OUTERPLANAR COARSENESS OF PLANAR GRAPHS

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Abstract. The (outer) planar coarseness of a graph is the largest number of pairwise-edge-disjoint non-(outer)planar subgraphs. It is shown that the maximum outerplanar coarseness, over all n-vertex planar graphs, lies in the interval $\big[\lfloor (n-2)/3 \rfloor, \lfloor (n-2)/2 \rfloor\big]$.

1. Introduction

A graph H is *outerplanar* if the graph $H * K_1$, consisting of the join of H with an isolated vertex, is planar. Some invariants related to outerplanarity are bounded on the family of all planar graphs; e.g., Yannakakis [6] showed that the book thickness of a planar graph is at most 4, and the famous CGH-conjecture [1], that every planar graph has outerplanar thickness at most 2, may have at last been proven by Goncalves [2].

However, the worst-case outerplanar crossing number grows quadratically with the number of vertices of the planar graph. Indeed, as the number of edges in a planar graph is less than 3 times the number n of vertices, the outerplanar crossing number is less than $(9/2)n^2$, and we showed in [5] that the family G_n of n-vertex planar graphs has outerplanar crossing number $cr_{op}(G_n)$ asymptotically equal to $n^2/4$, where G_n is the join of two isolated vertices with an (n-2)-cycle, i.e., $G_n = C_{n-2} * \bar{K}_2$. In fact, [5] gives an exact formula for $n \geq 5$,

$$cr_{op}(G_n) = \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2n - 8.$$

We show that worst-case outerplanar coarseness of planar graphs grows linearly with n.

2. Outerplanar coarseness of G_n

The *outerplanar coarseness* of a graph G is the largest number of pairwise-edge-disjoint non-outerplanar subgraphs of G. As G is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$ [4, p. 107], it follows (Guy [3]) that G with m edges has outerplanar coarseness $\xi_{op}(G)$ at most $m/6$. This gives the upper bound for n-vertex planar graphs asserted in the abstract. The following theorem provides the lower bound.

Theorem 2.1. *Let $n \geq 5$. Then $\xi_{op}(G_n) = \lfloor (n-2)/3 \rfloor$.*

Proof. The $\geq$ inequality is obvious: Take any family of the form $\{\bar{K}_2 * W : W \in \mathcal{W}\}$, where $\mathcal{W}$ is a maximal collection of pairwise-vertex-disjoint 3-element subsets of $V(C_{n-2})$ and $\bar{K}_2$ denotes the same pair of vertices given in the definition of G_n. For the reverse inequality, note that a K_4-homeomorph in G_n includes the entire n-2-cycle so G_n contains only one such subgraph. Hence, for $n \geq 5$, to maximize the number of pairwise edge-disjoint non-outerplanar subgraphs, one can use homeomorphs of $K_{2,3}$ so $\leq$ holds as well. □

We conjecture that G_n maximizes $cr_{op}(G)$ and $\xi_{op}(G)$ over all n-vertex planar graphs G.

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