RATES OF UNIFORM CONVERGENCE FOR RIEMANN INTEGRALS

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Abstract. A function $f\colon [0,1] \to \mathbb{R}$ is Riemann integrable if and only if its Riemann sums $f(T)$ and $f(T')$ get closer to each other as $\delta \to 0$, uniformly over all δ-fine tagged divisions T and T'. We show that $\delta^{-1}|f(T) - f(T')| \asymp \mathrm{Var}(f)$. We also give an example of a function $f \notin \mathrm{BV}$ with $|f(T) - f(T')| = \mathcal{O}(\delta|\ln\delta|)$. As a lemma, we show that any $f \in \mathrm{BV}$ can be approximated uniformly by a step function g with $\mathrm{Var}(g) \approx \mathrm{Var}(f)$.

1. Introduction and Notation

Definition 1.1. *A division of the interval* $[0,1]$ *is a finite partition*

$$0 = s_0 < s_1 < s_2 < \cdots < s_m = 1.$$

A tagged division is a division together with selected points $\sigma_j \in [s_{j-1}, s_j]$*; the number* σ_j *is called the tag of the subinterval* $[s_{j-1}, s_j]$*. We shall denote a typical tagged division by* $T = \{(\sigma_j, [s_{j-1}, s_j])\}_{j=1}^{m}$*. For any function* $f\colon [0,1] \to \mathbb{R}$*, the Riemann sum over the tagged division* T *is*

$$f(T) \quad = \quad \sum_{j=1}^{m} f(\sigma_j)(s_j - s_{j-1}).$$

Let δ *be a positive number; a tagged division* T *is called* δ*-fine, written* $T \ll \delta$*, if* $\max_i(s_i - s_{i-1}) < \delta$*. (Some of the ideas in this paper will be generalized in [2], where* δ *may be a positive function, not just a positive number.)*

Definition 1.2. *(The following is equivalent to the usual definitions.) A number* v *is the Riemann integral of a function* $f : [0,1] \to \mathbb{R}$ *if*

> *for each number* $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that, whenever* T *is a* δ*-fine tagged division, then* $|f(T) - v| < \varepsilon$*.*

Observation 1.3. *(Cauchy condition) A function* $f\colon [0,1] \to \mathbb{R}$ *is Riemann integrable if it has a Riemann integral. In other words, a function* f *is Riemann integrable if and only if*

> *for each number $\varepsilon > 0$ there exists a number $\delta > 0$ such that, whenever T and T' are δ-fine tagged divisions, then $|f(T) - f(T')| < \varepsilon$.*

(That follows from the fact that the real number system is a complete metric space.)

Remark 1.4. *For any function $f\colon [0,1] \to \mathbb{R}$ and any number $\delta > 0$, let us denote*

$$\theta_\delta(f) = \sup_{T,T' \ll \delta} |f(T) - f(T')|.$$

Here the supremum is over all tagged divisions T and T' that are δ-fine. It is easy to show that $\theta_\delta(f) < \infty$ if and only if f is bounded, and that θ_δ is a seminorm on the linear space of bounded (not necessarily measurable) functions from $[0,1]$ into $\mathbb{R}$. The Cauchy criterion for integrability is that $\lim_{\delta \downarrow 0} \theta_\delta(f) = 0$.

On the Riemann integrable functions, we may also define this seminorm:

$$\psi_\delta(f) = \sup_{T \ll \delta} \left| f(T) - \int_0^1 f(s)ds \right|.$$

It is evident that both seminorms, θ_δ and ψ_δ, vanish on constant functions f. It is shown in [2] that these two seminorms vanish only on constant functions. The two seminorms are equivalent on integrable functions; we have

$$\psi_\delta(f) \le \theta_\delta(f) \le 2\psi_\delta(f).$$

(To prove $\psi \le \theta$, hold T fixed and let $f(T') \to \int f$.)

This paper's main theorems state that

$$\sup_{\delta > 0} \frac{\psi_\delta(f)}{\delta} \le \mathrm{Var}(f) \le \liminf_{\delta \downarrow 0} \frac{\theta_\delta(f)}{\delta}$$

for any function $f\colon [0,1] \to \mathbb{R}$. Consequently, a function f has bounded variation if and only if its Riemann sums converge to its integral at a rate of $\mathcal{O}(\delta)$, and that rate cannot be improved even for functions that have greater smoothness properties. In Example 4.1 we give an example of a Riemann integrable function with unbounded variation; its approximations converge at the slower rate of $\mathcal{O}(\delta|\ln\delta|)$. Lemma 2.1, on the approximation of bounded variation functions by step functions, may also be of interest in its own right.

Our results should be contrasted with those of Chui [3], who investigates the rate at which $R_n(f;a) \to \int f$ as $n \to \infty$, where

$$R_n(f;a) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k-a}{n}\right) \qquad \text{for } a \in [0,1].$$

Chui's sum $R_n(f;a)$ is just *one* of the many sums $f(T)$ that can be obtained with a choice of $T \ll 1/n$; hence

$$\left|R_n(f;a) - \textstyle\int f\right| \le \psi_{1/n}(f).$$

The quantities on the two sides of this inequality need not be close. For instance, Chui shows that

$$R_n\left(f;\frac{1}{2}\right) - \textstyle\int f = o(1/n) \quad \text{for } f \text{ absolutely continuous, and}$$
$$R_n\left(f;\frac{1}{2}\right) - \textstyle\int f = \mathcal{O}(1/n^2) \quad \text{if } f \text{ is differentiable with } f' \in BV.$$

But our own Theorem 2.2 shows that $\psi_{1/n}(f)$ cannot converge to 0 any faster than $\mathcal{O}(1/n)$.

On the other hand, some of Chui's examples of slow convergence would also apply to our own functions. Chui's Theorem 2 shows that for any sequence (ε_n) decreasing to 0, there exists a function f satisfying $R_n(f;0) - \int f \ge \varepsilon_n$; hence also $\psi_{1/n}(f) \ge \varepsilon_n$.

This paper is based on results in the first author's doctoral dissertation [1].

2. Upper Bound for Errors

Lemma 2.1. *Suppose $f\colon [0,1] \to \mathbb{R}$ has bounded variation, and some number $\varepsilon > 0$ is given. Then there exists a step function $g\colon [0,1] \to \mathbb{R}$ such that*

$$\|f-g\|_{\sup} \le \varepsilon \qquad \text{and} \qquad |\mathrm{Var}(f) - \mathrm{Var}(g)| \le \varepsilon,$$

where "Var" denotes variation.

Remarks. We emphasize that the step function need not be left- or right-continuous.

One is tempted to make the stronger assertion that there exists a step function g satisfying $|\mathrm{Var}(f-g)| \le \varepsilon$. But that is not true, for instance when $f(s) = s$.

Proof. Since f has bounded variation, it has a left-hand limit $f(s-)$ at each point $s \in (0,1]$ and a right-hand limit $f(s+)$ at each point $s \in [0,1)$. It has only countably many discontinuities, and each of those is a jump. The size of a jump at s is the number $|f(s) - f(s+)| + |f(s) - f(s-)|$; the sum of the sizes of the jumps is less than or equal to the variation. Let $s_1, s_2, s_3, \dots, s_N$ be the locations of the largest jumps, chosen so that any jump not in this finite set has size less than ε.

Enlarge the set $\{s_1, s_2, \ldots, s_N\}$, adding finitely many points to obtain a partition

$$0 = v_0 < v_1 < v_2 < \cdots < v_M = 1$$

with the property that

$$\mathrm{Var}(f) - \varepsilon \le \sum_{j=1}^{P} |f(v_{j-1}) - f(v_j)| \le \mathrm{Var}(f).$$

This pair of inequalities will be preserved if we add still more points to the partition. We now add points to make the partition

$$\begin{aligned} 0 = v_0 < w_0 < u_1 < v_1 < w_1 < \cdots < u_{M-1} < v_{M-1} \\ < w_{M-1} < u_M < v_M = 1 \end{aligned}$$

as follows:

For each of $i = 0, 1, 2, \ldots, M-1$, choose some point w_i that is greater than v_i, and is close enough to v_i to satisfy these conditions:

$$w_i - v_i < \frac{1}{2}(v_{i+1} - v_i), \qquad \sup_{s \in (v_i, w_i]} |f(s) - f(w_i)| < \varepsilon.$$

Likewise, for each of $i = 1, 2, 3, \ldots, M$, choose some point u_i that is less than v_i, and is close enough to v_i to satisfy these conditions:

$$v_i - u_i < \frac{1}{2}(v_i - v_{i-1}), \qquad \sup_{s \in [u_i, v_i)} |f(s) - f(u_i)| < \varepsilon.$$

(The conditions involving $\frac{1}{2}$ ensure that we actually do have $w_{i-1} < u_i$.)

Finally, we add still a few more points to the partition, as follows: Subdivide each interval $[w_{i-1}, u_i]$ into finitely many subintervals

$$w_{i-1} = x_i^0 < x_i^1 < x_i^2 < \cdots < x_i^{p_i} = u_i$$

having the property that

$$\sup\left\{ |f(s) - f(s')| \ : \ s, s' \in [x_i^{j-1}, x_i^j] \right\} \ < \ \varepsilon.$$

That such a subdivision is possible follows via a compactness argument, using the fact that any jumps f has in $[w_{i-1}, u_i]$ are smaller than ε.

Now define a step-function $g : [0,1] \to \mathbb{R}$ as follows:

$$g(s) = \begin{cases} f(x_i^{j-1}) & \text{if} \quad x_i^{j-1} \le s < x_i^j, \\ f(u_i) & \text{if} \quad u_i \le s < v_i, \\ f(v_i) & \text{if} \quad s = v_i, \\ f(w_i) & \text{if} \quad v_i < s \le w_i. \end{cases}$$

It is now easy to verify that $\|f-g\|_{\sup} \le \varepsilon$ and that

$$\mathrm{Var}(g) = \sum_{i=1}^{M} \Big\{ |f(v_{i-1}) - f(w_{i-1})| \\ + |f(u_i) - f(v_i)| + \sum_{k=1}^{p_i} |f(x_i^{k-1}) - f(x_i^k)| \Big\},$$

hence, $\mathrm{Var}(f) - \varepsilon \le \mathrm{Var}(g) \le \mathrm{Var}(f)$. □

Theorem 2.2. *Suppose* $f\colon [0,1] \to \mathbb{R}$ *has bounded variation,* δ *is a positive number, and* T *is a* δ*-fine tagged division of* $[0,1]$*. Then*

$$|f(T) - \textstyle\int_0^1 f| \le \delta\, \mathrm{Var}(f).$$

Proof. By Lemma 2.1, it suffices to consider the case where f is a step function. Then we may describe f as taking constant values x_j on disjoint nonempty intervals J_j $(j = 1, 2, 3, \ldots, p)$ whose union is $[0,1]$. Each J_j may be open, closed, or half-open, and each J_j may have positive length or (in the case where J_j is a single point) zero length. We may assume the intervals J_j are arranged from left to right with increasing j. Then $\mathrm{Var}(f) = \sum_{j=1}^{p-1} |x_j - x_{j+1}|$.

Let r_j be the right endpoint of J_j (which may or may not be a member of J_j), and let $r_0 = 0$. Then

$$0 = r_0 \le r_1 \le r_2 \le \cdots \le r_p = 1,$$

with $r_{j-1} = r_j$ holding just in the case where J_j is a singleton. The length of J_j is $r_j - r_{j-1}$, and we have

$$\int_0^1 f \;=\; \sum_{j=1}^{p} (r_j - r_{j-1}) x_j \;=\; x_p + \sum_{j=1}^{p-1} (x_j - x_{j+1}) r_j.$$

Let $T = \{(\sigma_i, [s_{i-1}, s_i])\}_{i=1}^m$ be some tagged division that is δ-fine. Note that each $[s_{i-1}, s_i]$ has positive length.

For each $j \in \{1, 2, \ldots, p\}$, say that

- the integer j is *taggish* if at least one tag σ_i lies in the interval J_j, or
- j is untaggish if no tag lies in J_j.

If j is a taggish integer, then all the i's satisfying $\sigma_i \in J_j$ must be consecutive i's (since the σ_i's form a nondecreasing sequence). Hence the union of their $[s_{i-1}, s_i]$'s is an interval, which we shall denote by $[u_j, v_j]$; it has positive length.

For each untaggish integer j, it will be convenient to define an interval $[u_j, v_j]$ of length 0, i.e., a single point. That point is chosen so that all the intervals $[u_j, v_j]$ (taggish or not) are arranged from left to right with increasing j. That is,

$$0 = u_0 \leq v_0 = u_1 \leq v_1 = \cdots \leq v_{p-1} = u_p \leq v_p = 1.$$

Hence, $u_j = v_{j-1}$ for all $j = 1, 2, \ldots, p$. Observe that

$$\begin{aligned} f(T) \;&=\; \sum_{i=1}^{m}(s_i - s_{i-1})f(\sigma_i) \;=\; \sum_{j=1}^{p}\;\sum_{\{i\,:\,\sigma_i \in J_j\}}(s_i - s_{i-1})x_j \\ &=\; \sum_{j=1}^{p}(v_j - u_j)x_j \;=\; \sum_{j=1}^{p}(v_j - v_{j-1})x_j \;=\; x_p + \sum_{j=1}^{p-1}(x_j - x_{j+1})v_j. \end{aligned}$$

Subtracting that from our earlier expression for $\int f$ yields

$$\textstyle\int f - f(T) \;=\; \displaystyle\sum_{j=1}^{p-1}(x_j - x_{j+1})(r_j - v_j).$$

Since $\sum_j |x_j - x_{j+1}| = \mathrm{Var}(f)$, it suffices to show that $|r_j - v_j| < \delta$ for all j. We prove that in two parts. First, we shall show $v_j < r_j + \delta$ by considering two cases:

- First, suppose that j is lower than every taggish integer. Any intervals $[u, v]$ to the left of v_j have length zero, so $v_j = 0$. Hence $v_j < \delta \leq r_j + \delta$.
- In the remaining case, there exists at least one taggish $\widehat{\jmath}$ satisfying $\widehat{\jmath} \leq j$. Take the highest such $\widehat{\jmath}$. (Thus $\widehat{\jmath} = j$ if j itself is taggish.) Now, $v_j = v_{\widehat{\jmath}}$ since any untaggish integer has its $[u, v]$ with length zero. We have $\sigma_{\widehat{\imath}} \in J_{\widehat{\jmath}}$ for some $\widehat{\imath}$, and any such $\widehat{\imath}$ satisfies $\sigma_{\widehat{\imath}} \leq r_{\widehat{\jmath}}$. More specifically, let $\widehat{\imath}$ be the highest integer for which $\sigma_{\widehat{\imath}}$ lies in $J_{\widehat{\jmath}}$; then $s_{\widehat{\imath}} = v_{\widehat{\jmath}}$. Hence,
$$v_j - \delta = s_{\widehat{\imath}} - \delta < s_{\widehat{\imath}-1} \leq \sigma_{\widehat{\imath}} \leq r_{\widehat{\jmath}} \leq r_j.$$

Finally, we shall show $r_j < v_j + \delta$ by considering two cases:

- First, suppose that there are no taggish integers higher than j. Then any intervals $[u, v]$ to the right of v_j have length 0, so $v_j = 1$. Therefore, $r_j \leq v_j < v_j + \delta$.
- On the other hand, suppose that there does exist a taggish integer $\widehat{\jmath}$ with $j < \widehat{\jmath}$. Choose the smallest such $\widehat{\jmath}$. Then any intervals $[u, v]$ between v_j and $u_{\widehat{\jmath}}$ correspond to untaggish integers, and have length 0, so $v_j = u_{\widehat{\jmath}}$. Let $\widehat{\imath}$ be the smallest integer for which $\sigma_{\widehat{\imath}} \in J_{\widehat{\jmath}}$. Thus, $\sigma_{\widehat{\imath}}$ is the lowest tag that lies to the right of R_j. The interval $[s_{\widehat{\imath}-1}, s_{\widehat{\imath}}]$ is the leftmost of the intervals whose union makes up $[u_{\widehat{\jmath}}, v_{\widehat{\jmath}}]$, so $s_{\widehat{\imath}-1} = u_{\widehat{\jmath}}$. Finally,
$$r_j \leq \sigma_{\widehat{\imath}} \leq s_{\widehat{\imath}} < s_{\widehat{\imath}-1} + \delta = v_j + \delta$$

as required.

□

3. Lower Bound for Worst-Case Errors

Theorem 3.1. *Let any function $f\colon [0,1] \to \mathbb{R}$ and any real number $\rho <$ $\mathrm{Var}(f)$ be given. Then for every number $\delta > 0$ sufficiently small,*

$$\sup_{T,T' \ll \delta} |f(T) - f(T')| > \delta\rho.$$

Remark. We do not require that f has bounded variation; the following argument is valid even in the case where $\mathrm{Var}(f) = +\infty$.

Proof. Choose some number θ slightly greater than ρ, and some large integer k, so that

$$\rho < \frac{k-1}{k}\,\theta < \theta < \mathrm{Var}(f).$$

Since $\mathrm{Var}(f) > \theta$, we may choose a partition of $[0,1]$,

$$0 = r_0 < r_1 < r_2 < \cdots < r_p = 1,$$

such that $\sum_{i=1}^{p} |f(r_i) - f(r_{i-1})| \geq \theta$. Fix any positive number δ less than $\min_i(r_i - r_{i-1})/k$. It suffices to exhibit tagged divisions T, T', both δ-fine for this choice of δ, satisfying

$$|f(T) - f(T')| \geq \frac{k-1}{k}\,\delta\theta.$$

Our tagged divisions $T = \{(\sigma_j, [s_{j-1}, s_j])\}_{j=1}^{m}$ and $T' = \left\{\left(\sigma_j', [s_{j-1}, s_j]\right)\right\}_{j=1}^{m}$ will both have the same division points

$$0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$$

and will differ only in their tags σ_j and σ_j'. Choose the division points s_j as follows.

For $1 \leq i \leq p$, let n_i be the integer part of $1 + \delta^{-1}(r_i - r_{i-1})$. Then arithmetic yields

$$\frac{k-1}{k}\,\delta < \frac{n_i - 1}{n_i}\,\delta \leq \frac{r_i - r_{i-1}}{n_i} < \delta.$$

Divide each interval $[r_{i-1}, r_i]$ into n_i subintervals of equal length; the subintervals obtained in this fashion will be the intervals $[s_{j-1}, s_j]$ of our tagged divisions T and T'. Each of those intervals has length $s_j - s_{j-1}$ between $(k-1)\delta/k$ and δ. Hence, T and T' are δ-fine. Since the partition (s_j) is a refinement of the partition (r_i), we have $\sum_{j=1}^{m} |f(s_j) - f(s_{j-1})| \geq \theta$.

For each j, we now define the tags σ_j and σ_j' by this rule:

$$\begin{array}{lll} & \sigma_j = s_{j-1}, \quad \sigma_j' = s_j & \text{if} \quad f(s_{j-1}) \geq f(s_j); \\ \text{or} & \sigma_j' = s_{j-1}, \quad \sigma_j = s_j & \text{if} \quad f(s_{j-1}) < f(s_j). \end{array}$$

It follows that $f(\sigma_j) - f(\sigma'_j) = |f(s_j) - f(s_{j-1})|$. Hence,

$$\begin{aligned} f(T) - f(T') &= \sum_{j=1}^{m} [f(\sigma_j) - f(\sigma'_j)](s_j - s_{j-1}) \\ &= \sum_{j=1}^{m} |f(s_j) - f(s_{j-1})|(s_j - s_{j-1}) \geq \frac{k-1}{k}\,\delta\theta \end{aligned}$$

as required. □

4. Example With Unbounded Variation

Example 4.1. *Let $r_n = 1 - e^{-n}$. Define $f\colon [0,1] \to \mathbb{R}$ by*

$$f(t) = \begin{cases} 0 & \text{if} \quad t = r_n \qquad (n = 0, 1, 2, 3, \ldots), \\ (-1)^n & \text{if} \quad r_{n-1} < t < r_n \quad (n = 1, 2, 3, \ldots). \end{cases}$$

Then f is Riemann integrable but does not have bounded variation. Moreover, if $\delta \in (0, 1/e)$ and $T \ll \delta$, then

$$\left| f(T) - \textstyle\int f \right| < 8\delta \ln(1/\delta).$$

Proof. Note that $0 = r_0 < r_1 < r_2 < \cdots$ with $\lim_{n\to\infty} r_n = 1$. The variation of f on $[0, r_n]$ is equal to $2n$; the variation of f on $[0, 1]$ is infinite. The function f is Riemann integrable, since it is bounded and has discontinuities in a set of measure 0.

Now suppose that $\delta \in (0, 1)$, and $T = \{(\sigma_i, [s_{i-1}, s_i])\}_{i=1}^{m}$ is a δ-fine tagged division of $[0, 1]$. We shall estimate $|f(T) - \int f|$.

Let n be the integer part of $1 + \ln(1/\delta)$. Then n is a positive integer, so $0 < r_n < 1$. Arithmetic yields $n > \ln(1/\delta)$, hence, $e^{-n} < \delta$. Also, since $\delta < 1/e$, we have $1 < \ln(1/\delta)$.

Choose the largest value of k that satisfies $s_k < r_n$; then $s_{k+1} \geq r_n$. Since T is δ-fine, we have $r_n - s_k < \delta$. Now compute

$$\begin{aligned} |f(T) - \textstyle\int f| &= \left| \sum_{i=1}^{m} f(\sigma_i)(s_i - s_{i-1}) - \int_0^1 f(s)ds \right| \\ &\leq \left| \sum_{i=1}^{k} f(\sigma_i)(s_i - s_{i-1}) - \int_0^{s_k} f(s)ds \right| \\ &\qquad + \left| \sum_{i=k+1}^{m} f(\sigma_i)(s_i - s_{i-1}) - \int_{s_k}^1 f(s)ds \right|. \end{aligned}$$

For $\sum_{i=1}^{k}$ we shall apply Theorem 2.2. For $\sum_{i=k+1}^{m}$ we shall use the fact that $|f(t)| \leq 1$ for all t. Thus we obtain

$$\begin{aligned} |f(T) - \textstyle\int f| &\leq \delta \operatorname{Var}(f; [0, s_k]) + 2(1 - s_k) \\ &\leq \delta \operatorname{Var}(f; [0, r_n]) + 2(\delta + 1 + -r_n) \\ &= 2\delta n + 2(\delta + e^{-n}) \\ &\leq 2\delta\Big(1 + \ln(1/\delta)\Big) + 2(\delta + \delta) \\ &< 8\delta \ln(1/\delta). \end{aligned}$$

□

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