AN EXCEEDINGLY SHORT PROOF THAT THE HARMONIC SERIES DIVERGES

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Abstract. In this short note we give *an exceedingly short proof that the harmonic series diverges.* The proof is virtually a one-liner.

1. Introduction

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is one of the most famous or well-known infinite series in elementary mathematical analysis. The series diverges—a fact first demonstrated by Nicole'd Oresme [1, ca. 1323-1382]. There are a number of proofs that the harmonic series diverges, some of them well-known and elementary. In [2], Steven J. Kifowit and Terra A. Stamps give a survey of 20 proofs of the divergence of the harmonic series, covering simple popular proofs, up to more advanced proofs.

2. The Exceedingly Short Proof

The author of the present note discovered the following elementary, almost one-line proof that the harmonic series diverges—this proof is not found in [2].

Theorem 2.1. *The harmonic series diverges.*

Proof. For all $x \geq 0$, $x \geq \ln(1+x)$. Therefore, we have the following exceedingly short proof that the harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{\infty} \big[\ln(n+1) - \ln n\big] = \lim_{n\to\infty} \ln(n+1) = \infty.$$

□

References

[1] N. Oresme, *Quastiones Super Geometriam Euclidis (Questions Concerning Euclid's geometry)*, (ca. 1360).
[2] S. J. Kifowit and T. A. Stamps, *The Harmonic Series Diverges Again and Again*, AMATYC Review, **27.2** (2006), 31–43.

MSC2010: 40A05

Key words and phrases: harmonic series, divergent series, real functions, summability, Oresme

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