SUCCESS RUNS IN SYMMETRIC BERNOULLI PROCESS

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ABSTRACT. It has been found that a particular sequence in a symmetric Bernoulli process can be squarely linked to a sequence of Fibonacci numbers.

1. SEQUENCE OF FIBONACCI NUMBERS: ITS HISTORY

Fibonacci numbers are a sequence of numbers, named after Leonardo da Pisa, better known as Fibonacci (son of Bonaccio), who introduced Fibonacci numbers in his 1202 book, *Liber Abaci*. Long before Fibonacci, ancient Indian mathematicians described this sequence of numbers, that was well-known in India [2, 4, 7].

Consider the study of Prosody in classical *Sanskrit* literature. In *Sanskrit* as well as *Prakrit* (and also in all other Indian languages having these languages as their origin) vowels are of two kinds - long (L) and short(S). Virahanka's and other Prosidicists' analysis amounts to computation of number of metres (*mātrās*) of a given overall length that can be composed of these syllables. Denoting a short vowel S by 1, and a long vowel L by 2, the solutions would be:

Patterns of length k	{Types}	No. of patterns
1	{S}	[1]
2	{SS and L}	[2]
3	{SSS, SL; LS}	[3]
4	{SSSS, SSL, SLS; LSS, LL}	[5]
5	{SSSSS, SSSL, SSLS, SLSS, SLL; LSSS, LSL, LLS }	[8]
6	{SSSSSS, SSSSL, SSSLS, SSLSS, SSLL SLSSS, SLSL, SLLS; LSSSS, LSSL, LSLS, LLSS, LLL}	[13]

and so on. The patterns of length n arising out of those of length $(n-1)$ and those of length $(n-2)$ are differentiated by the semi-colon (;). It can be seen that a pattern of length n can be formed by prefixing S to a pattern of length $(n-1)$ and prefixing L to a pattern of length $(n-2)$. Thus, the nth Fibonacci number is

$$F_n = F_{n-1} + F_{n-2}\,,\ n \geq 2,\ \ F_0 = 0,\ F_1 = 1. \tag{1.1}$$

The Fibonacci sequence has been observed in many real life situations, e.g., the bee ancestry code, phyllotaxi, and so on.

Apart from the fact that the sequence appears in many mathematical studies, i.e., number theory, occurrence of sequences of patterns in the Bernoulli process has been of interest for quite a long time (e.g. [1, 6]). Probabilists often meet Fibonacci (e.g. [3, 5] and so on). Here is another instance.

2. Success Runs of Length Two

Here we consider another Fibonacci distribution related to patterns of a symmetric Bernoulli process.

Let us consider a pattern of length 2, HH, (two consecutive Heads *for the first time*). Let X be the r.v. (random variable) representing the number of throws of getting this pattern for the **first time**; the experiment ends when this happens (success run of length 2). As can be seen, the *r.v.* X takes values n with probabilities:

$$P\{X = n\} \ = \ \{F_{n-1}/2^n\}\ \ ,\ \ n = 2, 3, 4, \dots . \tag{2.1}$$

This can easily be verified from the fact that the number of trials n required for pattern (HH to occur for the first time) can be obtained by adding the number of trials required $(n-1)$ with those of the number of trials required $(n-2)$ (as the case of the *Sanskrit* prosody example considered above).

The p.g.f. of X is given by

$$\begin{aligned}
P(s) &= \sum_{n=2}^{\infty} F_{n-1\cdot}\,(s/2)^n = (s/2)\left[\sum_{n=2} F_{n-1\cdot}\,(s/2)^{n-1}\right] \\
&= (s/2)\left[(s/2) + \sum_{n=3}^{\infty} F_{n-1\cdot}\,(s/2)^{n-1}\right] \\
&= (s/2)\left[(s/2) + \sum_{n=3} \{F_{n-2} + F_{n-3}\}\,(s/2)^{n-1}\right]
\end{aligned}$$

$$= (s/2)^2 + (s/2)\left[\sum_{m=2}^{\infty} F_{m-1\cdot}(s/2)^m + (s/2)\sum_{m=2}^{\infty} F_{m-1\cdot}(s/2)^m\right]$$
$$= (s/2)^2 + (s/2)\left[P(s) + (s/2)P(s)\right]$$

hence,

$$P(s) = \left(\frac{s}{2}\right)^2 \Big/ \left[1 - \frac{s}{2} - \left(\frac{s}{2}\right)^2\right]. \tag{2.2}$$

3. Success Runs of Higher Orders: Bernoulli - Fibonacci Distribution

Define

$$\begin{aligned} F_{n,\,r} &= F_{n-1,\,r} + F_{n-2,\,r} + \cdots + F_{n-r,r}, && n > r \geq 3 && (3.1)\\ &= F_n, && r \geq n \geq 1,\ r \geq 3 && (3.2)\\ &= F_n && r < n,\ r = 2 && (3.3)\\ &= F_1 && r = 1. && (3.4)\end{aligned}$$

For $r = 2$,

$$P\{X = n\} = F_{n-1,\,2}/2^n = F_{n-1}/2^n \tag{3.5}$$

as considered in Section 2 above.

By constructing tables (as in [6]) it can be seen that, for a success run of length $r (\geq 3)$, the probability distribution of the r.v. X (recurrence time) can be expressed as

$$\begin{aligned} P\{X = n\} &= F_{n-r+1,\,r}/2^n \\ &= [F_{n-r,\,r} + F_{n-(r-1),\,r} + \cdots + F_{n-1,\,r}]/2^n. \end{aligned} \tag{3.6}$$

It may be noted that the number of arrangements of the two events at the nth trial can be linked to those of the preceding r trials.

The p.g.f. (for a success run of length $r \geq 3$) is given by

$$\begin{aligned}
P(s) &= \sum_{n=r}^{\infty} P\{X=n\} s^n \\
&= \sum_{n=r}^{\infty} F_{n-r+1,\,r}, \quad (s/2)^n \\
&= (s/2)^{r-1} \sum_{n=r}^{\infty} F_{n-r+1,r} \ . \ (s/2)^{n-r+1} \\
&= (s/2)^{r-1} \left[F_{1,\,r}(s/2) + \sum_{n=r+1}^{\infty} F_{n-r+1,\,r} \ . \ (s/2)^{n-r+1} \right] \\
&= (s/2)^{r-1} \left[(s/2) + \sum_{m=r}^{\infty} F_{m+1-r+1,\,r} \ . \ (s/2)^{m+1-r+1} \right]. \qquad (3.7)
\end{aligned}$$

Expanding $F_{m-r+2,\,r}$ into r-term sum (using 3.1), it can be seen that (as in case of $r = 2$, in Section 2 above for positive integral r)

$$P(s) = (s/2)^r / \left[1 - s/2 \left\{1 + s/2 + \cdots + (s/2)^{r-1}\right\}\right]. \qquad (3.8)$$

This is the elegant result of Feller (in the symmetric case): it can be viewed through the prism of the Fibonacci sequence.

The distribution may be termed the *Bernoulli Fibonacci distribution*.

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