AN ELEMENTARY APPROACH TO THE DIOPHANTINE EQUATION $ax^m + by^n = z^r$ USING CENTER OF MASS

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Abstract. This paper takes an interesting approach to conceptualize some power sum inequalities and uses them to develop limits on possible solutions to some Diophantine equations. In this work, we introduce how to apply center of mass of a k-mass-system to discuss a class of Diophantine equations (with fixed positive coefficients) and a class of equations related to Fermat's Last Theorem. By a constructive method, we find a lower bound for all positive integers that are not the solution for these type of equations. Also, we find an upper bound for any possible integral solution for these type of equations. We write an alternative expression of Fermat's Last Theorem for positive integers in terms of the product of the centers of masses of the systems of two fixed points (positive integers) with different masses. Finally, by assuming the validity of Beal's conjecture, we find an upper bound for any common divisor of x, y, and z in the expression $ax^m + by^n = z^r$ in terms of a, b, m(or n)$, r$, and the center of mass of the k-mass-system of x and y.

1. Introduction

In physics, we usually apply mathematics to solve or express a physical phenomenon by finding a mathematical model (expression). In this work, our goal is to show the capability of physics to discuss (approximate) a mathematical problem (expression) via *center of mass of a k-mass-system* (Definition 2.1). For more discussion and a general overview of this matter, see [5]. Also, the author assumes that the reader is familiar with the concept of center of mass; and the problems in number theory that we use in this article. For a detailed study of center of mass, the reader is referred to [2, Chapter 5 p. 210] (for a general theory) and [7, p. 426] (for a special case that we use in this paper); and for the related problems in number theory, see [1] (on Diophantine equations), [3], and [6] (for a general overview in number theory).

This research was in part supported by grant no.95160041 from IPM.

The main goal of this paper is to study some properties of the equation $ax^m + by^n = z^r$. Besides many numerical examples (Section 3), the main results of this paper are Theorem 2.6 and Theorem 2.9 (whose proofs are mainly based on Proposition 2.2) which discusses some possible and impossible cases for a class of Diophantine equations and a class of equations related to Fermat's Last Theorem, respectively. Also, as a corollary to Theorem 2.9 (Corollary 2.11), we find an upper bound for any common factor of the *bases* of the equation $ax^m + by^n = z^r$ (see below) with respect to other parameters for *Beal's Conjecture* by assuming its validity (see also [4]).

We now state Beal's Conjecture as follows.

Conjecture 1.1 (Beal's Conjecture). *If $x^m + y^n = z^r$ where x, y, z (bases of the expression), m, n, and r are all positive integers; and m, n, and r are greater than two, then x, y, and z have a common factor (greater than one).*

• Clearly Fermat's Last Theorem is a special case of $x^m + y^n = z^r$ (the above expression), so if we could find some (easy) way to transform this into Fermat's Last Theorem, then we can easily discuss it via Wiles's result on Fermat's Last Theorem, for which there is no integral solution [8] (see for example, Proposition 2.10). If we do not require that all the exponents be greater than two, then there are infinitely many solutions such as $1^1 + 2^3 = 3^2$, $2^5 + 7^2 = 3^4$, and all Pythagorean triples. Also, there are infinitely many solutions for which x, y, and z are not relatively prime such as $2^n + 2^n = 2^{n+1}$.

• We conclude this section with two simple examples from number theory regarding the center-of-mass method and discuss more general cases on k-mass-systems (Definition 2.1) with some more examples in the next two sections. Note that, in this work (unless otherwise indicated), each point (number) on the x-axis is assumed to have a mass besides its distance from the origin.

Example 1.2. *What is the sum $S = 1 + 2 + \cdots + n$? Let us think of each number as representing the location of a physical point where we place a mass of size one unit of mass. Suppose $x_i = i$ (on the x-axis) is the distance of the ith point (number) from the origin for each $i = 1, 2, \ldots, n$ and also assume each x_i has mass $m_i = 1$ unit of mass. Hence, by the definition of center of mass, $S = \sum_1^n m_i x_i = c \sum_1^n m_i$ implies $S = nc$, where $c = (n+1)/2$ is the center of mass of the n points on x-axis, which is precisely in the middle of the interval $[1, n]$ since all the points in the mass system* *have equal masses and distributed uniformly; and $m_i = 1$ unit of mass for each $i = 1, 2, \dots, n$. More generally, we can apply this method to find the sum of the terms of any arithmetic progression $a, a+d, a+2d, \dots, a+(n-1)d$ with $a \geq 1$, $d \geq 1$, and $n \geq 2$ (an integer), by assuming each term $a + kd$ ($0 \leq k \leq n-1$) of the sequence has one unit of mass. That is, $S = nc$ where $c = [a + (a + (n-1)d)]/2$.*

Example 1.3. *Let $x_1, x_2, \dots, x_n$ be a strictly increasing sequence of $n \geq 2$ positive integers. What is $S = \sum_1^n x_i$? Again, we assume that each number x_i represents the location of a physical point with mass $m_i = 1$ for each $i = 1, 2, \dots, n$. Now, we apply the above center of mass method and get $S = nc$, where c is the center of mass of the system, which is either between x_1 and $(x_1 + x_n)/2$ (or $(x_1 + x_n)/2$ and x_n) whenever the number of x_i's in the first [resp. second] half part of the interval $[x_1, x_n]$ are more than the number of x_i's in the second [resp. first] half part of the interval $[x_1, x_n]$, respectively (see also the following remark).*

Remark 1.4. *Note that in a mass system of points, the center of mass or equilibrium point of the system is closer to the heavier (more massive) part of the system. Hence, any method or means that helps us to get a closer value (approximation) of the center of mass yields a sharper (better) approximation of the unknown entity in the related mathematical expression of the system.*

2. Arithmetical Inequalities via Center of Mass

We begin this section with the definition of a k-mass-system of $n \geq 2$ distinct positive integers with n fixed coefficients. We write the sum of the kth power of the n points of a k-mass-system in terms of the product of the centers of masses of the t-mass-systems with $1 \leq t \leq k$ (Proposition 2.2) and apply it to write Fermat's Last Theorem in terms of the product of the centers of masses (Corollary 2.3). The proof of all results in this section are mainly based on Proposition 2.2. We discuss the lower and upper bounds of possible and impossible cases (solutions) for a class of Diophantine equations with fixed positive coefficients (Theorem 2.6 and Corollary 2.7) and a class of equations related to Fermat's Last Theorem (Theorem 2.9). Finally, by assuming the validity of Beal's conjecture, we find an upper bound for any common divisor of x, y, and z in the expression $ax^m + by^n = z^r$ (Corollary 2.11). Also, in the next section, we provide some number-theoretic examples as an application to the results of this section.

Definition 2.1. *Let $n \geq 2$, $k \geq 1$, and $a_1, a_2, \dots, a_n$ be positive fixed integers. A strictly increasing sequence $x_1 < x_2 < \cdots < x_n$ of n positive integers is called a k-mass-system with coefficients $a_1, a_2, \dots, a_n$, denoted by*

$M(k; a_1, a_2, \ldots, a_n)$, if we assume each x_i represents the location of a fixed physical point on x-axis with each point x_i having $m_i = a_i x_i^{k-1}$ unit(s) of mass for each $i = 1, 2, \ldots, n$. Note that for the sake of convenience, $M(k; a_1, a_2, \ldots, a_n)$ will simply be denoted by $M(k, n)$ whenever $a_1 = a_2 = \cdots = a_n = 1$.

We now show that the sum of the kth power of the points of a k-mass-system can be written in terms of the product of the centers of masses of the t-mass-systems with $1 \leq t \leq k$. Note that throughout, we will be using the notation $S_k = \sum_{i=1}^{n} a_i x_i^k$, $S_0 = a_1 + a_2 + \cdots + a_n$, and c_t is the center of mass of the t-mass-system of $x_1, x_2, \ldots, x_n$ with coefficients $a_1, a_2, \ldots, a_n$ for each fixed $t = 1, 2, \ldots, k$.

Proposition 2.2. *For fixed positive integers $n \geq 2$ and $k \geq 1$, let*

$$M(k; a_1, a_2, \ldots, a_n)$$

be a k-mass-system of the points $x_1, x_2, \ldots, x_n$ with fixed positive coefficients $a_1, a_2, \ldots, a_n$. Let $S_k = \sum_{i=1}^{n} a_i x_i^k$ and $S_0 = a_1 + a_2 + \cdots + a_n$. Then $S_k = S_0 c_1 c_2 \cdots c_k$, where c_t is the center of mass of the t-mass-system of $x_1, x_2, \ldots, x_n$ with coefficients $a_1, a_2, \ldots, a_n$ for each fixed $t = 1, 2, \ldots, k$.

Proof. From the definition of the center of mass of n distinct points, it follows that $\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = c_k$ is the center of mass of n points (x_i's on x-axis) with m_i unit(s) of mass for each $i = 1, 2, \ldots, n$. Hence, $S_k = \sum_{i=1}^{n} m_i x_i = c_k S_{k-1}$, where c_k is the center of mass of the k-mass-system $M(k; a_1, a_2, \ldots, a_n)$ of the points $x_1 < x_2 < \cdots < x_n$. Also, note that $S_0 = a_1 + a_2 + \cdots + a_n$ by assumption. Now, from the above, it is clear that

$$\frac{S_k}{S_{k-1}} \frac{S_{k-1}}{S_{k-2}} \cdots \frac{S_1}{S_0} = c_k c_{k-1} \cdots c_1.$$

Thus, $S_k = S_0 c_1 c_2 \cdots c_k$. □

The following corollary provides an alternative expression of Fermat's Last Theorem (for positive integers) in terms of the product of the centers of masses.

Corollary 2.3. *Suppose $x < y$ is a k-mass system of 2 positive fixed integers (points) on x-axis with coefficients $a_1 = a_2 = 1$. Then for any fixed integer $k \geq 3$, the statement "$x^k + y^k = S_k = 2c_1 c_2 \cdots c_k$ cannot be kth power of a positive integer" is equivalent to Fermat's Last Theorem for positive integers.*

Remark 2.4. *In the above corollary, suppose $S_k = z^k$ for some positive integer z and positive integer $k \geq 3$. Thus, if the product of c_i's (the center of masses) is an integer, then z cannot be an odd integer. Furthermore, if $c_1 c_2 \cdots c_k$ is a fraction with a denominator different from 1 and 2, then the equality $S_k = z^k$ is never valid for any integer z. Actually, $c_1 c_2 \cdots c_k$ must always be an integer or a fraction (of course, in reduced form) with denominator 2 since S_k is always a positive integer. But since we know there is no solution to Fermat's Last Theorem [8], $2c_1 c_2 \cdots c_k$ is never the kth power of a positive integer.*

We will use the following lemma in the proofs of Theorems 2.6 and 2.9, respectively.

Lemma 2.5. *For fixed positive integers $n \geq 2$ and $k \geq 1$, let*

$$M(k; a_1, a_2, \ldots, a_n)$$

be a k-mass-system of the positive integers $x_1 < x_2 < \cdots < x_n$ with fixed positive coefficients $a_1, a_2, \ldots, a_n$. Let $S_t = \sum_{i=1}^{n} a_i x_i^t$ and $S_0 = a_1 + a_2 + \cdots + a_n$, where $1 \leq t \leq k$. Suppose $c_t = S_t / S_{t-1}$ for each $t = 1, 2, \ldots, k$. Then $c_1 \leq c_2 \leq \cdots \leq c_k$.

Proof. We just show that $c_k \geq c_{k-1}$ and leave the other parts to the reader. That is, $\frac{S_k}{S_{k-1}} \geq \frac{S_{k-1}}{S_{k-2}}$. Thus for the proof it suffices to show that $S_k S_{k-2} \geq S_{k-1} S_{k-1}$ or equivalently $2S_k S_{k-2} \geq 2S_{k-1} S_{k-1}$. We note that for all i and j, $x_i^2 + x_j^2 \geq 2x_i x_j$. Clearly,

$$\begin{aligned}
2S_k S_{k-2} = S_k S_{k-2} + S_{k-2} S_k &= \sum_{i,j} [(a_i x_i^k)(a_j x_j^{k-2})] + \sum_{i,j} [(a_i x_i^{k-2})(a_j x_j^k)] \\
&= \sum_{i,j} [(a_i x_i^{k-2})(a_j x_j^{k-2}) x_i^2]` + \sum_{i,j} [(a_i x_i^{k-2})(a_j x_j^{k-2}) x_j^2] \\
&= \sum_{i,j} [(a_i x_i^{k-2})(a_j x_j^{k-2})(x_i^2 + x_j^2)] \\
&\geq 2 \sum_{i,j} [(a_i x_i^{k-2})(a_j x_j^{k-2})(x_i x_j)] \\
&= 2 \sum_{i,j} [(a_i x_i^{k-1})(a_j x_j^{k-1})] = 2S_{k-1} S_{k-1}.
\end{aligned}$$

□

We now consider a class of Diophantine equations via a k-mass-system to show some possible and impossible cases regarding the lower and upper bounds of solutions.

Theorem 2.6. *Let $n \geq 2$, $k \geq 2$, $a_1, a_2, \ldots, a_n$, and $x_1 < x_2 \cdots < x_n$ be any fixed positive integers. Let $S_0 = a_1 + a_2 + \cdots + a_n$. Then the*

Diophantine equation $a_1x_1^k + a_2x_2^k + \cdots + a_nx_n^k = z^k$ *is never valid for all integers* $z \geq S_0^{1/k}x_n$. *Furthermore, if there exists any integral solution* z *for this Diophantine equation, it must satisfy the inequality* $z \leq S_0^{1/k}c_k$, *where* $c_k = \frac{S_k}{S_{k-1}}$.

Proof. From Proposition 2.2, we have $S_k = S_0c_1c_2\cdots c_k = z^k$. Let $c = \max\{c_1, c_2, \ldots, c_k\}$ which by Lemma 2.5, $c = c_k$. Then $S_k = a_1x_1^k + a_2x_2^k + \cdots + a_nx_n^k = z^k = S_0c_1c_2\cdots c_k \leq S_0c^k$. Hence, $z \leq S_0^{1/k}c \leq S_0^{1/k}x_n$ since c_i the center of mass of each mass system is always less than x_n for any $1 \leq i \leq k$. □

We now apply the above theorem to a class of equations related to Fermat's Last Theorem regarding the lower and upper bounds of (non)-solutions.

Corollary 2.7. *For any given fixed positive integers* $x < y$ *and* $k \geq 2$, *the* $x^k + y^k = z^k$ *is impossible for all positive integers* $z \geq 2^{1/k}y$. *Moreover, if there is an integral solution* z *for* $x^k+y^k = z^k$, *it must satisfy* $y \leq z \leq 2^{1/k}c$, *where* $c = \frac{S_k}{S_{k-1}}$.

Remark 2.8. *In the above corollary, it is clear that computing* $2^{1/k}y$ *has an advantage in finding the* k*th root of* $x^k + y^k$ *from a computational point of view (especially for a large integer* k*).*

We now turn our attention to the equation $ax^m + by^n = z^r$ (with some restrictions on its parameters) and discuss it via a k-mass-system to show some possible and impossible cases regarding the lower and upper bounds of solutions.

Theorem 2.9. *For any fixed positive integers* $x < y$ *and* $a, b, m, n, r \geq 1$, *we have the following results:*

(a) *Suppose* $m > n$ *and* $t = m - n$. *Then the expression* $ax^m + by^n = z^r$ *is impossible for all positive integers* $z \geq ((ay^t + b)/y^t)^{1/r}y$. *Moreover, if there is an integral solution* z *for* $ax^m + by^n = z^r$, *it must satisfy* $z \leq ((ay^t + b)/y^t)^{1/r}c^{m/r}$, *where* $c = \frac{S_m}{S_{m-1}}$ *with* $S_m = ay^tx^m + by^m$ *and* $S_{m-1} = ay^tx^{m-1} + by^{m-1}$;

(b) *Suppose* $m < n$ *and* $t = n-m$. *Then the expression* $ax^m+by^n = z^r$ *is impossible for all positive integers* $z \geq ((bx^t + a)/x^t)^{1/r}c^{n/r}$, *where* $c = \frac{S_n}{S_{n-1}}$ *with* $S_n = ax^n + bx^ty^n$ *and* $S_{n-1} = ax^{n-1} + bx^ty^{n-1}$. *Moreover, if there is an integral solution* z *for* $ax^m+by^n = z^r$, *it must satisfy* $z \leq ((bx^t + a)/x^t)^{1/r}c^{n/r}$;

(c) *Let* $m = n$. *Then the expression* $ax^m + by^m = z^r$ *is impossible for all positive integers* $z \geq ((a + b)^{1/r}c^{m/r}$, *where* $c = \frac{S_m}{S_{m-1}}$ *with*

$S_m = ax^m + by^m$ and $S_{m-1} = ax^{m-1} + by^{m-1}$. Moreover, if there is an integral solution z for $ax^m + by^n = z^r$, it must satisfy $z \leq (a+b)^{1/r} c^{m/r}$;

(d) *For $m = n$, $ax^m + by^m = z^r$ has no integral solution whenever*

$$r \geq \frac{(\ln(a+b) + m \ln c)}{\ln 2}.$$

Proof. We just give a proof for Part (a). Suppose $m > n$ and $t = m - n$. Let $S_m = ay^t x^m + by^m$, $S_{m-1} = ay^t x^{m-1} + by^{m-1}$, $\cdots$, $S_1 = ay^t x + by$, and $S_0 = ay^t + b$. Similar to the proof of Proposition 2.2, $S_m = S_0 c_1 c_2 \cdots c_m$, where c_i is the center of mass of the i-mass-system for each fixed $i = 1, 2, \ldots, m$. Again, similar to Lemma 2.5, it is not difficult to show that $c_m \geq c_{m-1} \cdots \geq c_1$. Now by denoting $c = c_m = \max\{c_1, c_2, \ldots, c_m\}$, the proof is immediate. The proof of (b) is similar to (a) and Part (c) is a special case of (a) for $t = 0$. The proof of Part (d) follows from Part (c) if we assume $(a+b)^r c^{m/r} < 2$ since $z = 1$ is the only positive integer less than 2. ☐

We now discuss a very special case of the equation $x^m + y^n = z^r$.

Proposition 2.10. *Let $m \geq 3$ and $m > r$ be positive integers with $t = m - r$. If $z^t x^m = u^m$ and $z^t y^m = w^m$ for some integers u and w, then $x^m + y^m = z^r$ is never valid for any set of distinct integers x, y, and z.*

Proof. Suppose $x^m + y^m = z^r$ is valid for some distinct integers x, y, and z. Then $z^m = z^t x^m + z^t y^m = u^m + w^m$ which is impossible since there is no integral solution for Fermat's Last Theorem by [8]. ☐

We conclude this section by finding an upper bound for any common divisor of x, y, and z in the expression $ax^m + by^n = z^r$ with respect to other parameters by applying Theorem 2.9 and assuming the validity of Beal's conjecture (see also [4]).

Corollary 2.11. *Let $m, n, r \geq 3$ and a, b, x, y, and z ($x < y$) be positive integers. Suppose Beal's Conjecture is true and consequently, there exists a common factor (positive integer p) of the bases (x, y, and z) of the expression $ax^m + by^n = z^r$. Then we have the following results:*

(a) *Suppose $m > n$ and $t = m - n$. Then $p \leq z \leq ((ay^t + b)/y^t)^{1/r} c^{m/r}$, where $c = \frac{S_m}{S_{m-1}}$ with $S_m = ay^t x^m + by^m$ and $S_{m-1} = ay^t x^{m-1} + by^{m-1}$;*

(b) *Suppose $m < n$ and $t = n - m$. Then $p \leq z \leq ((bx^t + a)/x^t)^{1/r} c^{n/r}$, where $c = \frac{S_n}{S_{n-1}}$ with $S_n = ax^n + bx^t y^n$ and $S_{n-1} = ax^{n-1} + bx^t y^{n-1}$;*

(c) *Let* $m = n$*, then* $p \leq z \leq ((a+b)^{1/r} c^{m/r}$*, where* $c = \frac{S_m}{S_{m-1}}$ *with* $S_m = ax^m + by^m$ *and* $S_{m-1} = ax^{m-1} + by^{m-1}$.

3. Some Number-theoretic Examples

In this section we apply some of the (main) results of the previous section to some numerical cases.

We now provide a simple example for Corollary 2.7.

Example 3.1. *Let* $x = 3$*,* $y = 4$*, and* $k = 2$*. Clearly, by the above corollary,* $z \geq \sqrt{2}y = \sqrt{2}\,4 > 5$ *is not an integral solution for* $x^2 + y^2 = z^2$*. Moreover, if there exists an integral solution* z *for* $x^2 + y^2 = z^2$*,* z *must satisfy* $y = 4 < z \leq \sqrt{2}c$*, where* $c = (25/7)$ *and* $\sqrt{2}(25/7) > 5$*. Hence, in this case* $z = 5$ *is the only integer between 4 and* $\sqrt{2}(25/7)$*.*

In the following example, we illustrate (check) the validity of the results of Theorem 2.9 for some numerical cases.

Example 3.2. (a) *In the expression* $3^3 + 6^3 = 3^5$*, we have* $m = n = 3$*,* $t = m - n = 0$*,* $r = 5$*, and* $3 = z \leq 2^{1/5}c^{3/5} = 2^{1/5}(5.4)^{3/5} \approx 3.1596$*, where*

$$c = \frac{3^3 + 6^3}{3^2 + 6^2} = 5.4.$$

(b) *The expression* $7^3 + 13^2 = 2^9$ *shows that Beal's conjecture is false if one of the exponents is allowed to be 2. In this expression, we have* $m = 3 > n = 2$*,* $t = m - n = 1$*,* $r = 9$*, and* $2 = z \leq ((13+1)/13)^{1/9}c^{3/9} = (14/13)^{1/9}(256/31)^{1/3} \approx 2.0379$*, where*

$$c = \frac{7^3 + 13^2}{7^2 + 13} = (256/31).$$

(c) *In the expression* $27^4 + 162^3 = 9^7$*, we have* $m = 4 > n = 3$*,* $t = m - n = 1$*,* $r = 7$*, and* $9 = z \leq ((162+1)/162)^{1/7}c^{4/7} = (163/162)^{1/7}(729/7)^{4/7} \approx 14.2335$*, where*

$$c = \frac{27^4 + 162^3}{27^3 + 162^2} = (729/7).$$

Example 3.3. *(See* [9]*). Let* a*,* b *(*$a \neq b$*), and* m *be positive integers. Clearly, the expression*

$$[a(a^m + b^m)]^m + [b(a^m + b^m)]^m = (a^m + b^m)^{m+1}$$

is not a counterexample to Beal's Conjecture since $(a^m + b^m)$ *is a common factor of the bases in this expression. Now we use this solution to show an inequality by applying Theorem 2.9. Suppose* $u = (a^m + b^m)$. *Thus, we have* $(au)^m + (bu)^m = u^{m+1}$. *Hence, by Part (c) of Theorem 2.9,* u *should satisfy the inequality* $u^{m+1} \leq 2c^m$, *where*

$$c = \frac{(au)^m + (bu)^m}{(au)^{m-1} + (bu)^{m-1}}.$$

Remark 3.4. *It is noteworthy to mention that the proof of the inequality in the above example could be quite challenging without applying Theorem 2.9.*

Remark 3.5. *In the end, the author believes that the center-of-mass method, merely, or together with a probabilistic approach could be very useful and efficient for investigation and study of generalized Fermat's Last Theorem (Beal's Conjecture) or Diophantine inequalities (approximations), especially for those people who like to challenge these type of problems via a computer programming (simulation) or heuristic methods.*

Acknowledgment

The author expresses his sincere thanks to the anonymous referee for the careful reading and suggestions which improved the exposition of the paper.

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MSC2010: 11D41, 26D15, 70-08

Key words and phrases: k-Mass-System of n points (positive integers), Beal's Conjecture, Diophantine equation, center of mass

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