FAMILIES OF VALUES OF THE EXCEDENT FUNCTION $\sigma(n) - 2n$

RAVEN DEAN, RICK ERDMAN, DOMINIC KLYVE, EMILY LYCETTE,
MELISSA PIDDE, AND DEREK WHEEL

Abstract. The excedent function, $e(n) := \sigma(n) - 2n$, measures the amount by which the sum of the divisors of an integer exceeds that integer. Despite having been in the mathematical consciousness for more than 2000 years, there are many unanswered questions concerning the function. Of particular importance to us is the question of explaining and classifying values in the image of $e(n)$ – especially in understanding the "small" values. We look at extensive calculated data, and use them as inspiration for new results, generalizing theorems in the literature, to better understand a family of values in this image.

1. Introduction and Background

For at least 2000 years mathematicians have been interested in studying the sum of the divisors of an integer. Classically, they often excluded the number itself, so that they were summing the proper divisors – sometimes called the *aliquot parts*. If we let $\sigma(n)$ be the sum of the divisors of an integer n, it makes sense to want to compare $\sigma(n)$ with $2n$ by finding the difference of these two values. This paper is a study of these differences.

We will define the *excedent* function, $e(n)$, to be the amount by which the sum of the proper divisors of an integer exceeds the integer. That is, $e(n) := \sigma(n) - 2n$. Note that the sign of $e(n)$ immediately tells us something about n. If $e(n) = 0$, n is a perfect number. Similarly, if $e(n) > 0$ (or $e(n) < 0$), then n is abundant (respectively, deficient). In general, the excedent of an integer measures how close it is to being perfect. In a previous work, the third author studied some of the values that $e(n)$ can obtain [6]. In this work, we will build on the previous study, attempting to classify and explain more families of values taken by $e(n)$.

2. Previous Results

Although one could claim that the study of values of $e(n)$ dates back at least to Nicomachus's classification of integers as abundant, perfect, or deficient, the modern study of the excedent function dates only to 1951, when Cattaneo [3] defined what he called (in Italian) the *eccedenza* of an integer, from which we take the term "excedent". He was particularly interested in knowing whether the excedent of an integer was ever precisely 1. Such integers are today called "quasi-perfect." Cattaneo's original question remains unanswered, but today we do know that if an integer has an excedent of 1, then

(1) $n = k^2$ where k is odd [3];
(2) if m is a proper divisor of n, then $\sigma(m) < 2n$ [3];
(3) if $r|\sigma(n)$ then $r \equiv 1$ or $3 \pmod 8$ [3];
(4) n has at least seven prime factors [9];
(5) $n > 10^{35}$ [9].

In 1983, Cohen considered the more general question of whether the excedent of an integer ever takes any square as its value [5]. Finding no examples, he managed to prove that such integers must be larger than 10^{20} and have at least four distinct prime factors.

More generally still, we might ask the question of which integers appear in the range of $e(n)$. Although this question hasn't been directly studied in detail, some work has been done on the closely related question of the values of $\sigma(n)-n$. Erdős showed [7] that there are infinitely many numbers m for which $\sigma(n) - n = m$ has no solution, and furthermore that these m have positive lower density. Chen and Zhao [4] have recently improved this to show that the density of these m is at least 0.06. Pomerance [12] has considered a more general case, considering the set

$$S(a) = \{n : \sigma(n) \equiv a \pmod n\}.$$

He showed that for all a, the set $S(a)$ has at least two elements.

More recently, there have been several papers in the literature related to the values of $\sigma(n) - 2n$. For example, Anavi, Pollack, and Pomerance show in [1] that the number of elements not greater than x in $S(a)$ (not counting those in a certain "obvious" set involving multiples of perfect and multiply perfect numbers) is bounded by $x^{1/2+o(1)}$ for each $|a| \le x^{1/4}$. Since $\sigma(n) \equiv e(n) \pmod n$, this immediately gives an upper bound on the number of n up to x for which $e(n) \equiv a \pmod n$ as well. One conclusion is that there can be no more than $x^{1/2+o(1)}$ integers $n \le x$ for which $e(n) = k$ (outside of the obvious set) for any $k \le x^{1/4}$.

A related idea is studied in [11], in which Pollack and Shevelev consider "near perfect numbers". These are integers whose excedent is equal to one of their divisors. Our Theorem 4.1 is a generalization of one of their results. Finally, Pollack and Pomerance show in [10] that for odd k, the number of $n \le x$ for which $e(n) = k$ is at most $x^{1/4+o(1)}$ as $x \to \infty$.

2.1. **The excedent function.** Much of this recent work has concerned asymptotic densities of preimages of individual values of $e(n)$. Other work has looked at individual values. The primary motivation for this work comes from a paper in which Davis, Klyve, and Kraght sought to determine which specific integers lie in the image of $e(n)$ [6]. They studied the density of excedents, and some arithmetic progressions which contain only excedents, using extensive computational data. Their work left several questions unanswered, and much of the work in this current study is based on a study of their data.

Two results mentioned in their paper concerning the excedent function should be mentioned here, as they have an important role in our work. The first is a fairly straightforward fact about $e(n)$, of the type often proved by students in a first course in number theory (for example, see [2, Ch. 6, Problems 6.1, exercise 7]).

Theorem 2.1. *The excedent of n, $e(n)$, is odd if and only if $n = k^2$ or $n = 2k^2$ for some integer k.*

From this we see that odd values of $e(n)$ are rarer than even values – indeed, the number of $n \leq x$ for which $e(n)$ is odd is $O(\sqrt{x})$. In a sense, this makes odd values simultaneously more interesting and more provocative – trying to find and explain the odd values will be more difficult, we expect, than explaining the even values.

A good start to classifying odd values of the excedent function was made by Pollack and Shevelev [11], who found one family of odd values guaranteed to be in the image of $e(n)$.

Theorem 2.2. *Let $M_p = 2^p - 1$ be a Mersenne prime. Then $2^p - 1$ will be in the image of the excedent function. In particular,*

$$e(2^{p-1}M_p^2) = M_p.$$

In the remainder of this paper we will build on this work with the goal of identifying and classifying more families of values in the image of $e(n)$.

3. Families of Values of $e(n)$

There are several families of $e(n)$ which are easy to find, simply by considering the number and multiplicity of the prime factors of n. For example, for prime p, we have $e(p) = 1 - p = -(p-1)$, and thus all integers of the form $-(p-1)$ are in the image of $e(n)$. (Note that from this it follows immediately that there are infinitely many distinct values in the image of $e(n)$). Similarly, we have the following.

Theorem 3.1. *If p and q are distinct primes, then $-(pq - p - q - 1)$ is in the image of $e(n)$.*

Proof. Let $n = pq$. Then $e(n) = s(n) - n = (p + q + 1) - pq$, from which the theorem follows. □

Both of these families contain only even numbers (with the trivial exception of $e(2) = -1$), but it is not hard to find other families of odd values. In general, integers with known prime factorizations lend themselves well to this type of analysis. For example, we have the following theorem.

Theorem 3.2. *If p is prime, then $-(p^2 - p - 1)$ is in the image of $e(n)$.*

Proof. Let $n = p^2$. Then $e(n) = s(n) - n = p + 1 - p^2$, from which the theorem follows. □

Knowing that these families exist, and knowing that to the facts above we could add more families, based on the values of the excedent function (values of the excedent function of integers of the form $n = p^3q^2r$, etc.), we next turn to values of $e(n)$ which we observe computationally, with the goal of seeing how many of these can be explained by such families.

4. Small Values of $e(n)$

Because $e(n)$ measures how close an integer n is to being perfect, it is especially interesting to consider small values of $e(n)$. The smallest value (in absolute value) is $e(n) = 0$, which corresponds to the perfect numbers, only 48 of which are known. Similarly, $e(n) = 1$ corresponds to quasiperfect numbers, none of which are known. We might ask, more generally, for solutions to $e(n) = m$ for other small m.

In [6], the authors studied the inverse image of small values of the excedent function, trying to find a list of the integers m for which there exists an n with $e(n) = m$. For quite a few small odd values of m, they found no such n, making the search for these particularly interesting. Specifically, they (and we) wanted to look for large values of n for which $|e(n)|$ is small. For this reason, the authors of [6] searched all $n \leq 10^{20}$ for which $e(n)$ is odd and $|e(n)| < 10^6$. They succeeded in classifying some of the values of $e(n)$ that they found, but many still remained. As an example of these values, consider Table 1, which gives all n in the interval $[10^{14}, 10^{16}]$ for which $|e(n)| < 10^6$ is odd. Note that since we already know that all such n must either have the form $n = k^2$ or $n = 2k^2$, we will, in an attempt to make integers of this size more comprehensible, write the n in this form.

We might believe that these n are special – after all, very few integers in this range have small values of $e(n)$. Perhaps all – or most – of them lie in a single family. On the other hand, perhaps they are more-or-less random, with no unifying explanation. We shall begin simply by studying the table of values and looking for patterns in what we've computed.

Table 1. Integers $10^{14} \le n \le 10^{16}$ for which $|e(n)| \le 10^6$ and $e(n)$ is odd.

n	**e(n)**
$2 \cdot 8386688^2$	982829
$2 \cdot 8388608^2$	−1
$2 \cdot 8388736^2$	−65539
$2 \cdot 8388992^2$	−196621
$2 \cdot 8389504^2$	−458809
$2 \cdot 8390528^2$	−983281
16777216^2	−1
$2 \cdot 16777216^2$	−1
21919274^2	−399359
33554176^2	131071
33554432^2	−1
$2 \cdot 33554432^2$	−1
67108864^2	−1
$2 \cdot 67108864^2$	−1
$2 \cdot 67109632^2$	−786445

Seven of the rows on this table are easy to explain. It is trivial to show that if $n = 2^k$, then $e(n) = -1$ This leaves us with seven other values – ideally, we could explain all (or most) of them as being members of the same family. Let's look again at the still-unexplained values.

Table 2. Table 1 rewritten, omitting value of n which are powers of 2.

n	**e(n)**
$2 \cdot 8386688^2$	982829
$2 \cdot 8388736^2$	−65539
$2 \cdot 8388992^2$	−196621
$2 \cdot 8389504^2$	−458809
$2 \cdot 8390528^2$	−983281
21919274^2	−399359
33554176^2	131071
$2 \cdot 67109632^2$	−786445

Recall that our goal is to find some pattern in the values of $e(n)$, so we ask if anything about Table 2 stands out. In fact, at least two of the values in the $e(n)$ column jump out at us fairly quickly, as they are close to powers of 2. In particular, 65539 is $2^{16}+3$, and 131071 is $2^{17}-1$. None of the other values are very close to a power of 2, but the appearance of near-powers is unlikely to be a coincidence.

Factoring the values of $e(n)$ is of little help, but trying an "approximate" factorization reveals something interesting. Namely, each of these values is close to a small multiple of a power of 2. We also note that many of the values of n are quite close to a power of 2. Combining these observations allows us to rewrite the table above in the following way.

TABLE 3. Another version of Table 2, writing values of n and $e(n)$ as values close to small multiples of powers of 2.

n	**e(n)**
$2(2^{23} - 15 \cdot 2^7)^2$	$15 \cdot (2^{16} - 211)$
$2(2^{23} + 2^7)^2$	$-(2^{16} + 3)$
$2(2^{23} + 3 \cdot 2^7)^2$	$-(3 \cdot 2^{16} + 13)$
$2(2^{23} + 7 \cdot 2^7)^2$	$-(7 \cdot 2^{16} + 57)$
$2(2^{23} + 15 \cdot 2^7)^2$	$-(15 \cdot 2^{16} + 241)$
21919274^2	$-(195 \cdot 2^{11} - 1)$
$(2^{25} - 2^8)^2$	$2^{17} - 1$
$2(2^{26} + 6 \cdot 2^7)^2$	$-(3 \cdot 2^{18} + 13)$

We seem to be getting closer to seeing some structure in these values. Perhaps we can rewrite this table one more time in a way that makes things still clearer. Indeed, an obvious possibility suggests itself – let's try to factor out the powers of two from every term on the left-hand column.

TABLE 4. Table 3 rewritten, factoring out powers of 2 from the values of n.

n	**e(n)**
$2^{15} \cdot (2^{16} - 15)^2$	$15 \cdot (2^{16} - 211)$
$2^{15} \cdot (2^{16} + 1)^2$	$-(2^{16} + 3)$
$2^{15} \cdot (2^{16} + 3)^2$	$-(3 \cdot 2^{16} + 13)$
$2^{15} \cdot (2^{16} + 7)^2$	$-(7 \cdot 2^{16} + 57)$
$2^{15} \cdot (2^{16} + 15)^2$	$-(15 \cdot 2^{16} + 241)$
21919274^2	$-(195 \cdot 2^{11} - 1)$
$2^{16}(2^{17} - 1)^2$	$2^{17} - 1$
$2^{17} \cdot (2^{18} + 3)^2$	$-(3 \cdot 2^{18} + 13)$

At this point, we note that a fairly distinctive pattern does become apparent. Seven of the values of n listed are of the form $2^{m-1}(2^m + k)^2$ for integers m and small k. Additionally, all of the values of $2^m + k$ are prime. These values seem, therefore, to come from an interesting generalization of even perfect numbers, which can be written in the form $2^{m-1}(2^m - 1)$. Furthermore, the corresponding value of $e(n)$ seems to be of the form $-(k \cdot 2^m + l)$ for an integer l (which may depend on k). Such observations from this table led us to the following theorem.

Theorem 4.1. *If $2^m - k$ is prime, then the value $k \cdot 2^m - (k^2 - k + 1)$ is in the image of the excedent function. In particular,*

$$e(2^{m-1}(2^m - k)^2) = k \cdot 2^m - (k^2 - k + 1).$$

(Note that Theorem 2.2 is a special case of Theorem 4.1.)

Proof. In general, we can use the multiplicativity of $\sigma(n)$ to find the excedent of the product of a power of 2 and a prime squared as follows:

$$\begin{aligned} e(2^{m-1}p^2) &= \sigma(2^{m-1}p^2) - 2 \cdot 2^{m-1}p^2 \\ &= \sigma(2^{m-1})\sigma(p^2) - 2^m p^2 \\ &= (2^m - 1)(p^2 + p + 1) - 2^m p^2 \\ &= 2^m p + 2^m - p^2 - p - 1. \end{aligned}$$

In our theorem, the prime p takes the value $2^m - k$, so we write

$$\begin{aligned} e(2^{m-1}p^2) &= 2^m(2^m - k) + 2^m - (2^m - k)^2 - (2^m - k) - 1 \\ &= 2^{2m} - k2^m + 2^m - 2^{2m} + 2k2^m - k^2 - 2^m + k - 1 \\ &= k2^m - k^2 + k - 1 \\ &= k \cdot 2^m - (k^2 - k + 1), \end{aligned}$$

as desired. □

Not only does this theorem explain most of the values of $e(n)$ in our tables, but it gives us something more. Every prime can be written in the form $2^m - k$ for some k and m, and once we find that k and m, we can use Theorem 4.1 to find a value of the excedent function. We will say that such values are *generated* by the prime $p = 2^m - k$. In fact, we can show something stronger.

Theorem 4.2. *Each prime p generates infinitely many distinct odd values of the excedent function.*

Proof. To find these odd values, we need only recognize that every prime p can be written in the form $p = 2^m - k$ in infinitely many distinct ways, simply by first choosing a value of m, and then setting $k = 2^m - p$. Once we have chosen these m and k, we generate a value in the image of $e(n)$ using Theorem 4.1. □

Example 4.3. Let $p = 17$. Letting m take values in the range $\{4, 5, \ldots, 10\}$, we find k as $k = 2^m - p$. Table 5 gives the values of m and k, together with the equation involving the excedent function generated by Theorem 4.1.

TABLE 5. Values of $e(n)$ generated by the prime 17.

m	**k**	$\mathbf{e(2^{m-1}p^2)}$	**=**	$\mathbf{k \cdot 2^m - (k^2 - k + 1)}$
4	-1	$e(2312)$	$=$	-19
5	15	$e(4624)$	$=$	269
6	47	$e(9248)$	$=$	845
7	111	$e(18496)$	$=$	1997
8	239	$e(36992)$	$=$	4301
9	495	$e(73984)$	$=$	8909
10	1007	$e(147968)$	$=$	18125

We see that for any given prime, each of the terms in this infinite family of values is distinct. Unfortunately, the sets of values so generated by two primes may not be disjoint. This overlap seems to limit our ability to use this observation to speculate about the density of odd integers in the image of the excedent function $e(n)$. Some values in the image of $e(n)$ are generated by more than one prime. For example, consider the following.

Example 4.4. The value -19 is generated by at least two primes: 5 and 17.

First, writing 5 as $2^1 + 3$, we have $m = 1$ and $k = -3$. In this case, we have $n = 2^{m-1}(2^m - k)^2 = 2^0(2^1 + 3)^2 = 25$, and

$$e(25) = \sigma(25) - 2 \cdot 25 = 31 - 50 = -19,$$

so -19 is generated by prime $p = 5$.

Next, we write 17 as $2^4 + 1$, giving us $m = 4$ and $k = -1$. We now have $n = 2^{m-1}(2^m - k)^2 = 2^3(2^4 + 1)^2 = 2312$, and

$$e(2312) = \sigma(2312) - 2 \cdot 2312 = 4605 - 4624 = -19,$$

and thus -19 is generated by prime $p = 17$.

Although it is not clear to us how to do this, it is quite possible that a method of counting those integers which can be represented in the form $k \cdot 2^n - (k^2 - k + 1)$ in more than one way would enable us to find good bounds on the density of odd integers in the image of $e(n)$.

5. That Which Remains

By using Theorem 4.1, we are able to explain – that is, to classify as being part of a family – all but one of the "small" values of $e(n)$ for n

in the range $[10^{14}, 10^{16}]$. Indeed, though it is not shown here, we can classify the rest of the small values of $e(n)$ in $[10^{14}, 10^{20}]$ (the limit of our computed data) in the same way. We are left with the example of 21919274^2, as $e(21919274^2) = -399359$. This doesn't seem to fall into the family described above, and without any other such sporadic examples, it's difficult to look for a pattern into which it may fall. The fact that -399359 can be written as $-(195 \cdot 2^{11} - 1)$ is tantalizing, however, and suggests there may be more structure to the values of $e(n)$ than we have yet found.

6. Conclusion

The type of experimental nature of investigation used in this article has a long history in mathematics. One can easily imagine Nicomachus himself playing with lists of integers as he probed their secrets. Euler once wrote an entire article dedicated to explaining the value of looking at examples [8], and rewriting the results until patterns become clear. The ultimate goal, for him and for us, was to prove new theorems. We are left with the happy fact that our work does not explain everything about the values of the excedent function, however – more mathematical play remains to be done, and more theorems remain to be proven.

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Department of Mathematics, Oregon State University, Corvallis, OR 97331
E-mail address: deanra@math.oregonstate.edu

Central Washington University, Ellensburg, WA 98926
E-mail address: erdmanri@cwu.edu

Department of Mathematics, Central Washington University, Ellensburg, WA 98926
E-mail address: klyved@cwu.edu

Todd Beamer High School, 35999 16th Ave S, Federal Way, WA 98003
E-mail address: Emily-Lycette@fwps.org

Central Washington University, Ellensburg, WA 98926
E-mail address: piddem@cwu.edu

Department of Mathematics, Western Washington University, Bellingham, WA 98225
E-mail address: derek.wheel@gmail.com