CUBIC IMPLICATIVE IDEALS OF BCK-ALGEBRAS

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Abstract. In this paper, we apply the concept of cubic sets to implicative ideals of BCK-algebras, and then characterize their basic properties. We discuss relations among cubic implicative ideals, cubic subalgebras and cubic ideals of BCK-algebras. We provide a condition for a cubic ideal to be a cubic implicative ideal. We define inverse images of cubic implicative ideals and establish how the inverse images of a cubic implicative ideal become a cubic implicative ideal. Finally we introduce products of cubic BCK-algebras.

1. Introduction

BCK/BCI-algebras are two important classes of logical algebras introduced by Iseki et al. [2] and extensively investigated by several researchers. Combining the idea of fuzzy sets [16] and interval-valued fuzzy sets [17], Jun et al. [4] introduced the concept of cubic sets, and applied it to subalgebras, ideals and q-ideals in BCK/BCI-algebras [5, 6, 7]. Jun et al. [9] applied double-framed soft sets in BCK/BCI-algebras. Muhiuddin et al. [10, 11, 12] applied cubic soft sets and (α, β)-type fuzzy sets in BCK/BCI-algebras. Senapati, together with colleagues [3, 13, 14, 15], applied the notion of cubic sets in G-algebras, B-algebras, BF-algebras, and BG-algebras. Recently, Jun et al. [8] introduced cubic soft ideals in BCK/BCI-algebras.

The objective of this paper is to introduce the concept of cubic sets to implicative ideals of BCK-algebras. We prove that every cubic implicative ideal must be a cubic ideal and a cubic subalgebra. In addition, we observe that in an implicative BCK-algebra, every cubic ideal is a cubic implicative ideal. By using the concept of a cubic level set, some characterization theorems are given.

The remainder of this paper is organized as follows: in Section 2, we recall important preliminary definitions and properties. Section 3 contains definition and related results of cubic subalgebras and ideals of BCK-algebras. In Section 4, we propose concepts and operations of cubic implicative ideals and discuss their properties in details. In Section 5, we investigate properties of cubic implicative ideals under homomorphisms. In Section 6, we study products of cubic implicative ideals. Finally, in Section 7, conclusions and the scope for future research are given.

2. Preliminaries

To make this work self-contained, we briefly mention some of the definitions and results employed in the rest of the work.

An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BCI-algebra [12] if it satisfies the following axioms for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$.
(ii) $(x * (x * y)) * y = 0$.
(iii) $x * x = 0$.
(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra. Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:

(1) $(x * y) * z = (x * z) * y$.
(2) $((x * z) * (y * z)) * (x * y) = 0$.
(3) $x * 0 = x$.
(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0$.

Throughout this paper, X always means a BCK-algebra without any specification.

A BCK-algebra X is said to be implicative [2] if it satisfies the identity $x = x * (y * x)$ for all $x, y \in X$. A mapping $f : X \to Y$ of BCK-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. A non-empty subset S of X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$. A nonempty subset I of X is called an ideal of X if it satisfies

(I_1) $0 \in I$ and
(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

A nonempty subset I of X is called an implicative ideal of X if it satisfies (I_1) and (I_3) and $(x * (y * x)) * z \in I$ and $z \in I$ implies $x \in I$.

Our main objective is to investigate the idea of implicative ideals of cubic sets. The cubic set is a particular type of fuzzy set. A fuzzy set A in X is of the form $A = \{< x, \mu_A(x) >: x \in X\}$, where $\mu_A(x)$ is called the membership value of x in A and $0 \leq \mu_A(x) \leq 1$.

An interval-valued fuzzy set A over X is an object having the form $A = \{\langle x, \tilde{\mu}_A(x)\rangle : x \in X\}$, where $\tilde{\mu}_A(x) : X \to D[0, 1]$, where $D[0, 1]$ is the set of all subintervals of $[0, 1]$. The intervals $\tilde{\mu}_A(x)$ denote the intervals of the degree of membership of the element x to the set A, where $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ for all $x \in X$.

The determination of maximum and minimum between two real numbers is very simple but it is not simple for two intervals. Biswas [1] described a method to find max/sup and min/inf between two intervals or a set of intervals.

Definition 2.1. [1] *Consider two elements* $D_1, D_2 \in D[0,1]$. *If* $D_1 = [a_1^-, a_1^+]$ *and* $D_2 = [a_2^-, a_2^+]$, *then* $rmin(D_1, D_2) = [\min(a_1^-, a_2^-), \min(a_1^+, a_2^+)]$ *which is denoted by* $D_1 \wedge^r D_2$. *Thus, if* $D_i = [a_i^-, a_i^+] \in D[0,1]$ *for i=1,2,3,4,…, then we define* $rsup_i(D_i) = [\sup_i(a_i^-), \sup_i(a_i^+)]$, *i.e,* $\vee_i^r D_i = [\vee_i a_i^-, \vee_i a_i^+]$. *Now we call* $D_1 \geq D_2$ *if and only if* $a_1^- \geq a_2^-$ *and* $a_1^+ \geq a_2^+$. *Similarly, the relations* $D_1 \leq D_2$ *and* $D_1 = D_2$ *are defined.*

Based on the (interval valued) fuzzy sets, Jun et al. [4] introduced the notion of (internal, external) cubic sets, and investigated several properties.

Definition 2.2. [4] *Let* X *be a nonempty set. A cubic set* A *in* X *is a structure* $A = \{\langle x, \tilde{\mu}_A(x), \nu_A\ (x)\rangle : x \in X\}$ *which is briefly denoted by* $A = (\tilde{\mu}_A, \nu_A)$ *where* $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ *is an interval-valued fuzzy set in* X *and* ν_A *is a fuzzy set in* X.

3. Cubic Subalgebras and Ideals of BCK-Algebras

Combining the definition of subalgebra, ideal over crisp set, and the idea of cubic set, Jun et al. [5, 6, 7] defined a cubic subalgebra and ideal. This is defined below.

Definition 3.1. [5] Let $A = (\tilde{\mu}_A, \nu_A)$ be cubic set in X, then the set A is cubic subalgebra over the binary operator $*$ if it satisfies the following conditions for all $x, y \in X$:

(F1) $\tilde{\mu}_A(x * y) \geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$.
(F2) $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Definition 3.2. [6] A cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X is called a cubic ideal of X if it satisfies:

(T1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$,
(T2) $\nu_A(0) \leq \nu_A(x)$,
(T3) $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$,
(T4) $\nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\}$,

for all $x, y \in X$.

Lemma 3.3. [5] Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic ideal of X. If the inequality $x \leq y$ holds in X, then $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$ and $\nu_A(x) \leq \nu_A(y)$.

Theorem 3.4. [5] Let X be a BCK-algebra. Then every cubic ideal of X is a cubic subalgebra of X.

Proposition 3.5. [5] Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic ideal of X. If the inequality $x * y \leq z$ holds in X, then $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A(y), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \max\{\nu_A(y), \nu_A(z)\}$.

4. Cubic Implicative Ideals of BCK-Algebras

In this section, cubic implicative ideals of BCK-algebras are defined and proved in some related results.

Definition 4.1. *A cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X is called a cubic implicative ideal of X if it satisfies $(T1)$, $(T2)$ and*

(T5) $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x * (y * x)) * z), \tilde{\mu}_A(z)\}$,
(T6) $\nu_A(x) \leq \max\{\nu_A((x * (y * x)) * z), \nu_A(z)\}$, *for all $x, y, z \in X$.*

Let us illustrate Definition 4.1 using the following example.

Example 4.2. *Consider a BCK-algebra X={0, 1, a, b, c} with the following Cayley table*

$*$	0	1	a	b	c
0	0	0	0	0	0
1	1	0	1	0	0
a	a	a	0	0	0
b	b	b	b	0	0
c	c	b	c	1	0

Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic set of X defined as $\tilde{\mu}_A(0) = \tilde{\mu}_A(1) = \tilde{\mu}_A(a) = [0.6, 0.8]$, $\tilde{\mu}_A(b) = \tilde{\mu}_A(c) = [0.3, 0.4]$, $\nu_A(0) = \nu_A(1) = \nu_A(a) = 0.3$ and $\nu_A(b) = \nu_A(c) = 0.5$. Routine calculation gives that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X.

Now we give a relation between a cubic implicative ideal and a cubic ideal.

Theorem 4.3. *Any cubic implicative ideal of X must be a cubic ideal of X.*

Proof. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic implicative ideal of X. Substituting x for y in $(T5)$ and $(T6)$, we get $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x * (x * x)) * z), \tilde{\mu}_A(z)\} = rmin\{\tilde{\mu}_A((x * 0) * z), \tilde{\mu}_A(z)\} = rmin\{\tilde{\mu}_A(x * z), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \max\{\nu_A((x * (x * x)) * z), \nu_A(z)\} = \max\{\nu_A((x * 0) * z), \nu_A(z)\} = \max\{\nu_A(x * z), \nu_A(z)\}$. This shows that $A = (\tilde{\mu}_A, \nu_A)$ satisfies (T3) and (T4). Combining (T1) and (T2), A is cubic ideal of X, proving the theorem. □

By applying Theorem 3.4 and 4.3, we get the following corollary.

Corollary 4.4. *Every cubic implicative ideal of X must be a cubic subalgebra of X.*

Theorem 4.5. *Let A be a cubic ideal of X. Then A is a cubic implicative ideal of X if and only if it satisfies the conditions $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(x * (y * x))$ and $\nu_A(x) \leq \nu_A(x * (y * x))$ for all $x, y \in X$.*

Proof. Assume that A is a cubic implicative ideal of X. Taking $z = 0$ in (T5) and (T6), and using (T1) and (T2) we get the conditions.

Conversely, suppose A satisfies the above two conditions. As A is a cubic ideal hence, $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(x * (y * x)) \geq rmin\{\tilde{\mu}_A((x * (y * x)) * z), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \nu_A(x * (y * x)) \leq \max\{\nu_A((x * (y * x)) * z), \nu_A(z)\}$. Then A is a cubic implicative ideal of X. □

The converse of Theorem 4.3 may not be true as shown in the following example.

Example 4.6. *Let X be a BCK-algebra as in Example 4.2. Define a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X by $\tilde{\mu}_A(0) = \tilde{\mu}_A(a) = [1, 1]$, $\tilde{\mu}_A(1) = \tilde{\mu}_A(b) = \tilde{\mu}_A(c) = [t_1, t_2]$ and $\nu_A(0) = \nu_A(a) = 0$, $\nu_A(1) = \nu_A(b) = \nu_A(c) = s$, where $[t_1, t_2] \in D[0, 1]$ and $s \in [0, 1]$. It is easy to check that A is a cubic ideal of X, but it is not a cubic implicative ideal of X because $\tilde{\mu}_A(1) \ngeq rmin\{\tilde{\mu}_A((1*(b*1))*a), \tilde{\mu}_A(a)\}$ and $\nu_A(1) \nleq \max\{\mu_A((1*(b*1))*a), \nu_A(a)\}$.*

In the following theorem, we can see that the converse of Theorem 4.3 also holds in an implicative BCK-algebra.

Theorem 4.7. *In an implicative BCK-algebra X, every cubic ideal of X is a cubic implicative of X.*

Proof. Since X is an implicative BCK-algebra, it follows that $x = x*(y*x)$ for all $x, y \in X$. Let A be a cubic ideal of X. Then by (T3) and (T4), $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A(x * z), \tilde{\mu}_A(z)\} = rmin\{\tilde{\mu}_A((x * (y * x)) * z), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \max\{\nu_A(x * z), \nu_A(z)\} = \max\{\nu_A((x * (y * x)) * z), \nu_A(z)\}$, for all $x, y, z \in X$. Hence, A is a cubic implicative ideal of X. This completes the proof. □

Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic set in X. For any $r \in [0, 1]$ and $[s, t] \in D[0, 1]$, we define $U(A; [s, t], r)$ as follows

$$U(A; [s, t], r) = \{x \in X | \tilde{\mu}_A(z) \geq [s, t], \nu_A(x) \leq r\}$$

and say it is a cubic level set of $A = (\tilde{\mu}_A, \nu_A)$.

Theorem 4.8. *For a cubic set $A = (\tilde{\mu}_A, \nu_A)$ in X, the following are equivalent:*

(i) *$A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X.*
(ii) *Every nonempty cubic level set of $A = (\tilde{\mu}_A, \nu_A)$ is an implicative ideal of X.*

Proof. Assume that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X. Let $x, y \in X$, $r \in [0, 1]$ and $[s, t] \in D[0, 1]$. If $x \in U(A; [s, t], r)$, then $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) \geq [s, t]$ and $\nu_A(0) \leq \nu_A(x) \leq r$. Thus, $0 \in U(A; [s, t], r)$. Let $x, y, z \in X$ be such that $(x*(y*x))*z \in U(A; [s, t], r)$ and $z \in U(A; [s, t], r)$.

Then $\tilde{\mu}_A((x*(y*x))*z) \geq [s,t]$, $\nu_A((x*(y*x))*z) \leq r$, $\tilde{\mu}_A(z) \geq [s,t]$, and $\nu_A(z) \leq r$. It follows from (T5) that $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\} \geq rmin\{[s,t],[s,t]\} = [s,t]$ and $\nu_A(x) \leq \max\{\mu_A((x*(y*x))*z), \mu_A(z)\} \leq \{r,r\} = r$ so that $x \in U(A;[s,t],r)$. Hence, $U(A;[s,t],r)$ is a cubic implicative ideal of X.

Conversely, suppose that (ii) is valid, that is, $U(A;[s,t],r)$ is non-empty and is an implicative ideal of X for all $r \in [0,1]$ and $[s,t] \in D[0,1]$. Assume that $\tilde{\mu}_A(0) < \tilde{\mu}_A(a)$, that is, $[\tilde{\mu}_A^-(0), \tilde{\mu}_A^+(0)] < [\tilde{\mu}_A^-(a), \tilde{\mu}_A^+(a)]$, or $\nu_A(0) > \nu_A(b)$ for some $a,b \in X$. If we take $s_a = \frac{1}{2}(\tilde{\mu}_A^-(0)+\tilde{\mu}_A^-(a))$, $t_a = \frac{1}{2}(\tilde{\mu}_A^+(0)+\tilde{\mu}_A^+(a))$, and $r_b = \frac{1}{2}(\nu_A(0)+\nu_A(b))$, then $\tilde{\mu}_A(0) = [\tilde{\mu}_A^-(0), \tilde{\mu}_A^+(0)] < [s_a,t_a] < [\tilde{\mu}_A^-(a), \tilde{\mu}_A^+(a)] = \tilde{\mu}_A(a)$, or $\nu_A(0) > r_b > \nu_A(b)$. Hence, $0 \notin U(A;[s,t],r)$. This is a contradiction. So $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$.

Now, suppose that there exist $a,b,c \in X$ such that $\tilde{\mu}_A(a) < rmin\{\tilde{\mu}_A((a*(b*a))*c), \tilde{\mu}_A(c)\}$ and $\nu_A(a) > \max\{\nu_A((a*(b*a))*c), \nu_A(c)\}$. Let $\tilde{\mu}_A(a) = [a^-,a^+]$, $\tilde{\mu}_A((a*(b*a))*c) = [((a*(b*a))*c)^-, ((a*(b*a))*c)^+]$ and $\tilde{\mu}_A(c) = [c^-,c^+]$. Take

$$s_0 = \frac{1}{2}[a^- + \min\{((a*(b*a))*c)^-, c^-\}],$$

$t_0 = \frac{1}{2}[a^+ + \min\{((a*(b*a))*c)^+, c^+\}]$, and $r_0 = \frac{1}{2}[\nu_A(a) + \max\{\nu_A((a*(b*a))*c), \nu_A(c)\}]$. Then $a^- < s_0 < \min\{((a*(b*a))*c)^-, c^-\}$ and $a^+ < t_0 < \min\{((a*(b*a))*c)^+, c^+\}$, which implies that

$$\begin{aligned}
\tilde{\mu}_A(a) &= [a^-,a^+] < [s_0,t_0] \\
&< [\min\{((a*(b*a))*c)^-, c^-\}, \min\{((a*(b*a))*c)^+, c^+\}] \\
&= rmin\{\tilde{\mu}_A((a*(b*a))*c), \tilde{\mu}_A(c)\}
\end{aligned}$$

and $\nu_A(a) > r_0 > \max\{\nu_A((a*(b*a))*c), \nu_A(c)\}$. Thus, $(a*(b*a))*c \in U(A;[s_0,t_0],r_0)$ and $c \in U(A;[s_0,t_0],r_0)$, but $a \notin U(A;[s_0,t_0],r_0)$. This is a contradiction. Therefore, $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \max\{\mu_A((x*(y*x))*z), \mu_A(z)\}$ for all $x,y,z \in X$. Hence, $A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X. □

Theorem 4.9. *If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X, then the set*

$$I_A = \{x \in X | \tilde{\mu}_A(x) = \tilde{\mu}_A(0), \nu_A(x) = \nu_A(0)\}$$

is an implicative ideal of X.

Proof. Obviously, $0 \in I$. Let $x,y,z \in X$ such that $(x*(y*x))*z \in I_A$ and $z \in I_A$. Then $\tilde{\mu}_A((x*(y*x))*z) = \tilde{\mu}_A(0) = \tilde{\mu}_A(z)$ and $\nu_A((x*(y*x))*z) = \nu_A(0) = \nu_A(z)$, and so $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\} = \tilde{\mu}_A(0)$ and $\nu_A(x) \leq \max\{\nu_A((x*(y*x))*z), \nu_A(z)\} = \nu_A(0)$. It follows from (T1) and (T2) that $\tilde{\mu}_A(x) = \tilde{\mu}_A(0)$ and $\nu_A(x) = \nu_A(0)$ so that $x \in I$. Therefore, I_A is an implicative ideal of X. □

Theorem 4.10. *Suppose that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic ideal of X. Then the following are equivalent:*

(i) *A is a cubic implicative ideal of X.*
(ii) *$\tilde{\mu}_A(x) \geq \tilde{\mu}_A(x*(y*x))$ and $\nu_A(x) \leq \nu_A(x*(y*x))$ for all $x, y \in X$.*
(iii) *$\tilde{\mu}_A(x) = \tilde{\mu}_A(x*(y*x))$ and $\nu_A(x) = \nu_A(x*(y*x))$ for all $x, y \in X$.*

Proof. (i) $\Rightarrow$ (ii). Let A be a cubic implicative ideal of X. Then by (T5), we get

$$\begin{aligned}\tilde{\mu}_A(x) &\geq rmin\{\tilde{\mu}_A((x*(y*x))*0), \tilde{\mu}_A(0)\}\\ &= rmin\{\tilde{\mu}_A(x*(y*x)), \tilde{\mu}_A(0)\}\\ &= \tilde{\mu}_A(x*(y*x))\end{aligned}$$

and $\nu_A(x) \leq \max\{\nu_A((x*(y*x))*0), \nu_A(0)\} = \max\{\nu_A(x*(y*x)), \nu_A(0)\} = \nu_A(x*(y*x))$. Hence, condition (ii) holds.
(ii) $\Rightarrow$ (iii). Observe that in X, $x*(y*x) \leq x$. Applying Lemma 3.3 we have $\tilde{\mu}_A(x) \leq \tilde{\mu}_A(x*(y*x))$ and $\nu_A(x) \geq \mu_A(x*(y*x))$. It follows from (ii) that $\tilde{\mu}_A(x) = \tilde{\mu}_A(x*(y*x))$ and $\nu_A(x) = \nu_A(x*(y*x))$. Hence the condition (iii) holds.
(iii) $\Rightarrow$ (i). Suppose the condition (iii) holds. Since A is a cubic ideal, by (T3) and (T4), we get $\tilde{\mu}_A(x*(y*x)) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$ and $\nu_A(x*(y*x)) \leq \max\{\nu_A((x*(y*x))*z), \nu_A(z)\}$. Combining (iii) we obtain, $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \max\{\nu_A((x*(y*x))*z), \nu_A(z)\}$. Thus, $\tilde{\mu}_A$, ν_A satisfies (T3) and (T4), respectively. Obviously, $\tilde{\mu}_A$ satisfies (Tl) and ν_A satisfies (T2). Therefore, A is a cubic implicative ideal of X. Hence, condition (i) holds. The proof is complete. $\square$

Theorem 4.11. *If P is an implicative ideal of X, then there is a cubic implicative ideal $A = (\tilde{\mu}_A, \nu_A)$ of X such that $U(A; [s,t], r) = P$ for any $r \in [0,1]$ and $[s,t] \in D[0,1]$.*

Proof. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic set in X defined by

$$\tilde{\mu}_A(x) = \begin{cases} [s,t], & \text{if } x \in P;\\ [0,0], & \text{otherwise;}\end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in P;\\ r, & \text{otherwise.}\end{cases}$$

Now we aim to verify that A is a cubic implicative ideal of X. If $(x*(y*x))*z \in P$ and $z \in P$, then $x \in P$ by (I_3); hence, $\tilde{\mu}_A((x*(y*x))*z) = \tilde{\mu}_A(z) = \tilde{\mu}_A(x) = [s,t]$ and $\nu_A((x*(y*x))*z) = \nu_A(z) = \nu_A(x) = r$ and so $\tilde{\mu}_A(x) = rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$ and $\nu_A(x) = \max\{\nu_A((x*(y*x))*z), \nu_A(z)\}$. If at least one of $(x*(y*x))*z$ and z is not in P, then at least one of $\tilde{\mu}_A((x*(y*x))*z)$ and $\tilde{\mu}_A(z)$ is $[0,0]$, and also at most one of $\nu_A((x*(y*x))*z)$ and $\mu_A(z)$ is r. Hence, $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \max\{\nu_A((x*(y*x))*z), \nu_A(z)\}$. Summarizing the above results, we know that $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$ and

$\nu_A(x) \leq \max\{\nu_A((x*(y*x))*z), \nu_A(z)\}$ for all $x, y, z \in X$. Since $0 \in P$, $\tilde{\mu}_A(0) = [s,t] \geq \tilde{\mu}_A(x)$ and $\nu_A(0) = r \leq \nu_A(x)$ for all $x \in X$. Therefore, A is a cubic implicative ideal of X. Obviously, $U(A; [s,t], r) = P$. The proof is complete. □

Finally, we give an equivalent condition for which a cubic subalgebra of X is a cubic implicative ideal of X.

Theorem 4.12. *A cubic subalgebra $A = (\tilde{\mu}_A, \nu_A)$ of X is a cubic implicative ideal of X if and only if it satisfies for all $x, y, z, u \in X$,*

(T7) $(x*(y*x))*z \leq u$ *implies* $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A(z), \tilde{\mu}_A(u)\}$ *and* $\nu_A(x) \leq \max\{\nu_A(z), \nu_A(u)\}$.

Proof. Assume that $A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X and let $x, y, z, u \in X$ be such that $(x*(y*x))*z \leq u$. Since A is also a fuzzy ideal of X by Theorem 4.3, it follows from Proposition 3.5 that $\tilde{\mu}_A(x*(y*x)) \geq rmin\{\tilde{\mu}_A(z), \tilde{\mu}_A(u)\}$ and $\nu_A(x*(y*x)) \leq \max\{\nu_A(z), \nu_A(u)\}$. Making use of the Theorem 4.10(iii) we obtain $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A(z), \tilde{\mu}_A(u)\}$ and $\nu_A(x) \leq \max\{\nu_A(z), \nu_A(u)\}$ namely, $A = (\tilde{\mu}_A, \nu_A)$ satisfies (T7).

Conversely, suppose that A satisfies (T7). Obviously A satisfies (T1) and (T2). Since, $(x*(y*x))*((x*(y*x))*z) \leq z$, it follows from (T7) that $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A((x*(y*x))*z), \tilde{\mu}_A(z)\}$ and $\nu_A(x) \leq \max\{\nu_A((x*(y*x))*z), \nu_A(z)\}$ which shows that A satisfies (T5) and (T6) and so A is a cubic implicative idea of X. The proof is complete. □

5. Images and Preimages of Cubic Implicative Ideals

In this section, homomorphisms of cubic implicative ideals are defined and some results are studied.

Definition 5.1. *Let f be a mapping from a set X into a set Y. Let $B = (\tilde{\mu}_B, \nu_B)$ be a cubic set in Y. Then the inverse image of B is defined as $f^{-1}(B) = (f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B))$ of B, where $f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x))$ and $f^{-1}(\nu_B)(x) = \nu_B(f(x))$.*

Theorem 5.2. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If $B = (\tilde{\mu}_B, \nu_B)$ is a cubic implicative ideal of Y, then the preimage $f^{-1}(B) = (f^{-1}(\tilde{\mu}_B), f^{-1}(\nu_B))$ of B under f is a cubic implicative ideal of X.*

Proof. Assume that $B = (\tilde{\mu}_B, \nu_B)$ is a cubic implicative ideal of Y. For all $x \in X$, $f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x)) \leq \tilde{\mu}_B(0) = \tilde{\mu}_B(f(0)) = f^{-1}(\tilde{\mu}_B)(0)$ and $f^{-1}(\nu_B)(x) = \nu_B(f(x)) \geq \nu_B(0) = \nu_B(f(0)) = f^{-1}(\nu_B)(0)$.

Let $x, y, z \in X$. Then

$$\begin{aligned} f^{-1}(\tilde{\mu}_B)(x) &= \tilde{\mu}_B(f(x)) \\ &\geq rmin\{\tilde{\mu}_B((f(x) * (f(y) * f(x))) * f(z)), \tilde{\mu}_B(f(z))\} \\ &= rmin\{\tilde{\mu}_B(f((x * (y * x)) * z), \tilde{\mu}_B(f(z))\} \\ &= rmin\{f^{-1}(\tilde{\mu}_B)((x * (y * x)) * z), f^{-1}(\tilde{\mu}_B)(z)\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\nu_B)(x) &= \nu_B(f(x)) \\ &\leq \max\{\nu_B((f(x) * (f(y) * f(x))) * f(z)), \nu_B(f(z))\} \\ &= \max\{\nu_B(f((x * (y * x)) * z), \nu_B(f(z))\} \\ &= \max\{f^{-1}(\nu_B)((x * (y * x)) * z), f^{-1}(\nu_B)(z)\}. \end{aligned}$$

Hence, $f^{-1}(B) = (f^{-1}(\tilde{\mu}_B), f^{-1}\ (\nu_B))$ is a cubic implicative ideal of X. □

Definition 5.3. *A cubic set $A = (\tilde{\mu}_A, \nu_A)$ of X has rsup-property and infimum property if for any T of X there exist $t_0 \in T$ such that $\tilde{\mu}_A(t_0) = rsup_{t_0 \in T}\tilde{\mu}_A(t)$ and $\nu_A(t_0) = \inf\limits_{t_0 \in T} \nu_A(t)$, respectively.*

Definition 5.4. *Let f be a mapping from the set X to the set Y. If $A = (\tilde{\mu}_A, \nu_A)$ is cubic set in X, then the cubic subset $B = (\tilde{\mu}_B, \nu_B)$ of Y is defined as*

$$f(\tilde{\mu}_A)(y) = \tilde{\mu}_B(y) = \begin{cases} rsup_{x \in f^{-1}(y)}\tilde{\mu}_A(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ [0, 0], & \text{otherwise;} \end{cases}$$

and

$$f(\nu_A)(y) = \nu_B(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \nu_A(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 1, & \text{otherwise;} \end{cases}$$

is said to be the images of $A = (\tilde{\mu}_A, \nu_A)$ under f.

Theorem 5.5. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If $A = (\tilde{\mu}_A, \nu_A)$ is a cubic implicative ideal of X, then the image $B = (\tilde{\mu}_B, \nu_B)$ of A under f is a cubic implicative ideal of Y.*

Proof. Let $A = (\tilde{\mu}_A, \nu_A)$ be a cubic implicative ideal of X with rsup-property and infimum property and $B = (\tilde{\mu}_B, \nu_B)$ be the images of A under f. Since A is a cubic implicative ideal it must be a cubic ideal by Theorem 4.3. Therefore, we have $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$.

Note that $0 \in f^{-l}(0')$, where 0 and $0'$ are the zero elements of X and Y, respectively. Thus, $\tilde{\mu}_B(0') = rsup_{t \in f^{-1}(0')}\tilde{\mu}_A(t) = \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\nu_B(0') = \inf\limits_{t \in f^{-1}(0')} \nu_A(t) = \nu_A(0) \leq \nu_A(x)$ for all $x \in X$, which implies that $\tilde{\mu}_B(0') \geq rsup_{t\in f^{-1}(x')}\tilde{\mu}_A(t) = \tilde{\mu}_B(x')$ and $\nu_B(0') \leq \inf\limits_{t\in f^{-1}(x')} \nu_A(t) = \nu_B(x')$ for any $x' \in Y$.

For any $x', y', z' \in Y$, let $x_0 \in f^{-1}(x')$, $y_0 \in f^{-1}(y')$ and $z_0 \in f^{-1}(z')$ be such that

$$\begin{aligned}
\tilde{\mu}_A(x_0) &= rsup_{t\in f^{-1}(x')}\tilde{\mu}_A(t),\\
\nu_A(x_0) &= \inf_{t\in f^{-1}(x')} \nu_A(t),\\
\tilde{\mu}_A(z_0) &= rsup_{t\in f^{-1}(z')}\tilde{\mu}_A(t),\\
\nu_A(z_0) &= \inf_{t\in f^{-1}(z')} \nu_A(t),\\
\tilde{\mu}_A((x_0 * (y_0 * x_0)) * z_0) &= \tilde{\mu}_B[f((x_0 * (y_0 * x_0)) * z_0)]\\
&= \tilde{\mu}_B((x' * (y' * x')) * z')\\
&= rsup_{((x_0*(y_0*x_0))*z_0)\in f^{-1}((x'*(y'*x'))*z')}\\
&\quad\ \tilde{\mu}_A((x_0 * (y_0 * x_0)) * z_0)\\
&= rsup_{t\in f^{-1}((x'*(y'*x'))*z')}\tilde{\mu}_A(t)
\end{aligned}$$

and

$$\begin{aligned}
\nu_A((x_0 * (y_0 * x_0)) * z_0) = \nu_B[f((x_0 * (y_0 * x_0)) * z_0)] &= \nu_B((x' * (y' * x')) * z')\\
&= \inf_{((x_0*(y_0*x_0))*z_0)\in f^{-1}((x'*(y'*x'))*z')} \nu_A((x_0 * (y_0 * x_0)) * z_0)\\
&= \inf_{t\in f^{-1}((x'*(y'*x'))*z')} \nu_A(t).
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{\mu}_B(x') &= rsup_{t\in f^{-1}(x')}\tilde{\mu}_A(t) = \tilde{\mu}_A(x_0)\\
&\geq rmin\{\tilde{\mu}_A((x_0 * (y_0 * x_0)) * z_0), \tilde{\mu}_A(z_0)\}\\
&= rmin\{rsup_{t\in f^{-1}((x'*(y'*x'))*z')}\tilde{\mu}_A(t), rsup_{t\in f^{-1}(z')}\tilde{\mu}_A(t)\}\\
&= rmin\{\tilde{\mu}_B((x' * (y' * x')) * z'), \tilde{\mu}_B(z')\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_B(x') &= \inf_{t\in f^{-1}(x')} \nu_A(t) = \nu_A(x_0)\\
&\leq \max\{\nu_A((x_0 * (y_0 * x_0)) * z_0), \nu_A(z_0)\}\\
&= \max\left\{\inf_{t\in f^{-1}((x'*(y'*x'))*z')} \nu_A(t), \inf_{t\in f^{-1}(z')} \nu_A(t)\right\}\\
&= \max\{\nu_B((x' * (y' * x')) * z'), \nu_B(z')\}.
\end{aligned}$$

Hence, $B = (\tilde{\mu}_B, \nu_B)$ is a cubic implicative ideal of Y. □

6. Product of Cubic Implicative Ideals of BCK-Algebras

In this section, product of cubic BCK-algebras are defined and some results are studied.

Definition 6.1. *Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be two cubic sets of X and Y, respectively. The cartesian product $A \times B = (X \times Y, \tilde{\mu}_A \times \tilde{\mu}_B, \nu_A \times \nu_B)$ is defined by $(\tilde{\mu}_A \times \tilde{\mu}_B)(x, y) = rmin\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\}$ and $(\nu_A \times \nu_B)(x, y) = \max\{\nu_A(x), \nu_B(y)\}$, where $\tilde{\mu}_A \times \tilde{\mu}_B : X \times Y \to D[0, 1]$ and $\nu_A \times \nu_B : X \times Y \to [0, 1]$ for all $(x, y) \in X \times Y$.*

Remark 6.2. *Let X and Y be BCK-algebras. We define $*$ on $X \times Y$ by $(x, y) * (z, p) = (x * z, y * p)$ for every (x, y) and $(z, p) \in X \times Y$. Then clearly, $X \times Y$ is a BCK-algebra.*

Definition 6.3. *A cubic subset $A \times B = (X \times Y, \tilde{\mu}_A \times \tilde{\mu}_B, \nu_A \times \nu_B)$ is called a cubic implicative ideal if*

(T7) $(\tilde{\mu}_A \times \tilde{\mu}_B)(0, 0) \geq (\tilde{\mu}_A \times \tilde{\mu}_B)(x, y)$, $(\nu_A \times \nu_B)(0, 0) \leq (\nu_A \times \nu_B)(x, y)$ *for all $(x, y) \in X \times Y$;*

(T8) $(\tilde{\mu}_A \times \tilde{\mu}_B)(x_1, y_1) \geq rmin\{(\tilde{\mu}_A \times \tilde{\mu}_B)((x_1, y_1) * ((x_2, y_2) * (x_1, y_1))) * (x_3, y_3)), (\tilde{\mu}_A \times \tilde{\mu}_B)(x_3, y_3)\}$; *and*

(T9) $(\nu_A \times \nu_B)(x_1, y_1) \leq \max\{(\nu_A \times \nu_B)((x_1, y_1) * ((x_2, y_2) * (x_1, y_1))) * (x_3, y_3)), (\nu_A \times \nu_B)(x_3, y_3)\}$, *for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.*

Theorem 6.4. *Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be cubic implicative ideals of X and Y, respectively. Then $A \times B$ is a cubic implicative ideal of $X \times Y$.*

Proof. For any $(x, y) \in X \times Y$, we have

$$\begin{aligned}(\tilde{\mu}_A \times \tilde{\mu}_B)(0, 0) &= rmin\{\tilde{\mu}_A(0), \tilde{\mu}_B(0)\} \\ &\geq rmin\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\} = (\tilde{\mu}_A \times \tilde{\mu}_B)(x, y)\end{aligned}$$

and

$$\begin{aligned}(\nu_A \times \nu_B)(0, 0) &= \max\{\nu_A(0), \nu_B(0)\} \\ &\leq \max\{\nu_A(x), \nu_B(y)\} \\ &= (\nu_A \times \nu_B)(x, y).\end{aligned}$$

Let (x_1, y_1), (x_2, y_2) and $(x_3, y_3) \in X \times Y$. Then

$$\begin{aligned}
&(\tilde{\mu}_A \times \tilde{\mu}_B)(x_1, y_1) = rmin\{\tilde{\mu}_A(x_1), \tilde{\mu}_B(y_1)\} \\
&\geq rmin\{rmin\{\tilde{\mu}_A((x_1 * (x_2 * x_1)) * x_3), \\
&\quad \tilde{\mu}_A(x_3)\}, rmin\{\tilde{\mu}_B((y_1 * (y_2 * y_1)) * y_3), \tilde{\mu}_B(y_3)\}\} \\
&= rmin\{rmin\{\tilde{\mu}_A((x_1 * (x_2 * x_1)) * x_3), \\
&\quad \tilde{\mu}_B((y_1 * (y_2 * y_1)) * y_3), rmin\{\tilde{\mu}_A(x_3), \tilde{\mu}_B(y_3)\}\} \\
&= rmin\{(\tilde{\mu}_A \times \tilde{\mu}_B)(((x_1 * (x_2 * x_1)) * x_3), \\
&\quad ((y_1 * (y_2 * y_1)) * y_3)), (\tilde{\mu}_A \times \tilde{\mu}_B)(x_3, y_3)\} \\
&= rmin\{(\tilde{\mu}_A \times \tilde{\mu}_B)(((x_1 * (x_2 * x_1)), \\
&\quad ((y_1 * (y_2 * y_1)))) * (x_3, y_3)), (\tilde{\mu}_A \times \tilde{\mu}_B)(x_3, y_3)\} \\
&= rmin\{(\tilde{\mu}_A \times \tilde{\mu}_B)(((x_1, y_1) * ((x_2 * x_1), \\
&\quad (y_2 * y_1))) * (x_3, y_3)), (\tilde{\mu}_A \times \tilde{\mu}_B)(x_3, y_3)\} \\
&= rmin\{(\tilde{\mu}_A \times \tilde{\mu}_B)(((x_1, y_1) * ((x_2, y_2), (x_1, y_1))) * (x_3, y_3)), \\
&\quad (\tilde{\mu}_A \times \tilde{\mu}_B)(x_3, y_3)\}
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_A \times \nu_B)(x_1, y_1) = \max\{\nu_A(x_1), \nu_B(y_1)\} \\
&\leq \max\{\max\{\nu_A((x_1 * (x_2 * x_1)) * x_3), \nu_A(x_3)\}, \\
&\quad \max\{\nu_B((y_1 * (y_2 * y_1)) * y_3), \nu_B(y_3)\}\} \\
&= \max\{\max\{\nu_A((x_1 * (x_2 * x_1)) * x_3), \nu_B((y_1 * (y_2 * y_1)) * y_3), \\
&\quad \max\{\nu_A(x_3), \nu_B(y_3)\}\} \\
&= \max\{(\nu_A \times \nu_B)(((x_1 * (x_2 * x_1)) * x_3), ((y_1 * (y_2 * y_1)) * y_3)), \\
&\quad (\nu_A \times \nu_B)(x_3, y_3)\} \\
&= \max\{(\nu_A \times \nu_B)(((x_1, y_1) * ((x_2, y_2), (x_1, y_1))) * (x_3, y_3)), \\
&\quad (\nu_A \times \nu_B)(x_3, y_3)\}
\end{aligned}$$

Hence, $A \times B$ is a cubic implicative ideal of $X \times Y$. □

Definition 6.5. *Let $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$ be cubic subset of X and Y respectively. For $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, the set $U(\tilde{\mu}_A \times \tilde{\mu}_B : [s_1, s_2]) = \{(x, y) \in X \times Y | (\tilde{\mu}_A \times \tilde{\mu}_B)(x, y) \geq [s_1, s_2]\}$ is called upper $[s_1, s_2]$-level of $A \times B$ and $L(\nu_A \times \nu_B : t) = \{(x, y) \in X \times Y | (\nu_A \times \nu_B)(x, y) \leq t\}$ is called lower t-level of $A \times B$.*

Theorem 6.6. *For any two cubic sets $A = (\tilde{\mu}_A, \nu_A)$ and $B = (\tilde{\mu}_B, \nu_B)$, $A \times B$ is a cubic implicative ideals of $X \times Y$ if and only if the non-empty upper $[s_1, s_2]$-level cut $U(\tilde{\mu}_A \times \tilde{\mu}_B : [s_1, s_2])$ and the non-empty lower t-level* *cut $L(\nu_A \times \nu_B : t)$ are implicative ideals of $X \times Y$ for any $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$.*

Proof. The proof is straightforward. □

7. Conclusions and Future Work

In this paper, cubic implicative ideals of BCK-algebras are introduced and their related properties are discussed in details. It is our hope that this work will provide a foundation for further study of the theory of BCK/BCI-algebras. In our future study of fuzzy structure of BCK/BCI-algebras, the following topics should be considered:

(i) to find cubic positive implicative ideals in BCK/BCI-algebras;
(ii) to find cubic commutative ideals in BCK/BCI-algebras;
(iii) to find the relationship between cubic implicative ideals, positive implicative ideals and commutative ideals in BCK/BCI-algebras.

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