HERMITIAN K-THEORY OF MONOID RINGS AND THE RING OF INTEGERS IN A FINITE EXTENSION OF $\mathbb{Q}_2$

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Abstract. In this work, we give two applications of Karoubi's fundamental theorem of hermitian K-theory. We prove some isomorphisms in L-theory of monoid rings and the ring of integers in a finite extension of $\mathbb{Q}_2$.

1. Introduction

Let A be a regular ring and M be a commutative cancellative torsion free and c-divisible monoid for some $c > 1$. In [1], Joseph Gubeladze showed that for $i \in \mathbb{N}$, if

$$K_i(A) \simeq K_i(A[M]),$$

then

$$K_i(A) \simeq K_i(A[M]/AI)$$

for all proper radical ideals I of M. He also showed that if in addition, there exists an integer p such that $\mathbb{Z}_+^p \subset M \subset \mathbb{Q}_+^p$, then

$$K_i(A) \simeq K_i(A[M])$$

for all $i \in \mathbb{Z}$.

Here, $\mathbb{Z}_+$ will denote the additive monoid of nonnegative integers and $\mathbb{Q}_+$ that of nonnegative rationals. In the first part of this work, we suppose that F is a commutative field of characteristic different from 2 provided with the trivial involution, M is a c-divisible monoid for some natural $c > 1$ such that $\mathbb{Z}_+^p \subset M \subset \mathbb{Q}_+^p$ ($p \in \mathbb{N}$), and J is a proper ideal of M such that $F.J$ is maximal in $F[M]$, and the field $F[M]/F.J$ has a characteristic different from 2. We prove that if

$${}_1L_0(F) \simeq {}_1L_0(F[M]/F.J) \quad \text{and} \quad {}_1L_1(F) \simeq {}_1L_1(F[M]/F.J),$$

then

$${}_\varepsilon L_n(F) \simeq {}_\varepsilon L_n(F[M]/F.J)$$

for all $n \in \mathbb{N}$ and $\varepsilon = \pm 1$.

In [5], Panin proved that for the ring of integers D in a local field $K/\mathbb{Q}_2$ with maximal ideal M, the natural homomorphism

$$K_i(D;\mathbb{Z}/2^n) \longrightarrow \varprojlim K_i(D/M^j;\mathbb{Z}/2^n)$$

is an isomorphism for all positive i and n. In the second part of this work, we prove that if there exists an integer $r \in \mathbb{N}$ such that the groups

$${}_\varepsilon L_r(D/M^j;\mathbb{Z}/2^n) \text{ and } {}_\varepsilon L_{r+1}(D/M^j;\mathbb{Z}/2^n)$$

are finite,

$${}_\varepsilon L_r(D;\mathbb{Z}/2^n) \simeq \varprojlim {}_\varepsilon L_r(D/M^j;\mathbb{Z}/2^n)$$
$$\text{and } {}_\varepsilon L_{r+1}(D;\mathbb{Z}/2^n) \simeq \varprojlim {}_\varepsilon L_{r+1}(D/M^j;\mathbb{Z}/2^n),$$

then

$${}_\varepsilon L_q(D;\mathbb{Z}/2^n) \simeq \varprojlim {}_\varepsilon L_q(D/M^j;\mathbb{Z}/2^n)$$

for all $q \geq r$.

2. Review of Known Facts

2.1. Here we recall some results obtained by using the algebraic suspension SA of a ring A.

Definition 2.1. *The cone of A, called CA, is the set of infinite matrices such that in each row and each column, we have a finite number of non-zero elements in A. Clearly, CA is a ring by matrix multiplication. We define the suspension SA of A as the quotient of CA by the two-sided ideal of finite matrices (i.e. whose entries are 0, except for a finite number). This definition may be iterated and $S^n(A)$ will denote the nth suspension of A.*

Remark 2.2. *If A is a hermitian ring, we endow SA with the following involution*

$$\overline{M} = {}^t\,\overline{M}.$$

Theorem 2.3. [6] *Let A be a unitary ring. We have a natural homotopy equivalence*

$$\Omega BGL(SA)^+ \sim K_0(A) \times BGL(A)^+.$$

The group $K_0(A)$ is endowed with the discrete topology. In particular, for every $n \geq 1$, we have

$$K_n(SA) \simeq K_{n-1}(A).$$

Theorem 2.4. [2] *Let A be a hermitian ring. We have a natural homotopy equivalence*

$$\Omega B_\varepsilon O(SA)^+ \sim {}_\varepsilon L_0(A) \times B_\varepsilon O(A)^+.$$

The group ${}_\varepsilon L_0(A)$ is endowed with the discrete topology. In particular, for every $n \geq 1$, we have

$${}_\varepsilon L_n(SA) \simeq {}_\varepsilon L_{n-1}(A).$$

These theorems are used to define groups K_n and ${}_\varepsilon L_n$ for all $n < 0$. For a unitary ring (resp. hermitian ring) A and $n < 0$, we set

$$K_n(A) = K_0(S^{-n}A) \quad (\text{resp. } {}_\varepsilon L_n(A) = {}_\varepsilon L_0(S^{-n}A)).$$

2.2. Let A be a hermitian ring. The hyperbolic functor [2] induces a group homomorphism

$$K_0(A) \longrightarrow_\varepsilon L_0(A)$$

and the homomorphism

$$GL_n(A) \longrightarrow_\varepsilon O_{n,n}(A)$$

defined by the following correspondence

$$M \longrightarrow \begin{pmatrix} M & 0 \\ 0 & {}^t\overline{M^{-1}} \end{pmatrix}$$

induces a map

$$BGL(A)^+ \longrightarrow_\varepsilon O(A)^+.$$

We denote ${}_\varepsilon\mathcal{U}(A)$ the homotopic fiber of the map

$$K_0(A) \times BGL(A)^+ \longrightarrow_\varepsilon L_0(A) \times B_\varepsilon O(A)^+.$$

Similarly, the forgetful functor [2] induces a group homomorphism

$${}_\varepsilon L_0(A) \longrightarrow K_0(A)$$

and the natural inclusions

$${}_\varepsilon O_{n,n}(A) \longrightarrow GL_{2n}(A)$$

induce a map

$$B_\varepsilon O(A)^+ \longrightarrow BGL(A)^+.$$

We denote ${}_\varepsilon\mathcal{V}(A)$ the homotopic fiber of the map

$${}_\varepsilon L_0(A) \times B_\varepsilon O(A)^+ \longrightarrow K_0(A) \times BGL(A)^+.$$

Theorem 2.5. [3] *Let A be a hermitian ring containing in its center an element λ, such that $\lambda + \overline{\lambda} = 1$. Then there exists a natural homotopy equivalence between $\Omega_\varepsilon\mathcal{U}(A)$ and ${}_{-\varepsilon}\mathcal{V}(A)$.*

For $n \geq 0$, we let

$${}_\varepsilon U_n(A) = \pi_n({}_\varepsilon\mathcal{U}(A)) \quad \text{and} \quad {}_\varepsilon V_n(A) = \pi_n({}_\varepsilon\mathcal{V}(A)).$$

For $n < 0$ we let

$${}_\varepsilon U_n(A) = {}_\varepsilon U_0(S^{-n}A) \quad \text{and} \quad {}_\varepsilon V_n(A) = {}_\varepsilon V_0(S^{-n}A).$$

For every $n \in \mathbb{Z}$, we have

$${}_\varepsilon U_{n+1}(A) = {}_{-\varepsilon} V_n(A).$$

We also have the following long exact sequences.

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow {}_{\varepsilon}V_n(A) \longrightarrow {}_{\varepsilon}L_n(A) \longrightarrow K_n(A) \longrightarrow {}_{\varepsilon}V_{n-1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow {}_{\varepsilon}L_{n+1}(A) \longrightarrow {}_{\varepsilon}U_n(A) \longrightarrow K_n(A) \longrightarrow {}_{\varepsilon}L_n(A) \longrightarrow {}_{\varepsilon}U_{n-1}(A) \longrightarrow \cdots$$

3. Hermitian K-theory of Monoid Rings

Definition 3.1. *Let M be a commutative monoid. M is called cancellative if, for all a, b and $c \in M$, $ab = ac$ implies that $b = c$.*

Definition 3.2. *A monoid M is called c-divisible for some $c \in \mathbb{N}$ if, for any $x \in M$, there exists $y \in M$ for which $cy = x$.*

Remark 3.3. *If A is a hermitian ring. We endow $A[M]$ with the following involution:*

$$\overline{\sum_{i=0}^{n} a_i m_i} = \sum_{i=0}^{n} \overline{a_i} m_i.$$

Later, $\mathbb{Z}_+$ will denote the additive monoid of nonnegative integers and $\mathbb{Q}_+$ that of nonnegative rationals.

Theorem 3.4. [1] *Let A be a unitary ring and M be a commutative cancellative torsion free and c-divisible monoid for some $c > 1$. The equality*

$$K_i(A) \simeq K_i(A[M])$$

implies

$$K_i(A) \simeq K_i(A[M]/AI)$$

where $i \in \mathbb{N}$ and I is an arbitrary proper radical ideal of M.

Theorem 3.5. [1] *Let A be a regular ring, $p \in \mathbb{N}$, and $c > 1$ a natural number. Then for an intermediate c-divisible monoid $\mathbb{Z}_+^p \subset M \subset \mathbb{Q}_+^p$, we have the natural isomorphisms*

$$K_i(A) \simeq K_i(A[M])$$

where $i \in \mathbb{Z}$.

Theorem 3.6. *Let A be a hermitian regular ring containing in its center an element λ, such that $\lambda + \overline{\lambda} = 1$, $c > 1$ a natural number, and M an intermediate c-divisible monoid $\mathbb{Z}_+^p \subset M \subset \mathbb{Q}_+^p$. Let $r \in \mathbb{Z}$, and suppose that*

$${}_{\varepsilon}L_r(A) \simeq {}_{\varepsilon}L_r(A[M]) \quad \text{and} \quad {}_{\varepsilon}L_{r+1}(A) \simeq {}_{\varepsilon}L_{r+1}(A[M]).$$

Then

$${}_{\varepsilon}L_n(A) \simeq {}_{\varepsilon}L_n(A[M])$$

for all $n \geq r$.

Proof. For all $n \in \mathbb{Z}$ the homomorphism

$$A \longrightarrow A[M]$$

induces the following diagrams of long exact sequences.

$$\begin{CD}
\cdots @>>> K_{n+1}(A) @>>> {}_\varepsilon V_n(A) @>>> {}_\varepsilon L_n(A) @>>> K_n(A) @>>> {}_\varepsilon V_{n-1}(A) @>>> \cdots \\
@. @VVV @VVV @VVV @VVV @VVV \\
\cdots @>>> K_{n+1}(A[M]) @>>> {}_\varepsilon V_n(A[M]) @>>> {}_\varepsilon L_n(A[M]) @>>> K_n(A[M]) @>>> {}_\varepsilon V_{n-1}(A[M]) @>>> \cdots
\end{CD}$$

$$\begin{CD}
\cdots @>>> {}_\varepsilon L_{n+1}(A) @>>> {}_\varepsilon U_n(A) @>>> K_n(A) @>>> {}_\varepsilon L_n(A) @>>> {}_\varepsilon U_{n-1}(A) @>>> \cdots \\
@. @VVV @VVV @VVV @VVV @VVV \\
\cdots @>>> {}_\varepsilon L_{n+1}(A[M]) @>>> {}_\varepsilon U_n(A[M]) @>>> K_n(A[M]) @>>> {}_\varepsilon L_n(A[M]) @>>> {}_\varepsilon U_{n-1}(A[M]) @>>> \cdots
\end{CD}$$

Consider the following diagram of exact sequences.

$$\begin{CD}
{}_\varepsilon L_{r+1}(A) @>>> K_{r+1}(A) @>>> {}_\varepsilon V_r(A) @>>> {}_\varepsilon L_r(A) @>>> K_r(A) \\
@V{\wr}VV @V{\wr}VV @VVV @V{\wr}VV @V{\wr}VV \\
{}_\varepsilon L_{r+1}(A[M]) @>>> K_{r+1}(A[M]) @>>> {}_\varepsilon V_r(A[M]) @>>> {}_\varepsilon L_r(A[M]) @>>> K_r(A[M])
\end{CD}$$

We deduce that for any ε,

$${}_\varepsilon V_r(A) \simeq {}_\varepsilon V_r(A[M])$$

Then we have

$${}_\varepsilon U_{r+1}(A) \simeq {}_\varepsilon U_{r+1}(A[M]).$$

We proceed now by induction on n. Assume that

$${}_\varepsilon L_n(A) \simeq {}_\varepsilon L_n(A[M]) \quad \text{and} \quad {}_\varepsilon U_n(A) \simeq {}_\varepsilon U_n(A[M]).$$

The diagram of exact sequences

$$\begin{CD}
K_{n+1}(A) @>>> {}_\varepsilon L_{n+1}(A) @>>> {}_\varepsilon U_n(A) @>>> K_n(A) \\
@V{\wr}VV @VVV @V{\wr}VV @V{\wr}VV \\
K_{n+1}(A[M]) @>>> {}_\varepsilon L_{n+1}(A[M]) @>>> {}_\varepsilon U_n(A[M]) @>>> K_n(A[M])
\end{CD}$$

prove that the homomorphism

$${}_\varepsilon L_{n+1}(A) \longrightarrow {}_\varepsilon L_{n+1}(A[M])$$

is surjective. Consider the following diagram.

$$\begin{CD}
{}_\varepsilon L_{n+1}(A) @>>> K_{n+1}(A) @>>> {}_\varepsilon V_n(A) @>>> {}_\varepsilon L_n(A) @>>> K_n(A) \\
@VVV @V{\wr}VV @VVV @V{\wr}VV @V{\wr}VV \\
{}_\varepsilon L_{n+1}(A[M]) @>>> K_{n+1}(A[M]) @>>> {}_\varepsilon V_n(A[M]) @>>> {}_\varepsilon L_n(A[M]) @>>> K_n(A[M])
\end{CD}$$

We deduce that for any ε

$$_{\varepsilon}V_n(A) \simeq {_{\varepsilon}V_n(A[M])}.$$

Consequently, we have

$$_{\varepsilon}U_{n+1}(A) \simeq {_{\varepsilon}U_{n+1}(A[M])}.$$

Finally, consider the diagram of exact sequences.

$$\begin{array}{ccccccccc}
{_{\varepsilon}U_{n+1}(A)} & \longrightarrow & K_{n+1}(A) & \longrightarrow & {_{\varepsilon}L_{n+1}(A)} & \longrightarrow & {_{\varepsilon}U_n(A)} & \longrightarrow & K_n(A) \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \downarrow \wr \\
{_{\varepsilon}U_{n+1}(A[M])} & \longrightarrow & K_{n+1}(A[M]) & \longrightarrow & {_{\varepsilon}L_{n+1}(A[M])} & \longrightarrow & {_{\varepsilon}U_n(A[M])} & \longrightarrow & K_n(A[M])
\end{array}$$

It follows that

$$_{\varepsilon}L_{n+1}(A) \simeq {_{\varepsilon}L_{n+1}(A[M])}.$$

The theorem follows. □

Theorem 3.7. *Let F be a field of characteristic different from 2 provided with the trivial involution, M an intermediate c-divisible monoid $\mathbb{Z}_+^p \subset M \subset \mathbb{Q}_+^p$, and J a proper ideal of M such that $F.J$ is maximal in $F[M]$, and the field $F[M]$ $F.J$ has a characteristic different from 2. If*

$$_1L_0(F) \simeq {_1L_0(F[M]/F.J)} \quad \textit{and} \quad {_1L_1(F)} \simeq {_1L_1(F[M]/F.J)},$$

then

$$_{\varepsilon}L_n(F) \simeq {_{\varepsilon}L_n(F[M]/F.J)}$$

for all $n \geq 0$ and $\varepsilon = \pm 1$.

Proof. Since

$$K_n(F) \simeq K_n(F[M])$$

we have, according to Theorem 3.4, an isomorphism

$$K_n(F) \simeq K_n(F[M]/F.J)$$

for all $n \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned}
{_{-1}L_0(F)} &\simeq \mathbb{Z} \simeq {_{-1}L_0(F[M]/F.J)} \quad \text{(see [2], p. 6)} \\
{_{-1}L_1(F)} &= {_{-1}L_1(F[M]/F.J)} = 0 \quad \text{(see [2], p. 96)}
\end{aligned}$$

Then we prove the result by proceeding as in Theorem 3.6. □

4. Hermitian K-theory with Coefficients of the Ring of Integers in a Finite Extension of $\mathbb{Q}_2$

Let X be a topological space. For $n \geq 2$, $\pi_n(X;\mathbb{Z}/l)$ will denote the nth homotopy group of X with coefficients in $\mathbb{Z}/l$.

Definition 4.1. *Let A be a unitary (resp. hermitian) ring. For all $n \geq 2$, we let*

$$K_n(A;\mathbb{Z}/l) = \pi_n(BGL(A)^+;\mathbb{Z}/l) \ \ (resp.\ {}_\varepsilon L_n(A;\mathbb{Z}/l) = \pi_n(B_\varepsilon O(A)^+;\mathbb{Z}/l)).$$

For $n < 2$, we let

$$K_n(A;\mathbb{Z}/l) = K_2(S^{2-n}A;\mathbb{Z}/l) \ \ \ (resp.\ {}_\varepsilon L_n(A;\mathbb{Z}/l) ={}_\varepsilon L_2(S^{2-n}A;\mathbb{Z}/l)).$$

Definition 4.2. *Let A be a hermitian ring. For $n \geq 2$, we let*

$${}_\varepsilon U_n(A;\mathbb{Z}/l) = \pi_n({}_\varepsilon\mathcal{U}(A);\mathbb{Z}/l) \ \ \ and \ \ {}_\varepsilon V_n(A;\mathbb{Z}/l) = \pi_n({}_\varepsilon\mathcal{V}(A);\mathbb{Z}/l).$$

For $n < 2$, we let

$${}_\varepsilon U_n(A;\mathbb{Z}/l) ={}_\varepsilon U_2(S^{2-n}A;\mathbb{Z}/l) \ \ \ and \ \ {}_\varepsilon V_n(A;\mathbb{Z}/l) ={}_\varepsilon V_2(S^{2-n}A;\mathbb{Z}/l).$$

Note that for all $n \in \mathbb{Z}$, we have

$${}_\varepsilon U_n(SA;\mathbb{Z}/l) \simeq{}_\varepsilon U_{n-1}(A;\mathbb{Z}/l), \ \ {}_\varepsilon V_n(SA;\mathbb{Z}/l) \simeq{}_\varepsilon V_{n-1}(A;\mathbb{Z}/l)$$

and

$${}_\varepsilon U_{n+1}(A;\mathbb{Z}/l) \simeq{}_{-\varepsilon} V_n(A;\mathbb{Z}/l).$$

We also have the following long exact sequences.

$$\cdots \longrightarrow K_{n+1}(A;\mathbb{Z}/l) \longrightarrow {}_\varepsilon V_n(A;\mathbb{Z}/l) \longrightarrow {}_\varepsilon L_n(A;\mathbb{Z}/l) \longrightarrow K_n(A;\mathbb{Z}/l) \longrightarrow {}_\varepsilon V_{n-1}(A;\mathbb{Z}/l) \longrightarrow \cdots$$

$$\cdots \longrightarrow {}_\varepsilon L_{n+1}(A;\mathbb{Z}/l) \longrightarrow {}_\varepsilon U_n(A;\mathbb{Z}/l) \longrightarrow K_n(A;\mathbb{Z}/l) \longrightarrow {}_\varepsilon L_n(A;\mathbb{Z}/l) \longrightarrow{}_\varepsilon U_{n-1}(A;\mathbb{Z}/l) \longrightarrow \cdots$$

Theorem 4.3. [5] *Let p be a prime integer and $K/\mathbb{Q}_p$ a finite field extension. Let $D \subset K$ be the ring of integers in K, and M its maximal ideal. Then for all $r \geq 0$ and $n \geq 1$, we have the following statements.*

1. *The group $K_r(D/M^j;\mathbb{Z}/p^n)$ is finite for all $j \in \mathbb{N}$.*
2. *The natural homomorphism*

$$K_r(D;\mathbb{Z}/p^n) \longrightarrow \varprojlim K_r(D/M^j;\mathbb{Z}/p^n)$$

is an isomorphism.

Theorem 4.4. *Let $K/\mathbb{Q}_2$ be a finite field extension with involution. Let $D \subset K$ be the ring of integers in K containing in its center an element λ, such that $\lambda + \overline{\lambda} = 1$, and M its maximal ideal such that D and M are invariant by the involution. Suppose that there exists $r \in \mathbb{N}$ such that the groups ${}_\varepsilon L_r(D/M^j;\mathbb{Z}/2^n)$ and ${}_\varepsilon L_{r+1}(D/M^j;\mathbb{Z}/2^n)$ are finite, $\varepsilon = \pm 1$, and $j \in \mathbb{N}$. Then for all $q \geq r$, the groups ${}_\varepsilon U_q(D/M^j;\mathbb{Z}/2^n)$, ${}_\varepsilon V_q(D/M^j;\mathbb{Z}/2^n)$ and ${}_\varepsilon L_q(D/M^j;\mathbb{Z}/2^n)$ are finite.*

Proof. Consider the following exact sequence.

$${}_\varepsilon L_{r+1}(D/M^j;\mathbb{Z}/2^n) \xrightarrow{\alpha'_{r+1}} {}_\varepsilon U_r(D/M^j;\mathbb{Z}/2^n) \xrightarrow{\alpha_r} K_r(D/M^j;\mathbb{Z}/2^n).$$

We also have

$$\begin{aligned} |{}_\varepsilon U_r(D/M^j;\mathbb{Z}/2^n)| &= |ker\alpha_r||Im\alpha_r| \\ &= |Im\alpha'_{r+1}||Im\alpha_r|. \end{aligned}$$

Since the groups ${}_\varepsilon L_{r+1}(D/M^j;\mathbb{Z}/2^n)$ and $K_r(D/M^j;\mathbb{Z}/2^n)$ are finite, the group ${}_\varepsilon U_r(D/M^j;\mathbb{Z}/2^n)$ is finite.

We proceed now by induction on q. Assume that ${}_\varepsilon L_q(D/M^j;\mathbb{Z}/2^n)$ is finite. The following exact sequence

$$K_{q+1}(D/M^j;\mathbb{Z}/2^n) \longrightarrow_\varepsilon V_q(D/M^j;\mathbb{Z}/2^n) \longrightarrow_\varepsilon L_q(D/M^j;\mathbb{Z}/2^n)$$

proves that for any ε the group ${}_\varepsilon V_q(D/M^j;\mathbb{Z}/2^n)$ is finite. Hence for any ε, the group ${}_\varepsilon U_{q+1}(D/M^j;\mathbb{Z}/2^n)$ is also finite. Finally, consider the exact sequence

$$K_{q+2}(D/M^j;\mathbb{Z}/2^n) \longrightarrow_\varepsilon L_{q+2}(D/M^j;\mathbb{Z}/2^n) \longrightarrow_\varepsilon U_{q+1}(D/M^j;\mathbb{Z}/2^n).$$

It follows that the group ${}_\varepsilon L_{q+2}(D/M^j;\mathbb{Z}/2^n)$ is finite. The theorem follows. $\square$

Theorem 4.5. *Under the conditions of Theorem 4.4, assume moreover that*

$$\begin{aligned} &{}_\varepsilon L_r(D;\mathbb{Z}/2^n) \simeq \varprojlim {}_\varepsilon L_r(D/M^j;\mathbb{Z}/2^n) \\ \textit{and}\quad &{}_\varepsilon L_{r+1}(D;\mathbb{Z}/2^n) \simeq \varprojlim {}_\varepsilon L_{r+1}(D/M^j;\mathbb{Z}/2^n). \end{aligned}$$

Then

$${}_\varepsilon L_q(D;\mathbb{Z}/2^n) \simeq \varprojlim {}_\varepsilon L_q(D/M^j;\mathbb{Z}/2^n)$$

for all $q \geq r$.

Proof. Since for all $j \in \mathbb{N}$ and $q \geq r$, the groups ${}_\varepsilon U_q(D/M^j;\mathbb{Z}/2^n)$, ${}_\varepsilon V_q(D/M^j;\mathbb{Z}/2^n)$, $K_q(D/M^j;\mathbb{Z}/2^n)$, and ${}_\varepsilon L_q(D/M^j;\mathbb{Z}/2^n)$ are finite, then the following long sequences

$$\cdots \longrightarrow \varprojlim K_{r+1}(D/M^j;\mathbb{Z}/2^n) \longrightarrow \varprojlim {}_\varepsilon V_r(D/M^j;\mathbb{Z}/2^n) \longrightarrow \varprojlim {}_\varepsilon L_r(D/M^j;\mathbb{Z}/2^n) \longrightarrow \varprojlim K_r(D/M^j;\mathbb{Z}/2^n)$$

$$\cdots \longrightarrow \varprojlim {}_{\varepsilon}L_{r+1}(D/M^j;\mathbb{Z}/2^n) \longrightarrow \varprojlim {}_{\varepsilon}U_r(D/M^j;\mathbb{Z}/2^n) \longrightarrow \varprojlim K_r(D/M^j;\mathbb{Z}/2^n) \longrightarrow \varprojlim {}_{\varepsilon}L_r(D/M^j;\mathbb{Z}/2^n).$$

are exact. □

The proof of this theorem is completely analogous to that of Theorem 3.6. We only have to replace $K_n(A)$, ${}_{\varepsilon}L_n(A)$, …, by $K_n(D;\mathbb{Z}/2^n)$, ${}_{\varepsilon}L_n(D;\mathbb{Z}/2^n)$, and $K_n(A[M])$, ${}_{\varepsilon}L_n(A[M])$, …, by $\varprojlim K_n(D/M^j;\mathbb{Z}/2^n)$, and $\varprojlim {}_{\varepsilon}L_n(D/M^j;\mathbb{Z}/2^n)$.

Remark 4.6. *The same statements of Theorem 4.4 and 4.5 are true for odd prime p, but the case $p = 2$ is more interesting. The case p odd is an immediate consequence of the periodicity theorem for odd torsion of the Witt groups [4].*

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