# ON GENERALIZED $\omega\beta$-CLOSED SETS

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Abstract. The aim of this paper is to introduce and study the class of $g\omega\beta$-closed sets. This class of sets is finer than $g$-closed sets and $\omega\beta-$closed sets. We study the fundamental properties of this class of sets. Further, we introduce and study $g\omega\beta$-open sets, $g\omega\beta$-neighborhoodsets, $g\omega\beta$-continuous functions, $g\omega\beta$-irresolute functions and $g\omega\beta$-closed functions.

## 1. Introduction

Through this work, a space will always mean a topological space on which no separation axiom is assumed unless explicitly stated. Let $(X, \tau)$ be a space and let $A$ be a subset of $X$. For $A \subseteq X$, the closure and the interior of $A$ in $X$ are denoted by $Cl(A)$ and $Int(A)$, respectively. It is well-known that a subset $A$ of a space $(X, \tau)$ is $\beta$-open [1] if $A \subseteq Cl(Int(Cl(A)))$. The complement of $\beta$-open set is called $\beta$-closed. $W$ is called $\omega\beta$-open [3](resp. $\omega$-open [5]) if for each $x \in W$, there exists a $\beta$-open set $U$ (resp. $U \in \tau$) such that $x \in U$ and $U - W$ is countable. The complement of an $\omega\beta$-open (resp. $\omega$-open) set is called $\omega\beta$-closed ($\omega$-closed). The intersection of all $\omega\beta$-closed sets of $X$ containing $A$ is called the $\omega\beta$-closure of $A$ and denoted by $\omega\beta Cl(A)$. And the union of all $\omega\beta$-open sets of $X$ contained in $A$ is called $\omega\beta$-interior of $A$ and is denoted by $\omega\beta Int(A)$.

In 1970, Levine [7] introduced the notion of generalized closed sets. He defined a subset $A$ of a space $(X, \tau)$ to be generalized closed (briefly, $g$-closed) if $Cl(A) \subseteq U$ whenever $U \in \tau$ and $A \subseteq U$. Generalized semi-closed [6] (resp. generalized $\beta$-closed [10], generalized $\omega$-closed [4]) sets are defined by replacing the closure operator in Levine's original definition by the semi-closure (resp. $\beta$-closure, $\omega$-closure) operator.

In this paper, we follow a similar line to introduce generalized $\omega\beta$-closed sets by utilizing the $\omega\beta$-closure operator. We define $g\omega\beta$-open sets and $g\omega\beta$-neighborhoods and study the properties of each one. The $g\omega\beta$-continuous, $g\omega\beta$-irresolute and $g\omega\beta$-closed functions are studied and we find the relationship between them and other well-known functions.

Now we begin to recall some known notions, definitions, and results which will be used in the work.

**Definition 1.1.** [3] *A space $(X, \tau)$ is called a $\beta$-anti locally countable space if each non-empty $\beta$-open set is an uncountable set.*

**Proposition 1.2.** [3] *Let $(X, \tau)$ be a topological space.*

*i. The intersection of an $\omega\beta$-open set and an $\omega$-open set is $\omega\beta$-open.*
*ii. The intersection of any family of $\omega\beta$-closed set is $\omega\beta$-closed.*

**Definition 1.3.** [2] *A function $f\colon (X, \tau) \to (Y, \sigma)$ is called*

*i. $\omega\beta$-continuous if $f^{-1}(V)$ is $\omega\beta$-open in $(X, \tau)$ for each open set $V \subseteq Y$.*
*ii. $\omega\beta$-irresolute if $f^{-1}(V)$ is $\omega\beta$-open in $(X, \tau)$ for each $\omega\beta$-open set $V$ in $(Y, \sigma)$.*
*iii. $\omega\beta$-open if $f(V)$ is $\omega\beta$-open in $(Y, \sigma)$ for each $\omega\beta$-open set $V$ in $(X, \tau)$.*
*iv. $\omega\beta$-closed if $f(V)$ is $\omega\beta$-closed in $(Y, \sigma)$ for each $\omega\beta$-closed set $V$ in $(X, \tau)$.*

**Definition 1.4.** [3] *A topological space $(X, \tau)$ is said to be*

*i. $\omega\beta$-regular if each pair of a point and a closed set not containing the point can be separated by disjoint $\omega\beta$-open sets.*
*ii. $\omega\beta$-normal if every two disjoint closed sets can be separated by $\omega\beta$-open sets.*

## 2. Generalized $\omega\beta$-Closed Sets

In this section we introduce $g\omega\beta$-closed sets in a topological space and study some of their properties.

**Definition 2.1.** *A subset $A$ of a space $(X, \tau)$ is called generalized $\omega\beta$-closed (briefly, $g\omega\beta$-closed) if $\omega\beta Cl(A) \subseteq U$ whenever $U \in \tau$ and $A \subseteq U$.*

We denote the family of all generalized $\omega\beta$-closed (resp. generalized closed) subsets of a space $(X, \tau)$ by $G\omega\beta C(X, \tau)$ (resp. $GC(X, \tau)$).

**Proposition 2.2.** *Let $(X, \tau)$ be a topological space. Then $G\omega\beta C(X, \tau) = \mathrm{P}(X)$ if one of the following properties holds.*

*i. $(X, \tau)$ is a countable space (i.e., $X$ is countable).*
*ii. $\omega\beta$-open and $\omega\beta$-closed coincide in $(X, \tau)$.*

*Proof.*
i) It is obvious.
ii) Suppose $A \subseteq U$, where $U$ is open in $X$. Since $U$ is $\omega\beta$-open, it is $\omega\beta$-closed by hypothesis. Hence, $\omega\beta Cl(A) \subseteq U$ and $A$ is $g\omega\beta$-closed. Then, $G\omega\beta C(X, \tau) = \mathrm{P}(X)$. $\square$

Every $\omega\beta$-closed set is $g\omega\beta$-closed. However, the converse is not true in general as the following example shows.

**Example 2.3.** *Let $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$ and let $A = [0, 1]$. Then $A$ is $g\omega\beta$-closed in $(X, \tau)$ since the only open set containing $A$ is $X$. However, $A$ is not $\omega\beta$-closed in $(X, \tau)$.*

**Example 2.4.** *Let $X = \{1, 2, 3\}$ with $\tau = \{\phi, X, \{1\}, \{1, 2\}\}$ and let $A = \{1\}$. Then $A$ is $g\omega\beta$-closed. But $A$ is not $g$-closed since $A \subseteq A$ and $Cl(A) = X \not\subseteq A$.*

**Example 2.5.** *Let $X = \{1, 2, 3, 4, 5\}$ with the topology $\tau = \{\phi, X, \{1\}, \{1, 2, 3\}\}$. Set $A = \{1\}$. Then $A$ is $g\omega\beta$-closed since the space $X$ is countable. However, $A$ is not $g\beta$-closed in $(X, \tau)$ since $\{1\} \subseteq \{1, 2, 3\} \in \tau$ but $X = \beta Cl(\{1\}) \not\subseteq \{1, 2, 3\}$.*

**Example 2.6.** *Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, X, \mathbb{R} - \mathbb{Q}\}$ and put $A = \mathbb{R} - \mathbb{Q}$. Then $A$ is $g\omega\beta$-closed. But $A$ is not $g\omega$-closed, since $A$ is open and $A \subseteq A$, $\omega Cl(A) \not\subseteq A$ (because $A$ is not $\omega$-closed).*

We have the following relation for $g\omega\beta$-closed sets with other known sets.

$$\begin{array}{ccccccc}
 & & \text{Closed} & \rightarrow & \omega\text{-closed} & & \\
 & & \downarrow & & \downarrow & & \\
 & & g\text{-closed} & \rightarrow & g\omega\text{-closed} & & \\
 & & \downarrow & & \downarrow & & \\
 & & g\beta\text{-closed} & \rightarrow & g\omega\beta\text{-closed} & & \\
 & & \uparrow & & \uparrow & & \\
\text{Closed} & \rightarrow & \beta\text{-closed} & \rightarrow & \omega\beta\text{-closed} & \leftarrow & \omega\text{-closed}
\end{array}$$

**Theorem 2.7.** *Let $A$ be a $g\omega\beta$-closed subset of $(X, \tau)$. Then $\omega\beta Cl(A) - A$ does not contain any non-empty closed sets.*

*Proof.* Suppose by contrary that $\omega\beta Cl(A) - A$ contains a non-empty closed set $F$. Then $A \subseteq X - F$ and $X - F$ is open in $(X, \tau)$. Thus, $\omega\beta Cl(A) \subseteq X - F$ or equivalently, $F \subseteq X - \omega\beta Cl(A)$. This implies that $F \subseteq (X - \omega\beta Cl(A)) \cap (\omega\beta Cl(A) - A) = \phi$. □

**Corollary 2.8.** *Let $A$ be a $g\omega\beta$-closed subset of $(X, \tau)$. Then $A$ is $\omega\beta$-closed if and only if $\omega\beta Cl(A) - A$ is closed.*

*Proof.* Let $A$ be a $g\omega\beta$-closed set. If $A$ is $\omega\beta$-closed, then we have $\omega\beta Cl(A) - A = \phi$ which is a closed set. Conversely, let $\omega\beta Cl(A) - A$ be closed. Then by Theorem 2.7, $\omega\beta Cl(A) - A$ does not contain any non-empty closed subset and since $\omega\beta Cl(A) - A$ is a closed subset of itself, then $\omega\beta Cl(A) - A = \phi$. This implies that $A = \omega\beta Cl(A)$ and so $A$ is $\omega\beta$-closed. □

**Proposition 2.9.** *Let $(X, \tau)$ be a topological space. Then the following are equivalent.*

*i. Every open set of $X$ is $\omega\beta$-closed.*

*ii. Every subset of $X$ is $g\omega\beta$-closed.*

*Proof.* (i) $\rightarrow$ (ii) Let $A \subseteq U \in \tau$. Then by (i), $U$ is $\omega\beta$-closed, so $\omega\beta Cl(A) \subseteq \omega\beta Cl(U) = U$. Thus, $A$ is $g\omega\beta$-closed.
(ii) $\rightarrow$ (i) Let $U \in \tau$. Then by (ii), $U$ is $g\omega\beta$-closed and hence, $\omega\beta Cl(U) \subseteq U$. So $U$ is $\omega\beta$-closed. □

**Proposition 2.10.** *If $A$ is open and $g\omega\beta$-closed, then $\omega\beta Cl(A) - A = \phi$.*

*Proof.* It is obvious. □

**Theorem 2.11.** *If $A$ is a $g\omega\beta$-closed set and $B$ is any set such that $A \subseteq B \subseteq \omega\beta Cl(A)$, then $B$ is $g\omega\beta$-closed.*

*Proof.* Let $U \in \tau$ and $B \subseteq U$. Then $A \subseteq B \subseteq U$. Since $A \in g\omega\beta$-closed, $\omega\beta Cl(B) \subseteq \omega\beta Cl(\omega\beta Cl(A)) = \omega\beta Cl(A) \subseteq U$ and the result follows. □

**Definition 2.12.** *Let $A$ be a subset of a space $X$. A point $x \in X$ is said to be a $\omega\beta$-limit point of $A$ if for each $\omega\beta$-open set $U$ containing $x$, we have $U \cap (A - \{x\}) \neq \phi$. The set of all $\omega\beta$-limit points of $A$ is called the $\omega\beta$-derived set of $A$ and is denoted by $D_{\omega\beta}(A)$.*

Since every open set is $\omega\beta$-open, we have $D_{\omega\beta}(A) \subseteq D(A)$ for any subset $A \subseteq X$, where $D(A)$ is the derived set of $A$. Moreover, since every closed set is $\omega\beta$-closed, we have $A \subseteq \omega\beta Cl(A) \subseteq Cl(A)$.

The proof of the following result is straightforward and is omitted.

**Lemma 2.13.** *If $D(A) = D_{\omega\beta}(A)$, then we have $Cl(A) = \omega\beta Cl(A)$.*

**Corollary 2.14.** *If $D(A) \subseteq D_{\omega\beta}(A)$ for any subset $A$ of $X$. Then for any subsets $F$ and $B$ of $X$, we have $\omega\beta Cl(F \cup B) = \omega\beta Cl(F) \cup \omega\beta Cl(B)$.*

**Proposition 2.15.** *If $A$ and $B$ are $g\omega\beta$-closed sets such that $D(A) \subseteq D_{\omega\beta}(A)$ and $D(B) \subseteq D_{\omega\beta}(B)$. Then $A \cup B$ is $g\omega\beta$-closed.*

*Proof.* Let $U$ be an open set such that $A \cup B \subseteq U$. Since $A$ and $B$ are $g\omega\beta$-closed sets, we have $\omega\beta Cl(A) \subseteq U$ and $\omega\beta Cl(B) \subseteq U$. Since $D(A) \subseteq D_{\omega\beta}(A)$, $D(A) = D_{\omega\beta}(A)$ and by Lemma 2.13, $Cl(A) = \omega\beta Cl(A)$. Similarly, $Cl(B) = \omega\beta Cl(B)$. Thus, $\omega\beta Cl(A \cup B) = \omega\beta Cl(A) \cup \omega\beta Cl(B) \subseteq U$, which implies that $A \cup B$ is $g\omega\beta$-closed. □

The following example shows that the countable union of $g\omega\beta$-closed sets need not be $g\omega\beta$-closed.

**Example 2.16.** *Let $X = \mathbb{R}$ with the usual topology $\tau_u$. For each $n \in \mathbb{N}$, put $A_n = [\frac{1}{n}, 1]$ and $A = \bigcup_{n\in\mathbb{N}}^{\infty} A_n$. Then $A$ is a countable union of $g\omega\beta$-closed sets but $A$ is not $g\omega\beta$-closed since $U = (0, 2) \in \tau_u$, $A \subseteq U$ and $\omega\beta Cl(A) = [0, 1] \not\subseteq U$.*

**Proposition 2.17.** *Let $A$, $B$ be subsets of a topological space $(X, \tau)$. Then $A \cap B$ is $g\omega\beta$-closed whenever one of the following properties holds.*

*i. $A$ is open and $g\omega\beta$-closed, and $B$ is $\omega\beta$-closed.*
*ii. $A$ is $g\omega\beta$-closed and $B$ is closed.*

*Proof.* i) By Proposition 2.10, $A$ is $\omega\beta$-closed. Hence by Proposition 1.2, $A \cap B$ is $\omega\beta$-closed in $X$ which implies that $A \cap B$ is $g\omega\beta$-closed in $X$.
ii) Let $U$ be an open set in $(X, \tau)$ such that $A \cap B \subseteq U$. Put $W = X - B$. Then $A \subseteq U \cup W \in \tau$. Since $A \in g\omega\beta-$closed, $\omega\beta Cl(A) \subseteq U \cup W$. Now $\omega\beta Cl(A \cap B) \subseteq \omega\beta Cl(A) \cap \omega\beta Cl(B) \subseteq \omega\beta Cl(A) \cap Cl(B) = \omega\beta Cl(A) \cap B \subseteq (U \cup W) \cap B \subseteq U$. □

The finite intersection of $g\omega\beta$-closed sets need not be $g\omega\beta$-closed. Let $X$ be an uncountable set and let $A$ be a subset of $X$ such that $A$ and $X - A$ are uncountable. Let $\tau = \{\phi, X, A\}$. Choose $x_1, x_2 \notin A$ and $x_1 \neq x_2$. Then $A_1 = A \cup \{x_1\}$ and $A_2 = A \cup \{x_2\}$ are two $g\omega\beta$-closed subsets of $(X, \tau)$ (Since the only open set containing $A_1, A_2$ is $X$). But $A_1 \cap A_2 = A$ is not $g\omega\beta$-closed since $A \subseteq A \in \tau$ and $\omega\beta Cl(A) \neq A$.

**Theorem 2.18.** *A subset $A$ of a topological space $(X, \tau)$ is $g\omega\beta$-closed if and only if $Cl(\{x\}) \cap A \neq \phi$ for every $x \in \omega\beta Cl(A)$.*

*Proof.* Let $A$ be a $g\omega\beta$-closed set in $X$ and suppose, if possible, that there exists $x \in \omega\beta Cl(A)$ such that $Cl(\{x\}) \cap A = \phi$. Therefore, $A \subseteq (X \backslash Cl\{x\})$, and so $\omega\beta Cl(A) \subseteq (X \backslash Cl(\{x\}))$. Hence, $x \notin \omega\beta Cl(A)$ which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let $U$ be any open set containing $A$. Let $x \in \omega\beta Cl(A)$. Then by hypothesis $Cl(\{x\}) \cap A \neq \phi$, there exists $z \in Cl(\{x\}) \cap A$ and so $z \in A \subseteq U$. Thus, $\{x\} \cap U \neq \phi$. Hence, $x \in U$, which implies that $\omega\beta Cl(A) \subseteq U$. □

**Theorem 2.19.** *For an element $x \in X$, either $\{x\}$ is closed or $X \backslash \{x\}$ is $g\omega\beta$-closed.*

*Proof.* Suppose $\{x\}$ is not closed in $(X, \tau)$. Then $X \backslash \{x\}$ is not open and the only open set containing $X \backslash \{x\}$ is $X$. This implies $\omega\beta Cl(X \backslash \{x\}) \subseteq X$. Hence, $X \backslash \{x\}$ is a $g\omega\beta$-closed set in $X$. □

**Definition 2.20.** *A space $(X, \tau)$ is called an $\omega\beta$-$\mathrm{T}_{1/2}$ space if every generalized $\omega\beta$-closed set is $\omega\beta$-closed.*

**Example 2.21.** *Any set with indiscrete topology is an example for an $\omega\beta$-$\mathrm{T}_{1/2}$ space.*

**Theorem 2.22.** *A space $(X, \tau)$ is an $\omega\beta$-$\mathrm{T}_{1/2}$ space if and only if every singleton is either closed or $\omega\beta$-open.*

*Proof.* Necessity. Suppose $\{x\}$ is not a closed subset for some $x \in X$, hence by Theorem 2.19, $X - \{x\}$ is $g\omega\beta$-closed. By assumption, $X - \{x\}$ is $\omega\beta$-closed. Hence, $\{x\}$ is $\omega\beta$-open.
Sufficiency. Let $A$ be a $g\omega\beta$-closed subset of $(X, \tau)$ and $x \in \omega\beta Cl(A)$. We show that $x \in A$. If $\{x\}$ is closed and $x \notin A$, then $x \in (\omega\beta Cl(A) - A)$. Thus, $\omega\beta Cl(A) - A$ contains a nonempty closed set $\{x\}$, a contradiction to Theorem 2.7. So $x \in A$. If $\{x\}$ is $\omega\beta$-open, since $x \in \omega\beta Cl(A)$, then for every $\omega\beta$-open set $U$ containing $x$, we have $U \cap A \neq \phi$. Hence, $x \in A$. Therefore, $A$ is $\omega\beta$-closed. □

**Theorem 2.23.** *Let $(X, \tau)$ be a $\beta$-antilocally countable space. Then $(X, \tau)$ is a $T_1$-space if it is an $\omega\beta$-$T_{1/2}$ space.*

*Proof.* Let $x \in X$ and suppose that $\{x\}$ is not closed. Then by Theorem 2.19 $A = X - \{x\}$ is $g\omega\beta$-closed. Therefore, by assumption, $A$ is $\omega\beta$-closed, and thus, $\{x\}$ is $\omega\beta$-open. So there exists a $\beta$-open set $U$ such that $x \in U$ and $U - \{x\}$ is countable. It follows that $U$ is a nonempty countable $\beta$-open subset of $x \in X$, a contradiction. □

Recall that the kernel of a set $A$ [9], denoted $\ker(A)$, is the intersection of all open supersets of $A$.

**Proposition 2.24.** *A subset $A$ of $X$ is $g\omega\beta$-closed if and only if $\omega\beta Cl(A) \subseteq \ker(A)$.*

*Proof.* Since $A$ is $g\omega\beta$-closed, $\omega\beta Cl(A) \subseteq G$ for any open set $G$ with $A \subseteq G$ and hence, $\omega\beta Cl(A) \subseteq \ker(A)$. Conversely, let $G$ be an open set such that $A \subseteq G$. By hypothesis, $\omega\beta Cl(A) \subseteq \ker(A) \subseteq \ker(G) = G$ and hence, $A$ is $g\omega\beta$-closed. □

## 3. Generalized $\omega\beta$-Open Sets and Generalized $\omega\beta$-Neighborhoods

**Definition 3.1.** *A subset $A \subseteq X$ is called generalized $\omega\beta$-open (briefly, $g\omega\beta$-open) if its complement is generalized $\omega\beta$-closed. We denote the family of all generalized $\omega\beta$-open subsets of a space $(X, \tau)$ by $G\omega\beta O(X, \tau)$.*

**Remark 3.2.** $\omega\beta Cl(X - A) = X - \omega\beta Int(A)$.

**Corollary 3.3.** *A subset $A \subseteq X$ is $g\omega\beta$-open if and only if $F \subseteq \omega\beta Int(A)$, where $F$ is a closed set and $F \subseteq A$.*

*Proof.* Necessity. Let $A$ be $g\omega\beta$-open. Let $F$ be a closed set such that $F \subseteq A$. Then $X - A \subseteq X - F$, where $X - F$ is an open set. Since $A$ is $g\omega\beta$-open, $X - \omega\beta Int(A) = \omega\beta Cl(X - A) \subseteq X - F$. That is $F \subseteq \omega\beta Int(A)$.
Sufficiency. Suppose $F \subseteq \omega\beta Int(A)$ whenever $F$ is a closed set and $F \subseteq A$. Let $X - A \subseteq U$ where $U$ is open, then $X - U \subseteq A$ where $X - U$ is closed. By

hypothesis $X - U \subseteq \omega\beta Int(A)$. That is $\omega\beta Cl(X - A) \subseteq X - \omega\beta Int(A) \subseteq U$. This implies $X - A$ is $g\omega\beta$-closed and $A$ is $g\omega\beta$-open. □

**Proposition 3.4.** *If $\omega\beta Int(A) \subseteq B \subseteq A$ and $A$ is $g\omega\beta$-open, then $B$ is $g\omega\beta$-open.*

*Proof.* $\omega\beta Int(A) \subseteq B \subseteq A$ implies $X - A \subseteq X - B \subseteq X - \omega\beta Int(A)$. That is, $X - A \subseteq X - B \subseteq \omega\beta Cl(X - A)$. Since $X - A$ is $g\omega\beta$-closed, by Theorem 2.11, $X - B$ is $g\omega\beta$-closed and $B$ is $g\omega\beta$-open. □

**Proposition 3.5.** *If $A \subseteq X$ is $g\omega\beta$-closed, then $\omega\beta Cl(A) - A$ is $g\omega\beta$-open.*

*Proof.* Let $A$ be $g\omega\beta$-closed. Let $F$ be a closed set such that $F \subseteq \omega\beta Cl(A) - A$. Then by Theorem 2.7, $F = \phi$. So $F \subseteq \omega\beta Int(\omega\beta Cl(A) - A)$. This shows $\omega\beta Cl(A) - A$ is $g\omega\beta$-open. □

**Remark 3.6.** *For any $A \subseteq X$, $\omega\beta Int(\omega\beta Cl(A) - A) = \phi$.*

**Proposition 3.7.** *Let $A \subseteq B \subseteq X$ and let $\omega\beta Cl(A) \backslash A$ be $g\omega\beta$-open. Then $\omega\beta Cl(A) \backslash B$ is also $g\omega\beta-$open.*

*Proof.* Suppose $\omega\beta Cl(A) \backslash A$ is $g\omega\beta$-open and let $F$ be a closed subset of $(X, \tau)$ with $F \subseteq \omega\beta Cl(A) \backslash B$. Then $F \subseteq \omega\beta Cl(A) \backslash A$. By Corollary 3.3 and Remark 3.6 $F \subseteq \omega\beta Int(\omega\beta Cl(A) \backslash A) = \phi$. Thus, $F = \phi$ and hence, $F \subseteq \omega\beta Int(\omega\beta Cl(A) \backslash B)$. □

**Proposition 3.8.** *If a set $A$ is $g\omega\beta$-open in a topological space $(X, \tau)$, then $G = X$ whenever $G$ is open in $(X, \tau)$ and $\omega\beta Int(A) \cup A^c \subseteq G$.*

*Proof.* Suppose that $G$ is open and $\omega\beta Int(A) \cup A^c \subseteq G$. Now $G^c \subseteq \omega\beta Cl(A^c) \cap A = \omega\beta Cl(A^c) - A^c$. Since $G^c$ is closed and $A^c$ is $g\omega\beta$-closed, by Theorem 2.7 $G^c = \phi$ and hence, $G = X$. □

**Theorem 3.9.** *Let $(X, \tau)$ be a topological space and $A$, $B \subseteq X$. If one of the following conditions holds, then $A \cap B$ is $g\omega\beta$-open*

*i. $A$ is $g\omega\beta$-open and $B$ is $\omega$-open.*
*ii. $B$ is $g\omega\beta$-open and $\omega\beta Int(B) \subseteq A$.*

*Proof.* i) Let $F$ be any closed subset of $X$ such that $F \subseteq A \cap B$. Hence, $F \subseteq A$ and by Corollary 3.3, $F \subseteq \omega\beta Int(A) = \cup\{U : U$ is $\omega\beta$-open and $U \subseteq A\}$. Obviously, $F \subseteq \cup(U \cap B)$, where $U$ is an $\omega\beta$-open set in $X$ contained in $A$. By Theorem 1.2, $U \cap B$ is an $\omega\beta$-open set contained in $A \cap B$ for each $\omega\beta$-open set $U$ contained in $A$, so $F \subseteq \omega\beta Int(A \cap B)$, and by Corollary 3.3, $A \cap B$ is $g\omega\beta$-open in $X$.
ii) Since $B$ is $g\omega\beta$-open and $\omega\beta Int(B) \subset A \cap B \subseteq B$. By Proposition 3.4, $A \cap B$is $g\omega\beta$-open. □

Analogous to a neighborhood in a space $X$, we define a $g\omega\beta$-neighborhood in a space $X$ as follows.

**Definition 3.10.** *Let $X$ be a topological space and let $x \in X$. A subset $N$ of $X$ is called a $g\omega\beta$-neighborhood of $x$ if there exists a $g\omega\beta$-open set $G$ such that $x \in G \subseteq N$.*

**Definition 3.11.** *A subset $N$ of a space $X$ is called a $g\omega\beta$-neighbor-hood of $A \subseteq X$ if there exists a $g\omega\beta$-open set $G$ such that $A \subseteq G \subseteq N$.*

**Theorem 3.12.** *Every neighborhood $N$ of $x \in X$ is a $g\omega\beta$-neighbor-hood of $x$.*

*Proof.* Let $N$ be a neighborhood of a point $x \in X$, there exists an open set $G$ such that $x \in G \subseteq N$. Since every open set is a $g\omega\beta$-open set $G$, $N$ is a $g\omega\beta-$neighborhood of $x$. □

In general, a $g\omega\beta$-neighborhood $N$ of $x \in X$ need not be a neighborhood of $x \in X$, as seen from the following example.

**Example 3.13.** *Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $G\omega\beta O(X) = \mathrm{P}(X)$. The set $\{a, c\}$ is a $g\omega\beta$-neighborhood of the point $c$, since $\{c\}$ is the $g\omega\beta$-open set such that $c \in \{c\} \subseteq \{a, c\}$. However, the set $\{a, c\}$ is not a neighborhood of the point $c$, since there exists no open set $G$ such that $c \in G \subseteq \{a, c\}$.*

**Theorem 3.14.** *If a subset $N$ of a space $X$ is $g\omega\beta$-open, then $N$ is a $g\omega\beta$-neighborhood of each of its point.*

*Proof.* Suppose $N$ is $g\omega\beta$-open. Let $x \in N$. We claim that $N$ is a $g\omega\beta$-neighborhood of $x$. For $N$ is a $g\omega\beta$-open set such that $x \in N \subseteq N$. Since $x$ is an arbitrary point of $N$, it follows that $N$ is a $g\omega\beta$-neighborhood of each of its points. □

**Theorem 3.15.** *Let $X$ be a topological space. If $F$ is a $g\omega\beta$-closed subset of $X$ and $x \in F^c$. Then there exists a $g\omega\beta$-neighborhood $N$ of $x$ such that $N \cap F = \phi$.*

*Proof.* Let $F$ be a $g\omega\beta$-closed subset of $X$ and $x \in F^c$. Then $F^c$ is $g\omega\beta$-open set of $X$. So by Theorem 3.14, $F^c$ contains a $g\omega\beta$-neighborhood of each of its points. Hence there exists a $g\omega\beta$-neighbor-hood $N$ of $x$ such that $N \subseteq F^c$. That is $N \cap F = \phi$. □

**Definition 3.16.** *Let $x$ be a point in a space $X$. The set of all $g\omega\beta$-neighborhoods of $x$ is called the $g\omega\beta$-neighborhood system at $x$, and is denoted by $g\omega\beta - N(x)$.*

**Theorem 3.17.** *Let $X$ be a topological space and for each $x \in X$, let $g\omega\beta - N(x)$ be the collection of all $g\omega\beta$- neighborhoods of $x$. Then we have the following results.*

*i.* *For all $x \in X$, $g\omega\beta - N(x) \neq \phi$.*
*ii.* *If $N \in g\omega\beta - N(x)$, then $x \in N$.*
*iii.* *If $N \in g\omega\beta - N(x)$ and $N \subseteq M$, then $M \in g\omega\beta - N(x)$.*
*iv.* *If $N \in g\omega\beta - N(x)$, then there exists $M \in g\omega\beta - N(x)$ such that $M \subseteq N$ and $M \in g\omega\beta - N(y)$ for every $y \in M$.*

*Proof.*
i) Since $X$ is a $g\omega\beta$-open set, it is a $g\omega\beta$-neighborhood of every $x \in X$. Hence there exists at least one $g\omega\beta-$neighborhood (namely $X$) for each $x \in X$. Hence, $g\omega\beta - N(x) \neq \phi$ for every $x \in X$.
ii) If $N \in g\omega\beta - N(x)$, then $N$ is a $g\omega\beta$-neighborhood of $x$. So by Definition 3.10, $x \in N$.
iii) Let $N \in g\omega\beta - N(x)$ and $N \subseteq M$. Then there is a $g\omega\beta$-open set $G$ such that $x \in G \subseteq N$. Since $N \subseteq M$, $x \in G \subseteq M$ and so $M$ is a $g\omega\beta$-neighborhood of $x$. Hence, $M \in g\omega\beta - N(x)$.
iv) If $N \in g\omega\beta - N(x)$, then there exists a $g\omega\beta$-open set $M$ such that $x \in M \subseteq N$. Since $M$ is a $g\omega\beta$-open set, it is a $g\omega\beta$-neighborhood of each point of $M$. Therefore, $M \in g\omega\beta - N(y)$ for each $y \in M$. □

**Theorem 3.18.** *Let $X$ be a non-empty set, and for each $x \in X$, let $g\omega\beta - N(x)$ be a non-empty collection of subsets of $X$ satisfying the following conditions*

*i.* *If $N \in g\omega\beta - N(x)$ then $x \in N$.*
*ii.* *If $N, M \in g\omega\beta - N(x)$ then $M \cap N \in g\omega\beta - N(x)$.*

*Let $\tau$ consist of the empty set and all those non-empty subsets of $G$ of $X$ having the property that $x \in G$ implies that there exists an $N \in g\omega\beta - N(x)$ such that $x \in N \subseteq G$. Then $\tau$ is a topology for $X$.*

*Proof.* $\phi \in \tau$ by definition. We now show that $X \in \tau$. Let $x$ be any arbitrary element of $X$. Since $g\omega\beta - N(x)$ is non-empty, there is an $N \in g\omega\beta - N(x)$ such that $x \in N$ by (i). Since $N$ is a subset of $X$, we have $x \in N \subseteq X$. Hence, $X \in \tau$. Let $G_1$, $G_2 \in \tau$. If $x \in G_1 \cap G_2$, then $x \in G_1$ and $x \in G_2$. Since $G_1, G_2 \in \tau$, there exists $N, M \in g\omega\beta - N(x)$, such that $x \in N \subseteq G_1$ and $x \in M \subseteq G_2$. Then $x \in N \cap M \subseteq G_1 \cap G_2$, but $N \cap M \in g\omega\beta - N(x)$ by (ii). Hence, $G_1 \cap G_2 \in \tau$. Finally, let $G_\alpha \in \tau$ for every $\alpha \in \Delta$. If $x \in \cup\{G_\alpha : \alpha \in \Delta\}$, then $x \in G_{\alpha(x)}$ for some $\alpha(x) \in \Delta$. Since $G_{\alpha(x)} \in \tau$, there exists an $N \in g\omega\beta - N(x)$ such that $x \in N \subset G_{\alpha(x)} \subseteq \cup\{G_\alpha : \alpha \in \Delta\}$. Hence, $\cup\{G_\alpha : \alpha \in \Delta\} \in \tau$. It follows that $\tau$ is a topology for $X$. □

## 4. $g\omega\beta$-Continuity, $g\omega\beta$-Irresoluteness and $g\omega\beta$-Closedness

In this section we define generalized $\omega\beta$-continuity, generalized $\omega\beta$-irresoluteness and $g\omega\beta$-closed functions and some of the basic properties are studied. First we give some properties about the generalized $\omega\beta$-closure. The intersection of all $g\omega\beta$-closed (resp. $g$-closed [8]) sets of $X$ containing $A$ is called $g\omega\beta-$closure (resp. $g$-closure) of $A$, and it is denoted by $\omega\beta Cl^*(A)$ (resp. $Cl^*(A)$).

**Lemma 4.1.** *For an $x \in X$, $x \in \omega\beta Cl^*(A)$ if and only if $V \cap A \neq \phi$ for every $g\omega\beta$-open set $V$ containing $x$.*

*Proof.* It is trivial. □

**Lemma 4.2.** *Let $A$ and $B$ be subsets of $(X, \tau)$, then the following properties hold.*

*i.* $\omega\beta Cl^*(\phi) = \phi$ *and* $\omega\beta Cl^*(X) = X$.
*ii.* *If* $A \subseteq B$, *then* $\omega\beta Cl^*(A) \subseteq \omega\beta Cl^*(B)$.
*iii.* $A \subseteq \omega\beta Cl^*(A)$.
*iv.* $\omega\beta Cl^*(A) = \omega\beta Cl^*(\omega\beta Cl^*(A))$.
*v.* $\omega\beta Cl^*(A) \cup \omega\beta Cl^*(B) \subseteq \omega\beta Cl^*(A \cup B)$.
*vi.* $\omega\beta Cl^*(A) \cap \omega\beta Cl^*(B) \supseteq \omega\beta Cl^*(A \cap B)$.

**Definition 4.3.** *Let $(X, \tau)$ be a topological space*

*i.* $\tau^* = \{U \subseteq X | Cl^*(X - U) = X - U\}$ [8].
*ii.* $\tau_{\omega\beta}{}^* = \{W \subseteq X | \omega\beta Cl^*(X - W) = X - W\}$.

**Proposition 4.4.** *For a subset $A$ of $(X, \tau)$, the following implications hold.*

*i.* $A \subseteq \omega\beta Cl^*(A) \subseteq \omega\beta Cl(A) \subseteq Cl(A)$.
*ii.* $\tau \subseteq \omega\beta O(X, \tau) \subseteq \tau_{\omega\beta}{}^*$.
*iii.* $A \subseteq \omega\beta Cl^*(A) \subseteq Cl^*(A) \subseteq Cl(A)$.
*iv.* $\tau \subseteq \{g - open\ sets\} \subseteq \tau^* \subseteq \tau_{\omega\beta}{}^*$.

**Theorem 4.5.** *If $G\omega\beta O(X, \tau)$ is a topology, then $\tau_{\omega\beta}{}^*$ is a topology.*

*Proof.* Clearly $\phi, X \in \tau_{\omega\beta}{}^*$. Let $A, B \in \tau_{\omega\beta}{}^*$. Now $\omega\beta Cl^*(X - (A \cap B)) = \omega\beta Cl^*((X - A) \cup (X - B)) = \omega\beta Cl^*(X - A) \cup \omega\beta Cl^*(X - B) = (X - A) \cup (X - B) = X - (A \cap B)$. Hence, $A \cap B \in \tau_{\omega\beta}{}^*$. Let $\{A_i : i \in \Delta\} \in \tau_{\omega\beta}{}^*$. Then $\omega\beta Cl^*(X - \cup A_i) = \omega\beta Cl^*(\cap(X - A_i)) \subset \cap\omega\beta Cl^*(X - A_i) = \cap(X - A_i) = X - \cup A_i$. Since $X - \cup A_i \subset \omega\beta Cl^*(X - \cup A_i)$, $\omega\beta Cl^*(\cup(X - A_i)) = \cup\omega\beta Cl^*(X - A_i)$ and hence, $\cup A_i \in \tau_{\omega\beta}{}^*$. Thus, $\tau_{\omega\beta}{}^*$ is a topology. □

**Theorem 4.6.** *For a topological space $(X, \tau)$, the following properties hold.*

*i.* *A space* $(X, \tau)$ *is* $g\omega\beta - \mathrm{T}_{1/2}$ *if and only if* $\tau_{\omega\beta}{}^* = \omega\beta O(X, \tau)$.
*ii.* *Every* $g\omega\beta$*-closed is closed if and only if* $\tau_{\omega\beta}{}^* = \tau$.

*Proof.* i) Necessity. Let $A \in {\tau_{\omega\beta}}^*$. Then $\omega\beta Cl^*(X - A) = X - A$. Since $(X, \tau)$ is $g\omega\beta$-$\mathrm{T}_{1/2}$, $\omega\beta Cl(X - A) = \omega\beta Cl^*(X - A) = X - A$. Hence, $A \in \omega\beta O(X, \tau)$. By Proposition 4.4, ${\tau_{\omega\beta}}^* = \omega\beta O(X, \tau)$.
Sufficiency. Suppose ${\tau_{\omega\beta}}^* = \omega\beta O(X, \tau)$. Let $A$ be $g\omega\beta$-closed set. Then $\omega\beta Cl^*(A) = A$. This implies $X - A \in {\tau_{\omega\beta}}^* = \omega\beta O(X, \tau)$. So $A$ is $\omega\beta$-closed.
Proof of (ii) is similar to (i). □

**Definition 4.7.** *A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be $g\omega\beta$-continuous if $f^{-1}(V)$ is $g\omega\beta$-closed in $X$ for every closed set $V$ of $Y$.*

Continuity implies $g\omega\beta$-continuity but the converse need not be true.

**Example 4.8.** *Let $X = \mathbb{R}$ with the topology $\tau = \tau_u$ and let $Y = \{1, 2\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ be the function defined by*

$$f(x) = \begin{cases} 2, & x \in \mathbb{R} - \mathbb{Q}; \\ 1, & x \in \mathbb{Q}. \end{cases}$$

*Then $f$ is $g\omega\beta$-continuous but not continuous, since $f^{-1}(\{2\}) = \mathbb{R} - \mathbb{Q}$ is not closed in $(X, \tau)$.*

**Remark 4.9.**
*i) If ${\tau_{\omega\beta}}^* = \tau$ in $X$, then continuity and $g\omega\beta$-continuity coincide.*
*ii) Every $g\omega\beta$-continuous function defined on $g\omega\beta$-$\mathrm{T}_{1/2}$ space is $\omega\beta$-continuous.*
*iii) A function $f\colon (X, \tau) \to (Y, \sigma)$ is $g\omega\beta$-continuous if and only if the inverse image of every open set in $Y$ is $g\omega\beta-$open in $X$.*

**Theorem 4.10.** *If $f\colon (X, \tau) \to (Y, \sigma)$ is $g\omega\beta$-continuous, then $f(\omega\beta Cl^*(A)) \subseteq Cl(f(A))$ for every subset $A$ of $X$.*

*Proof.* Let $A \subseteq X$. Then $Cl(f(A))$ is closed in $Y$. By assumption $f^{-1}(Cl(f(A)))$ is $g\omega\beta$-closed in $X$. And $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(Cl(f(A)))$ implies $\omega\beta Cl^*(A) \subseteq f^{-1}(Cl(f(A)))$. Hence, $f(\omega\beta Cl^*(A)) \subseteq Cl(f(A))$. □

However, the converse does not hold.

**Example 4.11.** *Let $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$ and let $Y = \{1, 2\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ be the function defined by $f(x) = 2$ for all $x \in \mathbb{R} - \mathbb{Q}$. If we take $A = \mathbb{R} - \mathbb{Q}$, then $f(\omega\beta Cl^*(A)) \subseteq Cl(f(A))$. However $f$ is not $g\omega\beta$-continuous since $f^{-1}(\{2\}) = \mathbb{R} - \mathbb{Q}$ is not $g\omega\beta$-closed in $(X, \tau)$.*

**Theorem 4.12.** *Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function*
*a) If for each point $x \in X$ and each open set $V$ containing $f(x)$ there exists a $g\omega\beta$-open set $U$ containing $x$ such that $f(U) \subseteq V$, then for every subset*

*$A$ of $X$, $f(\omega\beta Cl^*(A)) \subseteq Cl(f(A))$.*
*b) The following statements are equivalent.*

*i. For every subset $A$ of $X$, $f(\omega\beta Cl^*(A)) \subseteq Cl(f(A))$.*
*ii. Suppose ${\tau_{\omega\beta}}^*$ is a topology. The function $f\colon (X, {\tau_{\omega\beta}}^*) \to (Y, \sigma)$ is continuous.*

*Proof.*
a) Let $y \in f(\omega\beta Cl^*(A))$. Let $V$ be an open set containing $y$. Then by hypothesis, there exists $x \in \omega\beta Cl^*(A)$ such that $f(x) = y$ and a $g\omega\beta$-open set $U$ containing $x$ such that $f(U) \subseteq V$. Therefore, by Lemma 4.1 $U \cap A \neq \phi$. Then $f(U \cap A) \neq \phi$. This implies $V \cap f(A) \neq \phi$. Hence, $y \in Cl(f(A))$.
b) (i) $\to$ (ii) Let $A$ be closed in $(Y, \sigma)$. By hypothesis,
$f(\omega\beta Cl^*(f^{-1}(A))) \subseteq Cl(f(f^{-1}(A))) \subseteq Cl(A) = A$. That is,
$\omega\beta Cl^*(f^{-1}(A)) \subseteq f^{-1}(A)$. Also, $f^{-1}(A) \subseteq \omega\beta Cl^*(f^{-1}(A))$. Thus, $f^{-1}(A)$ is closed in $(X, {\tau_{\omega\beta}}^*)$ and so $f$ is continuous.
(ii) $\to$ (i) For every subset $A$ of $X$, $Cl(f(A))$ is closed in $(Y, \sigma)$. Since $f\colon (X, {\tau_{\omega\beta}}^*) \to (Y, \sigma)$ is continuous, $f^{-1}(Cl(f(A)))$ is closed in $(X, {\tau_{\omega\beta}}^*)$ and hence, $\omega\beta Cl^*(f^{-1}(Cl(f(A)))) = f^{-1}(Cl(f(A)))$. Moreover, we have $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(Cl(f(A)))$ and by Lemma 4.2, $\omega\beta Cl^*(A) \subseteq$
$\omega\beta Cl^*(f^{-1}(Cl(f(A)))) = f^{-1}(Cl(f(A)))$. Therefore, we obtain
$f(\omega\beta Cl^*(A)) \subseteq Cl(f(A))$. □

**Theorem 4.13.** *If $f\colon (X, \tau) \to (Y, \sigma)$ is a continuous and $\omega\beta$-closed function, then $f(A)$ is $g\omega\beta$-closed in $Y$ for every $g\omega\beta$-closed set $A$ in $X$.*

*Proof.* Let $A$ be any $g\omega\beta$-closed set of $X$ and $U$ be any open set of $Y$ containing $f(A)$. Since $f$ is continuous, $f^{-1}(U)$ is open in $X$ and $A \subseteq f^{-1}(U)$. Therefore, we have $\omega\beta Cl(A) \subseteq f^{-1}(U)$ and hence, $f(\omega\beta Cl(A)) \subseteq U$. Since $f$ is $\omega\beta$-closed, $\omega\beta Cl(f(A)) \subseteq \omega\beta Cl(f(\omega\beta Cl(A))) = f(\omega\beta Cl(A)) \subseteq U$. Hence, $f(A)$ is $g\omega\beta$-closed in $Y$. □

**Definition 4.14.** *A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be $g\omega\beta$-irresolute if $f^{-1}(V)$ is $g\omega\beta$-closed in $X$ for every $g\omega\beta-$closed set $V$ of $Y$.*

It follows easily from the definition that a function $f$ is $g\omega\beta$-irresolute if and only if the inverse image of every $g\omega\beta$-open set in $Y$ is $g\omega\beta$-open in $X$.

Note that if a function is $g\omega\beta$-irresolute then it is $g\omega\beta$-continuous, but not conversely.

**Example 4.15.** *Let $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$ and let $Y = \{1, 2\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q}; \\ 2, & x \in \mathbb{Q}. \end{cases}$$

*Then $f$ is $g\omega\beta$-continuous but not $g\omega\beta$-irresolute, since $f^{-1}(\{1\}) = \mathbb{R} - \mathbb{Q}$ is not $g\omega\beta-$closed in $(X, \tau)$.*

**Proposition 4.16.** *If $f\colon (X, \tau) \to (Y, \sigma)$ is an $g\omega\beta$-continuous and $\sigma_{\omega\beta}{}^* = \sigma$ holds, then $f$ is $g\omega\beta$-irresolute.*

The proof follows from Remark 4.9.

**Theorem 4.17.** *If $f\colon (X, \tau) \to (Y, \sigma)$ is an $\omega\beta$-irresolute open bijection, then $f$ is $g\omega\beta$-irresolute.*

*Proof.* Let $F$ be any $g\omega\beta$-closed set of $Y$ and $U$ be an open set of $X$ containing $f^{-1}(F)$. Since $f$ is open, $f(U)$ is open in $Y$ and $F \subseteq f(U)$. Since $F$ is $g\omega\beta$-closed, $\omega\beta Cl(F) \subseteq f(U)$ and hence, $f^{-1}(\omega\beta Cl(F)) \subseteq U$. Since $f$ is $\omega\beta$-irresolute, $f^{-1}(\omega\beta Cl(F))$ is $\omega\beta$-closed. Hence, $\omega\beta Cl(f^{-1}(F)) \subseteq \omega\beta Cl(f^{-1}(\omega\beta Cl(F))) = f^{-1}(\omega\beta Cl(F)) \subseteq U$.
Therefore, $f^{-1}(F)$ is $g\omega\beta$-closed and $f$ is $g\omega\beta$-irresolute. □

The composition of two $g\omega\beta$-continuous functions need not be $g\omega\beta$-continuous as can be seen from the following example.

**Example 4.18.** *Consider $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$, $Y = \{1, 2\}$ with the topologies $\sigma = \{\phi, Y, \{1\}\}$ and $\rho = \{\phi, Y, \{2\}\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ be the function define by*

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q}; \\ 2, & x \in \mathbb{Q}. \end{cases}$$

*And $g\colon (Y, \sigma) \to (Y, \rho)$ be the identity function. Then $f$ and $g$ are $g\omega\beta$-continuous. However, $g \circ f$ is not $g\omega\beta-$continuous since $(g \circ f)^{-1}(1) = \mathbb{R} - \mathbb{Q}$ is not $g\omega\beta$-closed in $(X, \tau)$.*

**Theorem 4.19.** *Let $f\colon (X, \tau) \to (Y, \sigma)$ and $g\colon (Y, \sigma) \to (Z, \rho)$ be any two functions. Then*

- *i. $g \circ f$ is $g\omega\beta$-continuous, if $g$ is continuous and $f$ is $g\omega\beta$-contin-uous.*
- *ii. $g \circ f$ is $g\omega\beta$-irresolute, if $g$ is $g\omega\beta$-irresolute and $f$ is $g\omega\beta$-irresolute.*
- *iii. $g \circ f$ is $g\omega\beta$-continuous, if $g$ is $g\omega\beta$-continuous and $f$ is $g\omega\beta$-irresolute.*
- *iv. $g \circ f$ is $\omega\beta$-continuous, if $f$ is $\omega\beta$-irresolute and $g$ is $g\omega\beta$-contin-uous and $Y$ is a $g\omega\beta$-$T_{1/2}$ space.*
- *v. $g \circ f$ is $g\omega\beta$-continuous, if $f$, $g$ are $g\omega\beta$-continuous and $\sigma_{\omega\beta}{}^* = \sigma$.*

*Proof.*
i) Let $V$ be closed in $(Z, \rho)$. Then $g^{-1}(V)$ is closed in $(Y, \sigma)$, since $g$ is continuous. $g\omega\beta$-continuity of $f$ implies that $f^{-1}(g^{-1}(V))$ is $g\omega\beta$-closed in $(X, \tau)$. Hence, $g \circ f$ is $g\omega\beta$-continuous.
ii) Let $V$ be $g\omega\beta$-closed in $(Z, \rho)$. Then $g^{-1}(V)$ is $g\omega\beta$-closed in $(Y, \sigma)$, since $g$ is $g\omega\beta-$irresolute. $g\omega\beta$-irresoluteness of $f$ implies that $f^{-1}(g^{-1}(V))$ is $g\omega\beta$-closed in $(X, \tau)$. Hence, $g \circ f$ is $g\omega\beta$-irresolute.

iii) Let $V$ be closed in $(Z,\rho)$. Then $g^{-1}(V)$ is $g\omega\beta$-closed in $(Y,\sigma)$, since $g$ is $g\omega\beta$-continuous. $g\omega\beta-$irresoluteness of $f$ implies that $f^{-1}(g^{-1}(V))$ is $g\omega\beta$-closed in $(X,\tau)$. Hence, $g\circ f$ is $g\omega\beta$-continuous.
iv) Let $V$ be closed in $(Z,\rho)$. Then $g^{-1}(V)$ is $g\omega\beta$-closed in $(Y,\sigma)$, since $g$ is $g\omega\beta$-continuous. As $(Y,\sigma)$ is an $\omega\beta$-$\mathrm{T}_{1/2}$ space, $g^{-1}(V)$ is $\omega\beta$-closed in $(X,\tau)$. Hence, $g\circ f$ is $\omega\beta$-irresolute.
v) The proof follows from Remark 4.9. □

**Theorem 4.20.** *Let $f\colon (X,\tau)\to(Y,\sigma)$ be a function*

*i. If $f$ is $g\omega\beta$-irresolute and $X$ is $g\omega\beta$-$\mathrm{T}_{1/2}$, then $f$ is $\omega\beta$-irresolute.*
*ii. If $f$ is $g\omega\beta$-continuous and $X$ is $g\omega\beta$-$\mathrm{T}_{1/2}$, then $f$ is $\omega\beta$-continuous.*

*Proof.*
i) Let $V$ be $\omega\beta$-closed in $Y$. Since $f$ is $g\omega\beta$-irresolute and every $\omega\beta$-closed is $g\omega\beta$-closed, $f^{-1}(V)$ is $g\omega\beta$-closed in $X$. Since $X$ is $g\omega\beta$-$\mathrm{T}_{1/2}$, $f^{-1}(V)$ is $\omega\beta$-closed in $X$. Hence, $f$ is $\omega\beta$-irresolute.
ii) Let $V$ be closed in $Y$. Since $f$ is $g\omega\beta$-continuous, $f^{-1}(V)$ is $g\omega\beta$-closed in $X$. By assumption, it is $\omega\beta-$closed. Therefore, $f$ is $\omega\beta$-continuous. □

**Theorem 4.21.** *Let $f\colon (X,\tau)\to(Y,\sigma)$ be an $\omega\beta$-closed and $g\omega\beta$-irresolute surjection. If $(X,\tau)$ is an $\omega\beta$-$\mathrm{T}_{1/2}$ space, then $(Y,\sigma)$ is also an $\omega\beta$-$\mathrm{T}_{1/2}$ space.*

*Proof.* Let $F$ be any $g\omega\beta$-closed set of $Y$. Since $f$ is $g\omega\beta$-irresolute, $f^{-1}(F)$ is $g\omega\beta$-closed in $X$. Since $X$ is $\omega\beta$-$\mathrm{T}_{1/2}$, $f^{-1}(F)$ is $\omega\beta$-closed in $X$. As $f$ is $\omega\beta$-closed, $f(f^{-1}(F))=F$ is $\omega\beta-$closed in $Y$. This shows that $(Y,\sigma)$ is also $g\omega\beta$-$\mathrm{T}_{1/2}$ space. □

**Definition 4.22.** *A function $f\colon (X,\tau)\to(Y,\sigma)$ is said to be $g\omega\beta^*$-continuous if the inverse image of every $\omega\beta$-closed set in $Y$ is $g\omega\beta$-closed in $X$.*

**Remark 4.23.** *The class of all $g\omega\beta^*$-continuous functions lie inbetween the class of all $g\omega\beta$-irresolute functions and the class of all $g\omega\beta$-continuous functions, as seen in the following proposition.*

**Proposition 4.24.** *Let $f\colon (X,\tau)\to(Y,\sigma)$ be a function.*

*i. If $f$ is $g\omega\beta$-irresolute, then it is $g\omega\beta^*$-continuous.*
*ii. If $f$ is $g\omega\beta^*$-continuous, then it is $g\omega\beta$-continuous.*

The authors were unable to find an example to show that the converse of (i) in Proposition 4.24 is not always true. However, the function defined in Example 4.15 is $g\omega\beta$-continuous but not $g\omega\beta^*$-continuous.

**Proposition 4.25.** *If a bijection $f\colon (X,\tau)\to(Y,\sigma)$ is open and $g\omega\beta^*$-continuous, then it is $g\omega\beta$-irresolute.*

*Proof.* Let $A$ be $g\omega\beta$-closed in $Y$. Let $f^{-1}(A) \subseteq U$, where $U$ is open in $X$. Since $f$ is open, $f(U)$ is open in $Y$. $A \subseteq f(U)$ implies that $\omega\beta Cl(A) \subseteq f(U)$. That is, $f^{-1}(\omega\beta Cl(A)) \subseteq U$. Since $f$ is $g\omega\beta^*$-continuous, $\omega\beta Cl(f^{-1}(\omega\beta Cl(A))) \subseteq U$ and so $\omega\beta Cl(f^{-1}(A)) \subseteq U$. This shows that $f^{-1}(A)$ is $g\omega\beta$-closed in $X$. Hence, $f$ is $g\omega\beta$-irresolute. □

**Proposition 4.26.** *Let a bijection $f\colon (X,\tau) \to (Y,\sigma)$ be open $g\omega\beta^*$-continuous and $\omega\beta$-closed. If $X$ is $g\omega\beta$-$T_{1/2}$, then $Y$ is $g\omega\beta - T_{1/2}$.*

*Proof.* Let $A$ be $g\omega\beta$-closed in $Y$. By Proposition 4.25, $f^{-1}(A)$ is $g\omega\beta$-closed in $X$. By hypothesis, $f^{-1}(A)$ is $\omega\beta$-closed in $X$. Since $f$ is bijective and $\omega\beta$-closed, $A = f(f^{-1}(A))$ is $\omega\beta$-closed in $Y$. That is, $Y$ is an $\omega\beta$-$T_{1/2}$ space. □

**Definition 4.27.** *A function $f\colon (X,\tau) \to (Y,\sigma)$ is called a generalized $\omega\beta$-closed function (written as $g\omega\beta$-closed function) if for each closed set $F$ in $X$, $f(F)$ is a $g\omega\beta$-closed set of $Y$.*

Every closed function is a $g\omega\beta$-closed function, but not conversely.

**Example 4.28.** *Let $X = \{1,2\}$ with the topologies $\tau = \{\phi, X, \{1\}\}$ and $\sigma = \{\phi, X, \{2\}\}$. Let $f\colon (X,\tau) \to (X,\sigma)$ be the identity function. Then $f$ is $g\omega\beta$-closed but not closed, since $f(\{2\}) = 2$ is not closed in $(X,\sigma)$.*

**Theorem 4.29.** *A function $f\colon (X,\tau) \to (Y,\sigma)$ is $g\omega\beta$-closed if and only if for each subset $S$ of $Y$ and for each open set $U$ containing $f^{-1}(S)$, there is a $g\omega\beta$-open set $V$ of $Y$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.*

*Proof.* Necessity. Let $S$ be a subset of $Y$ and $U$ be an open set of $X$ such that $f^{-1}(S) \subseteq U$. Then $Y - f(X-U)$, say $V$, is a $g\omega\beta-$open set containing $S$ such that $f^{-1}(V) \subseteq U$.
Sufficiency. Let $F$ be a closed set of $X$, then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is open. By hypothesis, there is a $g\omega\beta$-open set $V$ of $Y$ such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, we have $F \subseteq X - f^{-1}(V)$ and hence, $Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$. This implies $f(F) = Y - V$, since $Y - V$ is $g\omega\beta$-closed, $f(F)$ is $g\omega\beta$-closed and thus, $f$ is a $g\omega\beta$-closed function. □

**Theorem 4.30.** *If a function $f\colon (X,\tau) \to (Y,\sigma)$ is $g\omega\beta$-closed, then $\omega\beta Cl^*(f(A)) \subseteq f(Cl(A))$ for every subset $A$ of $(X,\tau)$.*

*Proof.* Suppose that $f$ is $g\omega\beta$-closed and $A \subseteq X$. Then $Cl(A)$ is closed in $X$ and so $f(Cl(A))$ is $g\omega\beta$-closed in $(Y,\sigma)$. We have $f(A) \subseteq f(Cl(A))$ by Lemma 4.2, $\omega\beta Cl^*(f(A)) \subseteq \omega\beta Cl^*(f(Cl(A)))$. Since $f(Cl(A))$ is $g\omega\beta$-closed in $(Y,\sigma)$, $\omega\beta Cl^*(f(Cl(A))) = f(Cl(A))$, we have $\omega\beta Cl^*(f(A)) \subseteq f(Cl(A))$ for every subset $A$of $(X,\tau)$. □

**Theorem 4.31.** *If $f\colon (X,\tau) \to (Y,\sigma)$ is continuous, $g\omega\beta$-closed and $A$ is a $g$-closed subset of $X$, then $f(A)$ is $g\omega\beta$-closed.*

*Proof.* Let $f(A) \subseteq U$, where $U$ is an open subset of $Y$, then $f^{-1}(U)$ is an open set containing $A$. Since $A$ is $g$-closed, we have $Cl(A) \subseteq f^{-1}(U)$ and $f(Cl(A)) \subseteq U$. Since $f$ is $g\omega\beta$-closed, $f(Cl(A))$ is $g\omega\beta$-closed. Therefore, $\omega\beta Cl(f(Cl(A))) \subseteq U$ which implies that $\omega\beta Cl(f(A)) \subseteq U$. Hence, $f(A)$ is $g\omega\beta$-closed. □

**Theorem 4.32.** *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a bijective, open, and $g\omega\beta^* -$continuous function. Then $f$ is a $g\omega\beta$-irresolute function.*

*Proof.* Let $V$ be any $g\omega\beta$-closed subset of $Y$ and let $U$ be any open subset of $X$ such that $f^{-1}(V) \subseteq U$. Clearly $V \subseteq f(U)$, since $f$ is an open function, $f(U)$ is open and $V$ is $g\omega\beta$-closed. Hence, $\omega\beta Cl(V) \subseteq f(U)$ and $f^{-1}(\omega\beta Cl(V)) \subseteq U$. Since $f$ is $g\omega\beta^*-$continuous and $\omega\beta Cl(V)$ is $\omega\beta-$closed in $Y$, then $f^{-1}(\omega\beta Cl(V))$ is a $g\omega\beta$-closed subset of $U$ and so $\omega\beta Cl(f^{-1}(\omega\beta Cl(V))) \subseteq U$. So $\omega\beta Cl(f^{-1}(V)) \subseteq U$. Therefore, $f^{-1}(V)$ is a $g\omega\beta$-closed subset. Hence, $f$ is a $g\omega\beta$-irresolute function. □

**Proposition 4.33.** *If $f\colon (X,\tau) \to (Y,\sigma)$ is bijective, $\omega\beta$-closed and continuous, then the inverse function $f^{-1}\colon (Y,\sigma) \to (X,\tau)$ is $g\omega\beta$-irresolute.*

*Proof.* Let $A$ be $g\omega\beta$-closed in $(X,\tau)$. Let $(f^{-1})^{-1}(A) = f(A) \subseteq U$, where $U$ is open in $(Y,\sigma)$. Then $A \subseteq f^{-1}(U)$, since $f^{-1}(U)$ is open in $(X,\tau)$ and $A$ is $g\omega\beta$-closed in $(X,\tau)$, $\omega\beta Cl(A) \subseteq f^{-1}(U)$ and hence, $f(\omega\beta Cl(A)) \subseteq U$. Since $f$ is $\omega\beta$-closed, $f(\omega\beta Cl(A))$ is $\omega\beta$-closed in $(Y,\sigma)$ and $f(A) \subset f(\omega\beta Cl(A)$ and hence, $\omega\beta Cl(f(A)) \subseteq U$. Thus, $f(A)$ is $g\omega\beta$-closed in $(Y,\sigma)$ and so $f^{-1}$ is $g\omega\beta$-irresolute. □

**Theorem 4.34.** *If $f\colon (X,\tau) \to (Y,\sigma)$ is a continuous surjection and $g\colon (Y,\sigma) \to (Z,\rho)$ is a function such that $g \circ f\colon (X,\tau) \to (Z,\rho)$ is $g\omega\beta$-closed, then $g$ is $g\omega\beta$-closed.*

*Proof.* Let $V$ be a closed set of $Y$. Since $f^{-1}(V)$ is closed in $X$, $g(V) = (g \circ f)(f^{-1}(V))$ is $g\omega\beta$-closed in $Z$. Hence, $g$ is $g\omega\beta$-closed. □

**Theorem 4.35.** *If $f\colon (X,\tau) \to (Y,\sigma)$ is a continuous, onto and $g\omega\beta$-closed function from a normal space $(X,\tau)$ to a space $(Y,\sigma)$, then $(Y,\sigma)$ is $\omega\beta$-normal.*

*Proof.* Let $A$ and $B$ be disjoint closed sets of $Y$. Since $X$ is normal, then there exist disjoint open sets $U$ and $V$ in $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. By Theorem 4.29, there exist $g\omega\beta$-open sets $G$ and $H$ in $Y$ such that $A \subset G$, $B \subset H$, $f^{-1}(G) \subseteq U$, $f^{-1}(H) \subseteq V$ and $f^{-1}(G) \cap f^{-1}(H) = \phi$. Hence, $G \cap H = \phi$. Since $G$ is $g\omega\beta$-open and $A$ is a closed set such that $A \subseteq G$, $A \subseteq \omega\beta Int(G)$. Similarly, $B \subseteq \omega\beta Int(H)$. Hence, $\omega\beta Int(G) \cap \omega\beta Int(H) \subset G \cap H = \phi$. Therefore, $Y$ is $\omega\beta$-normal. □

**Theorem 4.36.** *If $f: (X, \tau) \to (Y, \sigma)$ is a continuous, $\omega\beta$-open and $g\omega\beta$-closed surjection from a regular space $(X, \tau)$ to a space $(Y, \sigma)$, then $(Y, \sigma)$ is $\omega\beta$-regular.*

*Proof.* Let $y \in Y$ and $U$ be an open set containing $y$ in $Y$, then there exists $x \in X$ such that $f(x) = y$. Now, $f^{-1}(U)$ is an open set in $X$ containing $x$. But $X$ is regular, then there exists an open set $V$ such that $x \in V \subseteq Cl(V) \subseteq f^{-1}(U)$ and $y \in f(V) \subseteq f(Cl(V)) \subseteq U$. But $f(Cl(V))$ is $g\omega\beta$-closed. Then we have $\omega\beta Cl(f(Cl(V))) \subseteq U$. Therefore, $y \in f(V) \subseteq \omega\beta Cl(f(V)) \subseteq U$ and $f(V)$ is $\omega\beta$-open in Y (because $f$ is $\omega\beta$-open). Hence, $Y$ is $\omega\beta$-regular. □

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