*y)_{\max(-0.5,-0.5)}=(x*y)_-0.5\in\vee q\mu_A^N.$$

Thus, $\mu_A^P(x*y)\geq 0.5$ and $\mu_A^N(x*y)\leq -0.5$. Otherwise, $\mu_A^P(x*y)+0.5<0.5+0.5=1$ and $\mu_A^N(x*y)-0.5>-0.5-0.5=-1$, a contradiction. Therefore,

$$\mu_A^P(x*y)\geq\min\{\mu_A^P(x),\mu_A^P(y),0.5\}$$
$$\mu_A^N(x*y)\leq\max\{\mu_A^N(x),\mu_A^N(y),-0.5\}$$

for all $x,y\in X$. Conversely, assume that the conditions of $(\in,\in\vee q)$-bipolar fuzzy subalgebras of X is valid. Let $x,y\in X$ and $t,s\in(0,1]$ and $m,n\in\neg A$ such that $x_t\in\mu_A^P$, $y_s\in\mu_A^P$ and $x_m\in\mu_A^N$, $y_n\in\mu_A^N$. Then, $\mu_A^P(x)\geq t$, $\mu_A^P(y)\geq s$ and $\mu_A^N(x)\leq m$, $\mu_A^N(y)\leq n$. If $\mu_A^P(x*y)<\min\{t,s\}$ and $\mu_A^N(x*y)>\max\{m,n\}$, then $\min\{\mu_A^P(x),\mu_A^P(y)\}\geq 0.5$ and $\max\{\mu_A^N(x),\mu_A^N(y)\}\leq -0.5$. Otherwise, we get

$$\mu_A^P(x*y)\geq\min\{\mu_A^P(x),\mu_A^P(y),0.5\}\geq\min\{\mu_A^P(x),\mu_A^P(y)\}\geq\min\{t,s\}$$

$$\mu_A^N(x*y)\leq\max\{\mu_A^N(x),\mu_A^N(y),-0.5\}\leq\max\{\mu_A^N(X),\mu_A^N(y)\leq\max\{m,n\},$$

a contradiction. It follows that

$$\mu_A^P(x*y)+\min\{t,s\}>2\mu_A^P(x*y)\geq 2\min\{\mu_A^P(x),\mu_A^P(y),0.5\}=1$$

$$\mu_A^N(x*y)+\max\{m,n\}<2\mu_A^N(x*y)\leq 2\max\{\mu_A^N(x),\mu_A^N(y),-0.5\}=-1.$$

Hence, $(x*y)_{\min(t,s)}\in\vee q\mu_A^P$ and $(x*y)_{\max(m,n)}\in\vee q\mu_A^N$. Therefore, the bipolar fuzzy set A is an $(\in,\in\vee q)$-bipolar fuzzy BCK/BCI-subalgebras of X. Hence, the proof is completed. □

Theorem 3.11. *A bipolar fuzzy set $A=(\mu_A^P,\mu_A^N)$ of X is an $(\in,\in\vee q)$-bipolar fuzzy subalgebras of X if and only if the level subset*

$$U(\mu_A^P,\mu_A^N;t,m)=\{x\in X\mid\mu_A^P(x)\geq t\text{ and }\mu_A^N(x)\leq m\}$$

is a bipolar fuzzy subalgebras of X *for all* $m \in [-0.5, 0)$ *and for all* $t \in (0, 0.5]$.

Proof. Assume that a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. Let $x, y \in U(\mu_A^P, \mu_A^N; t, m)$ with $m \in [-0.5, 0)$ and $t \in (0, 0.5]$. Then $\mu_A^P(x) \geq t$, $\mu_A^P(y) \geq t$ and $\mu_A^N(x) \leq m$, $\mu_A^N(y) \leq m$. Then from Theorem 3.10 that

$$\mu_A^P(x * y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} \geq \min\{t, 0.5\} = t$$
$$\mu_A^N(x * y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} \leq \{m, -0.5\} = m$$

so that $x * y \in U(\mu_A^P, \mu_A^N; t, m)$. Therefore, $U(\mu_A^P, \mu_A^N; t, m)$ is a subalgebra of X.

Conversely, let A be a bipolar fuzzy set of X such that the set

$$U(\mu_A^P, \mu_A^N; t, m) = \{x \in X | \mu_A^P(x) \geq t \ \ and \ \ \mu_A^N(x) \leq m\}$$

is a subalgebra of X for all $m \in [-0.5, 0)$ and for all $t \in (0, 0.5]$. If there exist $x, y \in X$ such that $\mu_A^P(x * y) < \min\{\mu_A^P(x), \mu_A^P(y), 0.5\}$ and $\mu_A^N(x * y) > \max\{\mu_A^N(x), \mu_A^N(y), -0.5\}$, then we take $m \in (-1, 0)$ and $t \in (0, 1)$ such that $\mu_A^P(x * y) < t < \min\{\mu_A^P(x), \mu_A^P(y), 0.5\}$ and $\mu_A^N(x * y) > m > \max\{\mu_A^N(x), \mu_A^N(y), -0.5\}$. Thus, $x, y \in U(\mu_A^P, \mu_A^N; t, m)$ with $t < 0.5$ and $m > -0.5$, and so $x * y \in U(\mu_A^P, \mu_A^N; t, m)$ i.e. $\mu_A^P(x * y) \geq t$ and $\mu_A^N(x * y) \leq m$ which is a contradiction. Therefore,

$$\mu_A^P(x * y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\}$$
$$\mu_A^N(x * y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\},$$

for all $x, y \in X$. Using Theorem 3.10, we conclude that A is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. □

Corollary 3.12. *Every bipolar fuzzy subalgebra of* X *is an* $(\in, \in \vee q)$*-bipolar fuzzy subalgebra of* X.

In general, the converse of the corollary 3.12 is not true as seen in the following example.

Example 3.13. *Consider a* BCK/BCI*-algebra* $X = \{0, a, b, c, d\}$ *with the following Caley Table 2. Let us define a bipolar fuzzy set* A *of* X *as follows:* $\mu_A^P(0) = 0.6$, $\mu_A^P(a) = \mu_A^P(c) = 0.7$, $\mu_A^P(b) = \mu_A^P(d) = 0.2$ *and* $\mu_A^N(0) = -0.7$, $\mu_A^N(a) = \mu_A^N(c) = -0.4$, $\mu_A^N(b) = -0.6$ *and* $\mu_A^N(d) = -0.3$. *This example gives a* $(\in, \in \vee q)$*-bipolar fuzzy subalgebra of* X *but not bipolar fuzzy subalgebra of* X *because* $\mu_A^P(a * c) = 0.6 \ngeq 0.7 = \min\{\mu_A^P(a), \mu_A^P(c)\}$.

Let M be a subset of X. Let us consider a bipolar fuzzy set $A_M = (\mu_M^P, \mu_M^N)$ in X defined as follows:

$$\mu_M^P(x) = \begin{cases} 1, & \text{if } \ x \in M; \\ 0, & \text{if } \ otherwise. \end{cases}$$

Table 2

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

$$\mu_M^N(x) = \begin{cases} -1, & \text{if } x \in M; \\ 0, & \text{if } otherwise, \end{cases}$$

for all $x \in X$.

Theorem 3.14. *Let M be a non-empty subset of X. Then M is a subalgebra of X if and only if the bipolar fuzzy set $A_M = (\mu_M^P, \mu_M^N)$ in X is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X.*

Proof. Let M be a subalgebra of X. Then $U(\mu_M^P, \mu_M^N; t, m)$ is a subalgebra of X for all $t \in (0, 0.5]$ and $m \in [-0.5, 0)$ by the Theorem 3.11.

Conversely, let A_M be an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. Let $x, y \in M$. Then

$$\mu_M^P(x * y) \geq \min\{\mu_M^P(x), \mu_M^N, 0.5\} = 1 \wedge 0.5 = 0.5$$
$$\mu_M^N(x * y) \leq \max\{\mu_M^P(x), \mu_M^N, 0.5\} = -1 \vee -0.5 = -0.5.$$

Here, $\mu_M^P(x * y) = 1$ and $\mu_M^N(x * y) = -1$ and thus, $x * y \in M$. Hence, M is a subalgebra of X. □

Theorem 3.15. *Let M be a subalgebra of X. Then for every $t \in (0, 0.5]$ and every $m \in [-0.5, 0)$, there exists a $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X such that $U(\mu_M^P, \mu_M^N; t, m) = M$.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set in X defined as follows:

$$\mu_A^P(x) = \begin{cases} t, & \text{if } x \in M; \\ 0, & \text{if } otherwise; \end{cases} \quad \text{and} \quad \mu_A^N(x) = \begin{cases} m, & \text{if } x \in M; \\ 0, & \text{if } otherwise; \end{cases}$$

for all $x \in X$, where $t \in (0, 0.5]$ and $m \in [-0.5, 0)$. Then $U(\mu_A^P, \mu_A^N; t, m) = M$ is obvious.

We assume that $\mu_A^P(p * q) < \min\{\mu_A^P(p), \mu_A^N(q), 0.5\}$ and $\mu_A^N(p * q) > \max\{\mu_A^N(p), \mu_A^N(q), -0.5\}$ for some $p, q \in X$. Since $Im(A) = 2$. Then it follows that $\mu_A^P(p * q) = 0$ and $\min\{\mu_A^P(p), \mu_A^P(q), 0.5\} = t$, and $\mu_A^N(p * q) = 0$ and $\max\{\mu_A^N(p), \mu_A^N(q), -0.5\} = m$. Thus, $\mu_A^P(p) = \mu_A^P(q) = t$ and $\mu_A^N(p) = \mu_A^N(q) = m$, and so $p, q \in M$. Since M is a subalgebra of X, $p * q \in M$. Then $\mu_A^P(p * q) = t$ and $\mu_A^N(p * q) = m$, these are the contradiction. Therefore, $\mu_A^P(x * y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\}$ and $\mu_A^N(x *$

$y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\}$ for all $x, y \in X$. Using Theorem 3.10, we conclude that $A = (\mu_A^P, \mu_A^N)$ is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. □

Theorem 3.16. *Let $A = (\mu_A^P, \mu^N)$ be an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X, where $\mu_A^P(x) < 0.5$ and $\mu_A^N > -0.5$ for all $x \in X$. Then $A = (\mu_A^P, \mu_A^N)$ is an $(\in, \in)$-bipolar fuzzy subalgebra of X.*

Proof. The proof is straightforward using Theorem 3.10. □

Theorem 3.17. *Let $\bigwedge$ be an index set and $\{(\mu_{A_i}^P, \mu_{A_i}^N) | \, i \in \bigwedge\}$ be a family of $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. Then $A = \bigcap\limits_{i \in \bigwedge} (\mu_{A_i}^P, \mu_{A_i}^N)$ is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X.*

Proof. Let us take $x, y \in X$ and $t_1, t_2 \in (0, 1]$, and $m_1, m_2 \in [-1, 0)$ be such that $\mu_A^P(x) \geq t_1$ and $\mu_A^P(y) \geq t_2$, and $\mu_A^N(x) \leq m_1$ and $\mu_A^N(y) \leq m_2$. Assume that $(x * y)_{\min(t_1,t_2)} \overline{\in \vee q} \mu^P$ and $(x * y)_{\max(m_1,m_2)} \overline{\in \vee q} \mu_A^N$. Then $\mu_A^P(x * y) < \min\{t_1, t_2\}$ and $\mu_A^P(x * y) + \min\{t_1, t_2\} \leq 1$, and $\mu_A^N(x * y) > \max\{m_1, m_2\}$ and $\mu_A^N(x * y) + \max\{m_1, m_2\} \geq -1$, which implies

$$\mu_A^P(x * y) < 0.5 \text{ and } \mu_A^N(x * y) > -0.5. \tag{3.1}$$

Now, we define $\Delta_1 = \{i \in \bigwedge | \, (x*y)_{\min\{t_1,t_2\}} \in \mu_{A_i}^P$ and $(x*y)_{\max\{m_1,m_2\}} \in \mu_{A_i}^N\}$ and $\Delta_2 = \{[i \in \bigwedge | \, (x*y)_{\min\{t_1,t_2\}} q \mu_{A_i}^P\} \cap \{j \in \wedge | \, (x*y)_{\min\{t_1,t_2\}} \overline{\in} \mu_j^P]$ and $[\{i \in \bigwedge | \, (x * y)_{\max\{m_1,m_2\}} q \mu_{A_i}^N\} \cap \{j \in \bigwedge | \, (x * y)_{\max\{m_1,m_2\}} \overline{\in} \mu_j^N]\}$. Then $\bigwedge = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \emptyset$. If $\Delta_2 = \emptyset$, then $(x*y)_{\min\{t_1,t_2\}} \in \mu_{A_i}^P$ and $(x * y)_{\max\{m_1,m_2\}} \in \mu_{A_i}^N$ for all $i \in \bigwedge$, i.e., $\mu_{A_i}^P(x * y) \geq \min\{t_1, t_2\}$ and $\mu_{A_i}^N(x * y) \leq \max\{m_1, m_2\}$ for all $i \in \bigwedge$, which indicate $\mu_A^P(x * y) \geq \min\{t_1, t_2\}$ and $\mu_A^N(x * y) \leq \max\{m_1, m_2\}$. This is a contradiction. Hence, $\Delta_2 \neq \emptyset$, and so for every $i \in \Delta_2$ we have $\mu_{A_i}^P(x * y) < \min\{t_1, t_2\}$ and $\mu_{A_i}^P(x*y) + \min\{t_1, t_2\} > 1$, and $\mu_{A_i}^N(x*y) > \max\{m_1, m_2\}$ and $\mu_{A_i}^N(x*y) + \max\{m_1, m_2\} < -1$. It follows that $\min\{t_1, t_2\} > 0.5$ and $\max\{m_1, m_2\} < -0.5$. Now, $x_{t_1} \in \mu_A^P$ and $x_{m_1} \in \mu_A^N$ implies that $\mu_A^P(x) \geq t_1$ and $\mu_A^N(x) \leq m_1$, and thus, $\mu_{A_i}^P(x) \geq \mu_A^P(x) \geq t_1 \geq \min\{t_1, t_2\} > 0.5$ and $\mu_{A_i}^N(x) \leq \mu_A^N(x) \leq m_1 \leq \max\{m_1, m_2\} < -0.5$ for all $i \in \bigwedge$. Similarly, we get $\mu_{A_i}^P(y) > 0.5$ and $\mu_{A_i}^N(y) < -0.5$ for all $i \in \bigwedge$. We suppose that $t = \mu_{A_i}^P(x * y) < 0.5$ and $m = \mu_{A_i}^N(x * y) > -0.5$. Taking that $t < r < 0.5$ and $m > n > -0.5$, we get $x_r \in \mu_{A_i}^P$ and $y_r \in \mu_{A_i}^P$, but $(x * y)_{\min(r,r)} = (x * y)_r \overline{\in \vee q} \mu_{A_i}^P$ and $x_n \in \mu_{A_i}^N$ and $y_n \in \mu_{A_i}^N$, but $(x * y)_{\max(n,n)} = (x * y)_n \overline{\in \vee q} \mu_{A_i}^N$. This contradicts that $A = (\mu_{A_i}^P, \mu_{A_i}^N)$ is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. Hence, $\mu_{A_i}^P(x * y) \geq 0.5$ and $\mu_{A_i}^N(x * y) \leq -0.5$ for all $i \in \bigwedge$, so $\mu_A^P(x * y) \geq 0.5$ and $\mu_A^N(x * y) \leq -0.5$ which contradicts (3.1). Therefore,

$(x*y)_{\min\{t_1,t_2\}} \in \vee q\mu^P$ and $(x*y)_{\max\{m_1,m_2\}} \in \vee q\mu_A^N$ and consequently, $A = (\mu_{A_i}^P, \mu_{A_i}^N)$ is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. □

Corollary 3.18. *Let $\{(\mu_{A_i}^P, \mu_{A_i}^N) |\ i \in \bigwedge\}$ be a family of $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. Then $\mu = \bigcap\limits_{i\in\bigwedge} (\mu_{A_i}^P, \mu_{A_i}^N)$ is a bipolar fuzzy subalgebra of X.*

The following example shows that the union of two $(\in, \in \vee q)$-bipolar fuzzy subalgebras of X may not be an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X.

Example 3.19. *Let $X = \{0, a, b, c\}$ be a BCI-algebra with the Caley Table 1 given in Example 3.5, and let $A = (\mu_A^P, \mu_A^N)$ be an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X is defined as $\mu_A^P(0) = 0.6$, $\mu_A^P(a) = 0.7$ and $\mu_A^P(b) = \mu_A^P(c) = 0.3$, and $\mu_A^N(0) = -0.8$, $\mu_A^N(b) = -0.7$ and $\mu_A^N(a) = \mu_A^N(c) = -0.3$. Then*

$$F[\mu_A^P](t) = \begin{cases} X, & \text{if } t \in (0, 0.3]; \\ \{0, a\}, & \text{if } t \in (0.3, 0.4]. \end{cases}$$

$$F[\mu_A^N](s) = \begin{cases} X, & \text{if } s \in [-0.3, 0); \\ \{0, b\}, & \text{if } s \in (-0.3, -0.4]. \end{cases}$$

Here X, $\{0, a\}$ and $\{0, b\}$ are subalgebra of X.

Let $B = (\mu_B^P, \mu_B^N)$ be an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X is defined as $\mu_B^P(0) = 0.4$, $\mu_B^P(a) = \mu_B^P(c) = 0.3$ and $\mu_B^P(b) = 0.5$, and $\mu_B^N(0) = -0.7$, $\mu_B^N(a) = \mu_B^N(b) = -0.3$ and $\mu_B^N(c) = -0.6$. Then

$$F[\mu_B^P](t) = \begin{cases} X, & \text{if } t \in (0, 0.3]; \\ \{0, b\}, & \text{if } t \in (0.3, 0.4]. \end{cases}$$

$$F[\mu_B^N](s) = \begin{cases} X, & \text{if } s \in [-0.3, 0); \\ \{0, c\}, & \text{if } s \in (-0.3, -0.4]. \end{cases}$$

Here X, $\{0, b\}$ and $\{0, c\}$ are bipolar fuzzy subalgebra of X.
Now, the union $(A \cup B)$ [where $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$] of A and B, respectively is given by $(\mu_{A\cup B}^P)(0) = 0.6$, $(\mu_{A\cup B}^P)(a) = 0.7$, $(\mu_{A\cup B}^P)(b) = 0.5$ and $(\mu_{A\cup B}^P)(c) = 0.3$, and $(\mu_{A\cup B}^N)(0) = -0.8$, $(\mu_{A\cup B}^N)(a) = -0.3$, $(\mu_{A\cup B}^N)(b) = -0.7$ and $(\mu_{A\cup B}^N)(c) = -0.6$. Hence,

$$F[\mu_{A\cup B}^P](t) = \begin{cases} X, & \text{if } t \in (0, 0.3]; \\ \{0, a, b\}, & \text{if} t \in (0.3, 0.4]. \end{cases}$$

$$F[\mu_{A\cup B}^N](s) = \begin{cases} X, & \text{if } s \in [-0.3, 0); \\ \{0, b, c\}, & \text{if } s \in (-0.3, -0.4]. \end{cases}$$

Since $\{0, a, b\}$ and $\{0, b, c\}$ are not bipolar fuzzy subalgebra of X. Therefore, $(A \cup B)$ is not an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X.

For any bipolar fuzzy set A in X, where $t \in (0,1]$ and $m \in [-1,0)$, we denote

$$\mu_t^P = \{x \in X | x_t \in q\mu_A^P\}$$

and

$$\mu_m^N = \{x \in X | x_m \in q\mu_A^N\},$$

and

$$[A]_{(t,m)} = \Big\{x \in X | x_t \in \vee q\mu_A^P \ \ and \ \ x_m \in \vee q\mu_A^N\Big\}.$$

Then it is obvious that $[A]_{(t,m)} = U(\mu_A^P, \mu_A^N; t, m) \cup \mu_t^P \cup \mu_m^N$. Here, $[A]_{(t,m)}$ is an $(\in \vee q)$-level subalgebra of A.

Theorem 3.20. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set in X. Then A is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X if and only if $[A]_{(t,m)}$ is a subalgebra of X for all $t \in (0,1]$ and for all $m \in [-1,0)$.*

Proof. Suppose that $A = (\mu_A^P, \mu_A^N)$ is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X and let $x, y \in [A]_{(t,m)}$ for $t \in (0,1]$, $m \in [-1,0)$. Then, $x_t \in \vee q\mu_A^P$, $y_t \in \vee q\mu_A^P$ and $x_m \in \vee q\mu_A^N$, $y_m \in \vee q\mu_A^N$, i.e., $\mu_A^P(x) \geq t$ or $\mu_A^P(x) + t > 1$, and $\mu_A^P(y) \geq t$ or $\mu_A^P(y) + t > 1$, and also $\mu_A^N(x) \leq m$ or $\mu_A^N(x) + m < -1$, and $\mu_A^N(y) \leq m$ or $\mu_A^N(y)+m < -1$. Using the Theorem 3.10, we get, $\mu_A^P(x* y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\}$ and $\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\}$.
Case 1. $\mu_A^P(x) \geq t$ and $\mu_A^P(y) \geq t$, and $\mu_A^N(x) \leq m$ and $\mu_A^N(y) \geq m$. If $t > 0.5$ and $m < -0.5$, then

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} = 0.5$$
$$\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} = -0.5.$$

Hence, $\mu_A^P(x*y) + t > 0.5 + 0.5 = 1$ and $\mu_A^N(x*y) + m < -0.5 - 0.5 = -1$, and so $x*y \in q\mu_A^P$ and $x*y \in q\mu_A^N$. If $t \leq 0.5$ and $m \geq -0.5$, then

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} \geq t$$
$$\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} \leq m,$$

thus $(x*y)_t \in \vee q\mu_A^P$ and $(x*y)_m \in \vee q\mu_A^P$. Therefore, $x*y \in [A]_{(t,m)}$.
Case 2. Let $\mu_A^P(x) \geq t$ and $\mu_A^P(y)+t > 1$, and $\mu_A^N(x) \leq m$ and $\mu_A^N(y)+m < -1$. If $t > 0.5$ and $m < -0.5$, then

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} = \mu_A^P(y) \wedge 0.5 = 1 - t \wedge 0.5 = 1 - t$$
$$\begin{aligned}\mu_A^N(x*y) &\leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} = \mu^N(y) \vee -0.5 \\ &= -1 - m \vee -0.5 = -1 - m\end{aligned}$$

and so $(x*y) \in q\mu_A^P$, and $(x*y)_m \in q\mu_A^N$. If $t \leq 0.5$ and $m \geq -0.5$, then

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} \geq \min\{t, 1-t, 0.5\} = t$$
$$\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} \geq \max\{m, -1-m, -0.5\} = m.$$

Hence, $(x*y)_t \in \vee q\mu_A^P$ and $(x*y)_m \in \vee q\mu_A^N$. Thus, $x*y \in [A]_{(t,m)}$.
Case 3. $\mu_A^P(x)+t > 1$ and $\mu_A^P(y) \geq t$, and $\mu_A^N(x)+m < -1$ and $\mu_A^N(y) \leq m$. If $t > 0.5$ and $m < -0.5$, then

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} = \mu_A^P(x) \wedge 0.5 = 1-t \wedge 0.5 = 1-t$$
$$\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} = \mu^N(x) \vee -0.5$$
$$= -1-m \vee -0.5 = -1-m$$

and so $(x*y)_t \in q\mu_A^P$, and $(x*y)_m \in q\mu_A^N$. If $t \leq 0.5$ and $m \geq -0.5$, then

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} \geq \min\{1-t, t, 0.5\} = t$$

$$\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} \geq \max\{-1-m, m, -0.5\} = m.$$

Hence, $(x*y)_t \in \vee q\mu_A^P$ and $(x*y)_m \in \vee q\mu_A^N$. Thus, $x*y \in [A]_{(t,m)}$.
Case 4. $\mu_A^P(x) + t > 1$ and $\mu_A^P(y) + t > 1$, and $\mu_A^N(x) + m < -1$ and $\mu_A^N(y) + m < -1$. If $t > 0.5$ and $m < -0.5$, then

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} > 1-t \wedge 0.5 = 1-t$$

$$\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} < -1-m \vee -0.5 = -1-m.$$

Thus, $(x*y)_t q\mu_A^P$ and $(x*y)_m q\mu_A^N$. If $t \leq 0.5$ and $m \geq -0.5$, then

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\} \geq 1-t \wedge 0.5 = 0.5 \geq t$$

$$\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\} leq -1-m \vee -0.5 = -0.5 \leq m.$$

Therefore, $(x*y)_t \in \mu_A^P$ and $(x*y)_m \in \mu_A^N$. Hence, $(x*y)_t \in \vee q\mu_A^P$ and $(x*y)_m \in \vee q\mu_A^N$, i.e., $x*y \in [A]_{(t,m)}$. Therefore, $[A]_{(t,m)}$ is a subalgebra of X.

Conversely, let $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set of X and $t \in (0,1]$ and $m \in [-1,0)$ be such that $[A]_{(t,m)}$ is a subalgebra of X. If possible, let

$$\mu_A^P(x*y) < t \leq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\}$$

$$\mu_A^N(x*y) > m \geq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\}$$

for some $t \in (0,t)$ and $m \in (-1,0)$. Then, $x, y \in U(\mu_A^P, \mu_A^N; t, m) \subseteq [A]_{(t,m)}$, which indicate $x*y \in [A]_{(t,m)}$. Thus, $\mu_A^P(x*y) \geq t$ or $\mu_A^P(x*y)+t > 1$, and $\mu_A^N(x*y) \leq m$ or $\mu_A^N(x*y)+m < -1$, and these are a contradiction. Hence,

$$\mu_A^P(x*y) \geq \min\{\mu_A^P(x), \mu_A^P(y), 0.5\}$$
$$\mu_A^N(x*y) \leq \max\{\mu_A^N(x), \mu_A^N(y), -0.5\}$$

for all $x, y \in X$. Now, by using the Theorem 3.10, we conclude that $A = (\mu_A^p, \mu_A^N)$ is an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X. □

A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X is said to be proper if $Im(A)$ has at least two elements. Two bipolar fuzzy sets are said to be equivalent if they have the same family of level subsets, otherwise they are non-equivalent.

Theorem 3.21. *Let $A = (\mu_A^P, \mu_A^N)$ be an $(\in, \in \vee q)$-bipolar fuzzy subalgebra of X, where $\Big\{A = (\mu_A^P, \mu_A^N) | \mu_A^P(x) < 0.5$ and $\mu_A^N(x) > -0.5\Big\} \geq 2$. Then there exists two proper non-equivalent $(\in, \in \vee q)$-bipolar fuzzy subalgebras of X such that A can be expressed as the union of them.*

Proof. Let $\{\mu_A^P(x) | \mu_A^P(x) < 0.5\} = \{t_1, t_2, \ldots, t_r\}$ such that $t_1 > t_2 > \cdots > t_r$, where $r \geq 2$, and $\{\mu_A^N(x) | \mu_A^N(x) > -0.5\} = \{m_1, m_2, \ldots, m_r\}$ such that $m_1 > m_2 > \cdots > m_r$, where $r \geq 2$. Then the chain of $(\in \vee q)$-level subalgebras of $A = (\mu_A^P, \mu_A^N)$ are given as follows:

$$[A]_{(0.5,-0.5)} \subseteq [A]_{(t_1,m_1)} \subseteq [A]_{(t_2,m_2)} \subseteq \cdots \subseteq [A]_{(t_r,m_r)} = X,$$

i.e.,

$$[\mu_A^P]_{0.5} \subseteq [\mu_A^P]_{t_1} \subseteq [\mu_A^P]_{t_2} \subseteq \cdots \subseteq [\mu_A^P]_{t_r}$$

$$[\mu_A^N]_{-0.5} \supseteq [\mu_A^N]_{m_1} \supseteq [\mu_A^N]_{m_2} \supseteq \cdots \supseteq [\mu_A^N]_{m_r}$$

Let (ν_A^P, ξ_A^P), and (ν_A^N, ξ_A^N) be fuzzy sets defined in X as follows:

$$\nu_A^P(x) = \begin{cases} t_1 & \text{if } x \in [\mu_A^P]_{t_1}; \\ t_2 & \text{if } x \in [\mu_A^P]_{t_2} \backslash [\mu_A^P]_{t_1}; \\ \cdots & \\ t_r & \text{if } x \in [\mu_A^P]_{t_r} \backslash [\mu_A^P]_{t_{r-1}}, \end{cases}$$

and

$$\xi_A^P(x) = \begin{cases} \mu_A^P(x) & \text{if } x \in [\mu_A^P]_{(0.5)}; \\ 0 & \text{if } x \in [\mu_A^P]_{t_2} \backslash [\mu_A^P]_{(0.5)}; \\ t_3 & \text{if } x \in [\mu_A^P]_{t_3} \backslash [\mu_A^P]_{t_2}; \\ \cdots & \\ t_r & \text{if } x \in [\mu_A^P]_{t_r} \backslash [\mu_A^P]_{t_{r-1}}. \end{cases}$$

$$\nu_A^N(x) = \begin{cases} m_r & \text{if } x \in [\mu_A^N]_{m_r}; \\ m_{r-1} & \text{if } x \in [\mu_A^N]_{m_{r-}} \backslash [\mu_A^N]_{m_r}; \\ m_r & \text{if } x \in [\mu_A^N]_{m_r} \backslash [\mu_A^N]_{m_{r-1}}; \\ \cdots & \\ m_3 & \text{if } x \in [\mu_A^N]_{m_3} \backslash [\mu_A^N]_{m_2}; \\ 0 & \text{if } x \in [\mu_A^N]_{m_2} \backslash [\mu_A^N]_{(-0.5)}; \\ \mu_A^N(x) & \text{if } x \in [\mu_A^N]_{(-0.5)}, \end{cases}$$

and

$$\xi_A^N(x) = \begin{cases} m_r & \text{if } x \in [\mu_A^N]_{m_r} \backslash [\mu_A^N]_{m_{r-1}}; \\ m_{r-1} & \text{if } x \in [\mu_A^P]_{m_{r-1}} \backslash [\mu_A^P]_{m_r}; \\ \dots & \\ m_2 & \text{if } x \in [\mu_A^N]_{m_2} \backslash [\mu_A^N]_{m_1}; \\ m_1 & \text{if } x \in [\mu_A^N]_{m_1}. \end{cases}$$

Then (ν_A^P, ν_A^N) and (ξ_A^P, ξ_A^N) are $(\in, \in \vee q)$-bipolar fuzzy subalgebras of X, and $\nu_A^P, \xi_A^P \leq \mu_A^P$ and $\nu_A^N, \xi_A^N \leq \mu_A^N$. The chain of $(\in \vee q)$-level subalgebras of ν_A^P,ξ_A^P and ν_A^N, ξ_A^N are respectively given by

$$[\mu_A^P]_{0.5} \subseteq [\mu_A^P]_{t_1} \subseteq [\mu_A^P]_{t_2} \subseteq \cdots \subseteq [\mu_A^P]_{t_r},$$

and

$$[\mu_A^P]_{t_1} \subseteq [\mu_A^P]_{t_2} \subseteq \cdots \subseteq [\mu_A^P]_{t_r},$$

and

$$[\mu_A^N]_{-0.5} \supseteq [\mu_A^N]_{m_1} \supseteq [\mu_A^N]_{m_2} \supseteq \cdots \supseteq [\mu_A^N]_{m_r},$$

and

$$[\mu_A^N]_{m_1} \supseteq [\mu_A^N]_{m_2} \supseteq \cdots \supseteq [\mu_A^N]_{m_r}.$$

Therefore, (ν_A^P, ξ_A^P) and (ν_A^N, ξ_A^N) are non-equivalent, and thus $A = (\nu_A^P \cup \xi_A^P, \nu_A^N \cup \xi_A^N)$. □

4. $(\in, \in \vee q)$-Bipolar Fuzzy BCK/BCI-Ideals

In this section, $(\in, \in \vee q)$-bipolar fuzzy ideals of BCK/BCI-algebras are defined and some important properties are presented.

Definition 4.1. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of X is called an $(\in, \in \vee q)$-bipolar fuzzy ideal of X if it satisfies the following conditions:*
(i) $x_t \in \mu_A^P \Rightarrow 0_t \in \vee q \mu_A^P$ and $0_m \in \mu_A^N \Rightarrow x_m \in \vee q \mu_A^N$, for all $x \in X$, $t \in (0, 1]$, $m \in [-1, 0)$
(ii) $(x * y)_t \in \mu_A^P, y_s \in \mu_A^P \Rightarrow x_{min(t,s)} \in \vee q \mu_A^P$ for all $x, y \in X$, $t, s \in (0, 1]$
(iii) $(x * y)_m \in \mu_A^N, y_n \in \mu_A^N \Rightarrow x_{\max(m,n)} \in \vee q \mu_A^N$ for all $x, y \in X$, $m, n \in [-1, 0)$.

Example 4.2. *Let $X = \{0, a, b, c, d\}$ be a BCK-algebra in Example 3.13 and a bipolar fuzzy set A of X defined by $\mu_A^P(0) = 0.7$, $\mu_A^P(a) = \mu_A^P(c) = 0.3$, $\mu_A^P(b) = \mu_A^P(d) = 0.2$ and $\mu_A^N(0) = -0.9$, $\mu_A^N(a) = -0.6$, $\mu_A^N(b) = -0.4$, $\mu_A^N(c) = -0.7$ and $\mu_A^N(d) = -0.3$ is an $(\in, \in \vee q)$-bipolar fuzzy ideal as well as a bipolar fuzzy ideal of X.*

Theorem 4.3. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of X is called a bipolar fuzzy ideal of X if and only if the following assertions are valid*
(i) $x_t \in \mu_A^P \Rightarrow 0_t \in \mu_A^P$ and $x_m \in \mu_A^N \Rightarrow 0_m \in \mu_A^N$, for all $x \in X$, $t \in [0, 1]$, $m \in [-1, 0]$)
(ii) $(x * y)_t \in \mu_A^P, y_s \in \mu_A^P \Rightarrow x_{\min(t,s)} \in \mu_A^P$, for all $x, y \in X$, $t, s \in [0, 1]$

(iii) $(x * y)_m \in \mu_A^N, y_n \in \mu_A^N \Rightarrow x_{\max(m,n)} \in \mu_A^N$, *for all* $x, y \in X$, $m, n \in [-1, 0]$).

Proof. Assume that Definition 2.6 (i) is valid and $x \in X$, $t \in [0,1]$, $m \in [-1,0]$ such that $x_t \in \mu_A^P$ and $x_m \in \mu_A^N$. Then $\mu_A^P(0) \geq \mu_A^P(x) \geq t$ and $\mu_A^N(0) \leq \mu_A^N(x) \leq m$, and so $0_t \in \mu_A^P$ and $0_m \in \mu_A^N$. Since $x_{\mu(x)} \in \mu_A^P$ and $x_{\mu(x)} \in \mu_A^N$ for all $x \in X$, it follows from (i) that $0_{\mu(x)} \in \mu_A^P$ and $0_{\mu(x)} \in \mu_A^N$ so that $\mu_A^P(0) \geq \mu_A^P(x)$ and $\mu_A^N(0) \leq \mu_A^N(x)$ for all $x \in X$. Assume that the condition (ii) and (iii) of Definition 2.6 holds. Let $x, y \in X$ and $t, s \in [0,1]$, $m, n \in [-1, 0]$ be such that $(x * y)_t \in \mu_A^P$, $y_s \in \mu_A^P$ and $(x * y)_m \in \mu_A^N$, $y_n \in \mu_A^N$. Then $\mu_A^P(x * y) \geq t$, $\mu_A^P(y) \geq s$ and $\mu_A^N(x * y) \leq m$, $\mu_A^N(y) \leq n$. It follows from (ii) and (iii) of Definition 2.6 that

$$\mu_A^P(x) \geq \min\{\mu_A^P(x * y), \mu_A^P(y)\} \geq \min\{t, s\}$$
$$\mu_A^N(x) \leq \max\{\mu_A^N(x * y), \mu_A^N(y)\} \leq \max\{m, n\}.$$

So, that $x_{\min(t,s)} \in \mu_A^P$ and $x_{\max(m,n)} \in \mu_A^N$. Again, suppose that (ii) and (iii) are valid. Also, for every $x, y \in X$, $(x * y)_{\mu_A^P(x*y)} \in \mu_A^P$, $y_{\mu_A^P(y)} \in \mu_A^P$ and $(x * y)_{\mu_A^N(x*y)} \in \mu_A^N$, $y_{\mu_A^N(y)} \in \mu_A^N$. Hence, $x_{\min\{\mu_A^P(x*y), \mu_A^P(y)\}} \in \mu_A^P$ and $x_{\max\{\mu_A^N(x*y), \mu_A^N(y)\}} \in \mu_A^N$ by (ii) and (iii), respectively and thus,

$$\mu_A^P(x) \geq \min\{\mu_A^P(x * y), \mu_A^P(y)\}$$
$$\mu_A^N(x) \leq \max\{\mu_A^N(x * y), \mu_A^N(y)\}.$$

Hence, the proof is completed. □

Remark 4.4. *Theorem 4.3 shows that every* $(\in, \in)$*-bipolar fuzzy ideal is precisely a bipolar fuzzy ideal and vice versa. Obviously, every* $(\in, \in)$*-bipolar fuzzy ideal is an* $(\in, \in \vee q)$*-bipolar fuzzy ideal.*

Theorem 4.5. *A bipolar fuzzy set* $A = (\mu_A^P, \mu_A^N)$ *of* X *is an* $(\in, \in \vee q)$*-bipolar fuzzy ideal of* X *if and only if it satisfies the following conditions:*
(i) $\mu_A^P(0) \geq \min\{\mu_A^P(x), 0.5\}$ *and* $\mu_A^N(0) \leq \max\{\mu_A^N(x), -0.5\}$ *for all* $x \in X$
(ii) $\mu_A^P(x) \geq \min\{\mu_A^P(x * y), \mu_A^P(y), 0.5\}$ *for all* $x, y \in X$
(iii) $\mu_A^N(x) \leq \max\{\mu_A^N(x * y), \mu_A^N(y), -0.5\}$ *for all* $x, y \in X$.

Proof. Suppose $A = (\mu_A^P, \mu_A^N)$ is an $(\in, \in \vee q)$-bipolar fuzzy ideal of X. Let $x \in X$ be such that $\mu_A^P(x) < 0.5$ and $\mu_A^N(x) > -0.5$. If $\mu_A^P(0) < \mu_A^P(x)$ and $\mu_A^N(0) > \mu_A^N(x)$, then $\mu_A^P(0) < t < \mu_A^P(x)$ and $\mu_A^N(0) > m > \mu_A^N(x)$ for some $t \in (0, 0.5)$ and for some $m \in (-0.5, 0)$, so we get $x_t \in \mu_A^P$ and $0_t \overline{\in} \mu_A^P$, and $x_m \in \mu_A^N$ and $0_m \overline{\in} \mu_A^N$. Since $\mu_A^P(0) + t < 1$ and $\mu_A^N(0) + m > -1$, so we have $0_t \overline{q} \mu_A^P$ and $0_m \overline{q} \mu_A^N$. It follows that $0_t \overline{\in \vee q} \mu_A^P$ and $0_m \overline{\in \vee q} \mu_A^N$, a contradiction. Hence, $\mu_A^P(0) \geq \mu_A^P(x)$ and $\mu_A^N(0) \leq \mu_A^N(x)$. Now if $\mu_A^P(x) \geq 0.5$ and $\mu_A^N(x) \leq -0.5$, then $x_{0.5} \in \mu_A^P$ and $x_{-0.5} \in \mu_A^N$ and thus, $0_{0.5} \in \vee q \mu_A^P$ and $0_{-0.5} \in \vee q \mu_A^N$. Thus, $\mu_A^P(0) \geq 0.5$ and $\mu_A^N(0) \leq -0.5$. Otherwise,

$\mu_A^P(x)+0.5 < 0.5+0.5 = 1$ and $\mu_A^N(x)+(-0.5) > -0.5-0.5 = -1$, a contradiction. Consequently, $\mu_A^P(0) \geq \{\mu_A^P(x), 0.5\}$ and $\mu_A^N(0) \leq \{\mu_A^N(x), -0.5\}$ for all $x \in X$. Let $x, y \in X$. Suppose that $\min\{\mu_A^P(x*y), \mu_A^P(y)\} < 0.5$ and $\max\{\mu_A^N(x*y), \mu_A^N(y)\} > -0.5$. Then $\mu_A^P(x) \geq \min\{\mu_A^P(x*y), \mu_A^P(y)\}$ and $\mu_A^N(x) \leq \max\{\mu_A^N(x*y), \mu_A^N(y)\}$. If not, then $\mu_A^P(x) < t < \min\{\mu_A^P(x*y), \mu_A^P(y)\}$ for some $t \in (0, 0.5)$ and $\mu_A^N(x) > m > \max\{\mu_A^N(x*y), \mu_A^N(y)\}$ for some $m \in (-0.5, 0)$. It follows that $(x*y)_t \in \mu_A^P$ and $y_t \in \mu_A^P$ but $x_{\min(t,t)} = x_t \overline{\in \vee q} \mu_A^P$ and $(x*y)_m \in \mu_A^N$ and $y_m \in \mu_A^N$ but $x_{\max(m,m)} = x_m \overline{\in \vee q} \mu_A^N$ which is a contradiction. Hence, $\mu_A^P(x) \geq \min\{\mu_A^P(x*y), \mu_A^P(y)\}$ whenever $\min\{\mu_A^P(x*y), \mu_A^P(y)\} < 0.5$ and $\mu_A^N(x) \leq \max\{\mu_A^N(x*y), \mu_A^N(y)\}$ whenever $\max\{\mu_A^N(x*y), \mu_A^N(y)\} > -0.5$. If $\min\{\mu_A^P(x*y), \mu_A^P(y)\} \geq 0.5$, then $(x*y)_{0.5} \in \mu_A^P$ and $y_{0.5} \in \mu_A^P$, which imply that $x_{0.5} = x_{\min(0.5,0.5)} \in \vee q \mu_A^P$ and if $\max\{\mu_A^N(x*y), \mu_A^N(y) \leq -0.5$, then $(x*y)_{-0.5} \in \mu_A^N$ and $y_{-0.5} \in \mu_A^N$, which imply that $x_{-0.5} = x_{\max(-0.5,-0.5)} \in \vee q \mu_A^N$. Therefore, $\mu_A^P(x) \geq 0.5$ and $\mu_A^N(x) \leq -0.5$, because if $\mu_A^P(x) < 0.5$ and $\mu_A^N(x) > -0.5$ then $\mu_A^P(x) + 0.5 < 0.5 + 0.5 = 1$ and $\mu_A^N(x) + (-0.5) > -0.5 - 0.5 = -1$, which is a contradiction. Hence,

$$\mu_A^P(x) \geq \min\{\mu_A^P(x*y), \mu_A^P(y), 0.5\}$$
$$\mu_A^N(x) \leq \max\{\mu_A^N(x*y), \mu_A^N(y), -0.5\}$$

for all $x, y \in X$.

Conversely, assume that A satisfies the conditions (i), (ii), and (iii). Let $x \in X$, $t \in (0,1]$ and $m \in [-1, 0)$ be such that $x_t \in \mu_A^P$ and $x_m \in \mu_A^N$. Then $\mu_A^P(x) \geq t$ and $\mu_A^N(x) \leq m$. Suppose that $\mu_A^P(0) < t$ and $\mu_A^N(0) > m$. If $\mu_A^P(x) < 0.5$ and $\mu_A^N(x) > -0.5$, then $\mu_A^P(0) \geq \min\{\mu_A^P(x), 0.5\} = \mu_A^P(x) \geq t$ and $\mu_A^N(0) \leq \max\{\mu_A^N(x), -0.5\} = \mu_A^N(x) \leq m$, a contradiction. Hence, we know that $\mu_A^P(x) \geq 0.5$ and $\mu_A^N(x) \leq -0.5$ and so we get

$$\mu_A^P(0) + t > 2\mu_A^P(0) \geq \min\{\mu_A^P(x), 0.5\} = 1$$
$$\mu_A^N(0) + m < 2\mu_A^N(0) \leq \max\{\mu_A^N(x). - 0.5\} = -1.$$

Thus, $0_t \in \vee q \mu_A^P$ and $0_m \in \vee q \mu_A^N$. Let $x, y \in X$, $t, s \in (0,1]$ and $m, n \in [-1, 0)$ be such that $(x*y)_t \in \mu_A^P$ and $y_s \in \mu_A^P$, and $(x*y)_m \in \mu_A^N$, $y_n \in \mu_A^N$. Then $\mu_A^P(x*y) \geq t$ and $\mu_A^P(y) \geq s$, and $\mu_A^N(x*y) \leq m$ and $\mu_A^N(y) \leq n$. Suppose that $\mu_A^P(x) < \min\{t, s\}$ and $\mu_A^N(x) > \max\{m, n\}$. If $\min\{\mu_A^P(x*y), \mu_A^P(y)\} < 0.5$ and if $\max\{\mu_A^N(x*y), \mu_A^N(y)\} > -0.5$. Then

$$\mu_A^P(x) \geq \min\{\mu_A^P(x*y), \mu_A^P(y), 0.5\} = \min\{\mu_A^P(x*y), \mu_A^P(y)\} \geq \min\{t, s\}$$
$$\mu_A^N(x) \leq \max\{\mu_A^N(x*y), \mu_A^N(y), -0.5\} = \max\{\mu_A^N(x*y), \mu_A^N(y)\} \leq \{m, n\}.$$

This is a contradiction. Hence, $\min\{\mu_A^P(x*y), \mu_A^P(y)\} \geq 0.5$ and $\max\{\mu_A^N(x*y), \mu_A^N(y)\} \leq -0.5$. It follows that

$$\mu_A^P(x) + \min\{t, s\} > 2\mu_A^P(x) \geq \min\{\mu_A^P(x*y), \mu_A^P(y), 0.5\} = 1$$

$$\mu_A^N(x) + \max\{m, n\} < 2\mu_A^N(x) \leq \max\{\mu_A^N(x * y), \mu_A^N(y), -0.5\} = -1$$

so that $x_{\min(t,s)} \in \vee q\mu_A^P$ and $x_{\max(m,n)} \in \vee q\mu_A^N$. Consequently, A is an $(\in, \in \vee q\mu)$-bipolar fuzzy ideal of a BCK/BCI-algebra of X. □

Theorem 4.6. *A bipolar fuzzy set A of X is an $(\in, \in \vee q)$-bipolar fuzzy ideal of X if and only if the set*

$$U(\mu_A^P, \mu_A^N; t, m) = \{x \in X | \ \mu_A^P(x) \geq t \text{ and } \mu_A^N(x) \leq m\}$$

is a bipolar fuzzy ideal of X for all $m \in [-0.5, 0)$ and for all $t \in (0, 0.5]$.

Proof. Assume that bipolar fuzzy set A is a bipolar $(\in, \in \vee q)$-fuzzy ideal of X with $m \in [-0.5, 0)$ and $t \in (0, 0.5]$. Now, using Theorem 4.5(i), we have $\mu_A^P(0) \geq \min\{\mu_A^P(x), 0.5\}$ and also $\mu_A^N(0) \leq \max\{\mu_A^N(x), -0.5\}$ for any $x \in U(\mu_A^P, \mu_A^N; t, m)$. It follows that $\mu_A^P(0) \geq \min\{t, 0.5\} = t$ and $\mu_A^N(0) \leq \max\{m, -0.5\} = m$. This implies that $0 \in U(\mu_A^P, \mu_A^N; t, m)$. Let $x, y \in X$ be such that $x * y \in U(\mu_A^P, \mu_A^N; t, m)$ and $y \in U(\mu_A^P, \mu_A^N; t, m)$ for $m \in [-0.5, 0)$ and for $t \in (0, 0.5]$. Then $\mu_A^P(x * y) \geq t$ and $\mu_A^P(y) \geq t$, and $\mu_A^N(x * y) \leq m$ and $\mu_A^N(y) \leq m$. Now using Theorem 4.5 (ii) and (iii), we get

$$\mu_A^P(x) \geq \min\{\mu_A^P(x * y), \mu_A^P(y), 0.5\} \geq \min\{t, 0.5\} = t$$
$$\mu_A^N(x) \leq \max\{\mu_A^N(x * y), \mu_A^N(y), -0.5\} \leq \max\{m, -0.5\} = m$$

and so $x \in U(\mu_A^P, \mu_A^N; t, m)$. Hence, $U(\mu_A^P, \mu_A^N; t, m)$ for $m \in [-0.5, 0)$ and for $t \in (0, 0.5]$, is a bipolar fuzzy ideal of X.

Conversely, let A be a bipolar fuzzy set of X such that $U(\mu_A^P, \mu_A^N; t, m) = \{x \in X | \ \mu_A^P(x) \geq t \ \textit{and} \ \mu_A^N(x) \leq m\}$ is a bipolar fuzzy ideal of X for all $m \in [-0.5, 0)$ and for all $t \in (0, 0.5]$. If there is $a \in X$ such that $\mu_A^P(0) < \min\{\mu_A^P(a), 0.5\}$ and $\mu_A^N(0) > \max\{\mu_A^N(a), -0.5\}$, then $\mu_A^P(0) < t < \min\{\mu_A^P(a), 0.5\}$ and $\mu_A^N(0) > m > \max\{\mu_A^N(a), -0.5\}$ for some $t \in (0, 0.5)$ and for some $m \in (-0.5, 0)$, so $0 \notin U(\mu_A^P, \mu_A^N; t, m)$. This is a contradiction. Hence, $\mu_A^P(0) \geq \min\{\mu_A^P(x), 0.5\}$ and $\mu_A^N(0) \leq \max\{\mu_A^N(x), -0.5\}$ for all $x \in X$. Assume that there exist $a', b' \in X$ such that $\mu_A^P(a') < \min\{\mu_A^P(a' * b'), \mu_A^P(b'), 0.5\}$ and $\mu_A^N(a') > \max\{\mu_A^N(a' * b'), \mu_A^N(b'), -0.5\}$. We take

$$t_0 = \frac{1}{2}(\mu_A^P(a') + \min\{\mu_A^P(a' * b'), \mu_A^P(b'), 0.5\})$$

and

$$m_0 = \frac{1}{2}(\mu_A^N(a') + \max\{\mu_A^N(a' * b'), \mu_A^N(b'), -0.5\}).$$

We get $t_0 \in (0, 0.5)$ and $m_0 \in (-0.5, 0)$, so that $\mu_A^P(a') < t_0 < min\{\mu_A^P(a' * b'), \mu_A^P(b'), 0.5\}$, and $\mu_A^N(a') > m_0 > \max\{\mu_A^N(a' * b'), \mu_A^N(b'), -0.5\}$. Thus, $a' * b' \in U(\mu_A^P, \mu_A^N; t, m)$ and $b' \in U(\mu_A^P, \mu_A^N; t, m)$, but $a' \notin U(\mu_A^P, \mu_A^N; t, m)$. This is a contradiction. Hence,

$$\mu_A^P(x) \geq \min\{\mu_A^P(x * y), \mu_A^P(y), 0.5\}$$

$$\mu_A^N(x) \leq \max\{\mu_A^N(x * y), \mu_A^N(y), -0.5\}$$

for all $x, y \in X$. It follows from Theorem 4.5 that A is an $(\in, \in \vee q)$-bipolar fuzzy ideal of X. $\square$

Corollary 4.7. *Every bipolar fuzzy ideal of X is an $(\in, \in \vee q)$-bipolar fuzzy ideal of X. The converse of Corollary 4.7 is not true in general, justified in the following example.*

Example 4.8. *Consider a BCI-algebra $X = \{0, a, b, c\}$ with Caley Table 1 given in Example 3.5, we define a bipolar fuzzy set A as follows $\mu_A^P(0) = 0.8$, $\mu_A^P(a) = \mu_A^P(b) = 0.7$, $\mu_A^P(c) = 0.6$ and $\mu_A^N(0) = -0.8$, $\mu_A^N(b) = -0.7$, and $\mu_A^N(a) = \mu_A^N(c) = -0.3$ is an $(\in, \in \vee q)$-bipolar fuzzy ideal of X but is not bipolar fuzzy ideal of X because $\mu_A^P(c) = 0.6 \ngeq 0.7 = min\{\mu_A^P(c*a), \mu_A^P(a)\}$.*

5. Conclusions

In this paper, the notion of $(\in, \in \vee q)$-bipolar fuzzy BCK/BCI-subalgebras and $(\in, \in \vee q)$-bipolar fuzzy BCK/BCI-ideals are introduced on BCK/BCI-algebras and characterized their useful properties. We investigated the relationship between $(\in, \in \vee q)$-bipolar fuzzy BCK/BCI-subalgebras and bipolar fuzzy BCK/BCI-subalgebras, and also the relation of their corresponding ideals.

In our future study of bipolar fuzzy structure of BCK/BCI-algebra, we may consider the following topics: (i) bipolar (T, S)-fuzzy soft BCK/BCI-algebra, where T and S are triangular norm and co-norm respectively, (ii) bipolar $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy soft BCK/BCI-algebra, (iii) $(\in, \in \vee q)$-bipolar fuzzy soft (p-, a- and q-)ideals and their relations. (iv) $(\in, \in \vee q)$-bipolar fuzzy relations.

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Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India
E-mail address: jana.chiranjibe7@gmail.com

Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India
E-mail address: mmaplvu@gmail.com

Department of pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran
E-mail address: arsham@mail.uk.ac.ir