A SYMMETRIC K-STEP METHOD FOR DIRECT INTEGRATION OF SECOND ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The design and implementation analysis of a K-step linear multistep method for direct integration of second order initial value problems of ordinary differential equations without reformulation into first order systems is discussed. The derivation of the method and analysis of its basic properties are adopted from the Taylor series expansion and Dahlquist stability model test methods. The result when examined with step-number $k = 6$ shows that the scheme is symmetric, consistent, zero-stable, and convergent.

1. INTRODUCTION

The general k-step method under consideration is of the form

$$\sum_{j=0}^{k} \alpha_j y_{m+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j}, m = 2 \tag{1.1}$$

where y_{n+j} is the sequence of values for $j = 0(1)k$ and $f_{n+j} \equiv y^n_{n+j}$. Operationally defined by [8] as $\rho(E)y_n = h^m \delta(E) f_n$, $\rho(E)$ and $\delta(E)$ are the first and second characteristic polynomials respectively, equation (1.1) is proposed for the solution of the second order initial value problems of ordinary differential equations of the form

$$y'' = f(x, y, y'), y(a) = \eta_0, y'(a) = \eta_1 \tag{1.2}$$

for $y, y', f \in \mathbb{R}^n$, $x \in [a, b]$. From (1.1) y_{n+1} is the approximate numerical solution obtained at $x_{n+j} f_{n+j} \equiv f(x_{n+j}, y_{n+j})$. It is assumed that 1.2 satisfies the existence and uniqueness theorem.

Differential equations are an important device used by scientists, engineers and business managers to extract information about systems they

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wish to analyze. Most differential equation problems encountered in science and engineering such as control theory, celestial mechanics, electrical network, transonic, air flow, radio-active process, and transverse motion to mention a few do not either admit a closed form solution or the analytical solution is too involved to be useful. In case of non-linear differential equations, a closed form solution is often ruled out. For these reasons, most problems in science and technology are approached numerically. In the literature, problems of the form (1.2) are reformulated as first order systems and solved conventionally. In [16], an attempt was made to solve 1.2 directly for step-number $k = 3$ when $m = 2$. In this paper, Problem 1.2 will be solved directly for any even $k \geq 6$ with $m = 2$. Different researchers have worked on the linear multistep method using different approaches (see [14, 15, 8, 19, 3, 4, 12, 13, 2, 6, 20, 9, 10, 11, 18, 17]). These approaches were adopted as a pilot guide in this paper.

2. THE ADOPTED METHOD WITH THE NEW SIX-STEP FORMULA

Consideration is given to linear multistep methods of the form

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}, k = 6 \tag{2.1}$$

subject to conditions $\alpha_j = \alpha_{k-j}$ and $\beta_j = \beta_{k-j}$ for $j = O(1)\frac{k}{2}$, α_j's and β_j's are real constants, and α_0 and β_0 are not both equal zero. The values of the coefficients are determined by the local truncation error.

Definition 1. *The truncation error is the quantity T which must be added to the true value representation of the computed quantity in order that the result be exactly equal to the quantity we are seeking to generate.*

Suppose

$$\mathbf{y}(true\ representation) + \mathbf{T} = \mathbf{y}(exact).$$

From the definition above using equation (2.1) the local truncation error becomes

$$T_{n+k} = y_{n+k} - \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j f_{n+j}. \tag{2.2}$$

Substituting the results obtained from the adoption of the Taylor series expansion of y_{n+k}, y_{n+j}, and f_{n+j} about x_n to yield

$$\begin{aligned}
T_{n+k} &= y_n + (kh)y_n^1 + \frac{(kh)^2}{2!}y_n^2 + \cdots + \frac{(kh)^p}{p!}y_n^p + \frac{(kh)^{p+1}}{(p+1)!}y_n^{p+1} + O(h^{p+2}) \\
&\quad - \sum_{j=0}^{k-1}\alpha_j\Big[y_n(jh)y_n^1 + \frac{(jh)^2}{2!}y_n^2 + \cdots + \frac{(jh)^p}{p!}y_n^p + \frac{(jh)^{p+1}}{(p+1)!}y_n^{p+1} \\
&\quad + O(h^{p+2})\Big] - h^2\sum_{j=0}^{k}\beta_j\Big[y_n^2 + (jh)y_n^3 + \frac{(jh)^2}{2!}y_n^4 + \cdots + \frac{(jh)^{p-2}}{(p-2)!}y_n^p \\
&\quad + \frac{(jh)^{p-1}}{(p-1)!}y_n^{p+1} + O(h^p)\Big]. \qquad (2.3)
\end{aligned}$$

Collecting terms in equal powers of h

$$\begin{aligned}
T_{n+k} =&\Big(1 - \sum_{j=0}^{j-1}\alpha_j\Big)y_n + \Big(k - \sum_{j=0}^{k-1}j\alpha_j\Big)hy_n^1 + \Big(\frac{k^2}{2!} - \sum_{j=0}^{k-1}\frac{(j)^2}{2!}\alpha_j \\
&- \sum_{j=0}^{k}\beta_j\Big)h^2y_n^2 + \Big(\frac{k^3}{3!} - \sum_{j=0}^{k-1}\frac{(j)^3}{3!}\alpha_j - \sum_{j=0}^{k}j\beta_j\Big)h^3y_n^3 + \cdots \\
&+ \Big(\frac{k^p}{p!} - \sum_{j=0}^{k-1}\frac{(j)^p}{p!}\alpha_j - \sum_{j=0}^{k}\frac{(j)^{p-2}}{(p-2)!}\beta_j\Big)h^py_n^p \\
&+ \Big(\frac{k^{p+1}}{(p+1)!} - \sum_{j=0}^{k-1}\frac{(j)^{p+1}}{(p+1)!}\alpha_j - \sum_{j=0}^{k}\frac{(j)^{p-1}}{(p-1)!}\beta_j\Big)h^{p+1}y_n^{p+1} \\
&+ O(h^{p+2}). \qquad (2.4)
\end{aligned}$$

Six-step Method.

Setting $k = 6$ in Equation (2.1) yields

$$\begin{aligned}
y_{n+6} &= \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + \alpha_5 y_{n+5} \qquad (2.5) \\
&+ h^2\{\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4} + \beta_5 f_{n+5} + \beta_6 f_{n+6}\}
\end{aligned}$$

with local truncation error.

$$
\begin{aligned}
T_{n+6} &= (1-\alpha_0-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)y_n + (6-\alpha_1-2\alpha_2-3\alpha_3\\
&- 4\alpha_4-5\alpha_5)hy_n^1 + \Big(\frac{36}{2}-\frac{1}{2!}(\alpha_1+4\alpha_2+9\alpha_3+16\alpha_4+25\alpha_5)\\
&- (\beta_0+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6)\Big)h^2y_n^2 + \Big(\frac{216}{6}-\frac{1}{3!}(\alpha_1\\
&+ 8\alpha_2+27\alpha_3+64\alpha_4+125\alpha_5)-(\beta_1+2\beta_2+3\beta_3+4\beta_4\\
&+ 5\beta_5+6\beta_6)\Big)h^3y_n^3 + \Big(\frac{1296}{24}-\frac{1}{4!}(\alpha_1+16\alpha_2+81\alpha_3+256\alpha_4\\
&+ 625\alpha_5)-\frac{1}{2!}(\beta_1+4\beta_2+9\beta_3+16\beta_4+25\beta_5+36\beta_6)\Big)h^4y_n^4\\
&+ \Big(\frac{7776}{120}-\frac{1}{5!}(\alpha_1+32\alpha_2+243\alpha_3+1024\alpha_4+3125\alpha_5)\\
&- \frac{1}{3!}(\beta_1+8\beta_2+27\beta_3+64\beta_4+125\beta_5+216\beta_6)\Big)h^5y_n^5\\
&+ \Big(\frac{46656}{720}-\frac{1}{6!}(\alpha_1+64\alpha_2+729\alpha_3+4096\alpha_4+15625\alpha_5)\\
&- \frac{1}{4!}(\beta_1+16\beta_2+81\beta_3+256\beta_4+625\beta_5+1296\beta_6)\Big)h^6y_n^6 \qquad (2.6)\\
&+ \Big(\frac{279936}{5040}-\frac{1}{7!}(\alpha_1+128\alpha_2+2187\alpha_3+16384\alpha_4+78125\alpha_5)\\
&- \frac{1}{5!}(\beta_1+32\beta_2+243\beta_3+1024\beta_4+3125\beta_5+7776\beta_6\Big)h^7y_n^7\\
&+ \Big(\frac{1679616}{40320}-\frac{1}{8!}(\alpha_1+256\alpha_2+6561\alpha_3+65536\alpha_4+390625\alpha_5)\\
&- \frac{1}{6!}(\beta_1+64\beta_2+72\beta_3+4096\beta_4+15625\beta_5+46656\beta_6\Big)h^8y_n^8\\
&+ \Big(\frac{10077696}{362880}-\frac{1}{9!}(\alpha_1+512\alpha_2+19683\alpha_3+262144\alpha_4+1953125\alpha_5)\\
&- \frac{1}{7!}(\beta_1+128\beta_2+2187\beta_3+16384\beta_4+78125\beta_5+279936\beta_6\Big)h^9y_n^9\\
&+ \Big(\frac{60466176}{3528800}-\frac{1}{10!}(\alpha_1+1024\alpha_2+59049\alpha_3+1048576\alpha_4\\
&+ 9765625\alpha_5)-\frac{1}{8!}(\beta_1+256\beta_2+6561\beta_3+65536\beta_4+390625\beta_5\\
&+ 1679616\beta_6\Big)h^{10}y_n^{10} + \Big(\frac{362797056}{39916800}-\frac{1}{11!}(\alpha_1+2048\alpha_2+177147\alpha_3\\
&+ 4914304\alpha_4+48828125\alpha_5)-\frac{1}{9!}(\beta_1+512\beta_2+19683\beta_3+262144\beta_4
\end{aligned}
$$

$$+ 1953125\beta_5 + 10077696\beta_6\Big)h^{11}y_n^{11} + \Big(\frac{21767782336}{479001600} - \frac{1}{12!}(\alpha_1$$
$$+ 4096\alpha_2 + 531441\alpha_3 + 16777216\alpha_4 + 244140625\alpha_5) - \frac{1}{10!}(\beta_1$$
$$+ 1024\beta_2 + 59049\beta_3 + 1048576\beta_4 + 9765625\beta_5 + 60466176\beta_6\Big)h^{12}y_n^{12}$$
$$+ O(h^{12}). \tag{2.7}$$

Definition 2. *A linear multistep method (2.1) and the local truncation error T_{n+k} is said to be of order p if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = 0$, but $C_{p+2} \neq 0$.*

Imposing accuracy of order P on truncation error T_{n+6}, we have

$$C_i = 0, i = O(1)p + 1,$$

and equation (2.7) now becomes

$$\begin{aligned}
&\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1,\\
&\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 = 6,\\
&\frac{1}{2!}(\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4 + 25\alpha_5) + (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5\\
&\quad + \beta_6) = \frac{36}{2}\\
&\frac{1}{3!}(\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 + 125\alpha_5) + (\beta_1 + 2\beta_2\\
&\quad + 3\beta_3 + 4\beta_4 + 5\beta_5 + 6\beta_6) = \frac{216}{6},\\
&\vdots\\
&\frac{1}{12!}(\alpha_1 + 4096\alpha_2 + 531441\alpha_3 + 16777216\alpha_4 + 244140625\alpha_5) + \frac{1}{10!}(\beta_1\\
&\quad + 1024\beta_2 + 59049\beta_3 + 104857\beta_4 + 9765625\beta_5 + 60466176\beta_6)\\
&= \frac{2176782336}{479001600}.
\end{aligned} \tag{2.8}$$

In [16, 17], it was shown that the coefficients of first characteristic polynomial $\rho(r)$ always agree with the rule of Pascal's triangle. Consequently, on adoption of Pascal's triangle, the values of α's in (2.8) are found to be

$$\alpha_0 = -1, \alpha_1 = 6, \alpha_2 = -15, \alpha_3 = 20, \alpha_4 = -15, \alpha_5 = 6 \tag{2.9}$$

which, by substitution, reduces (2.8) to

$$\begin{aligned}
\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 &= 0, \\
\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5 + 6\beta_6 &= 0, \\
\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4 + 25\beta_5 + 36\beta_6 &= 0, \\
\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4 + 125\beta_5 + 216\beta_6 &= 0, \\
\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4 + 625\beta_5 + 1296\beta_6 &= 24, \\
\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4 + 3125\beta_5 + 7776\beta_6 &= 360, \text{ and} \\
\beta_1 + 64\beta_2 + 729\beta_3 + 4096\beta_4 + 15625\beta_5 + 46656\beta_6 &= 3420. \qquad (2.10)
\end{aligned}$$

It follows from the Gaussian process that

$$\beta_0 = \frac{1}{12}, \beta_1 = \frac{6}{12}, \beta_2 = -\frac{33}{12}, \beta_3 = \frac{52}{13}, \beta_4 = -\frac{33}{12}, \beta_5 = \frac{6}{12}, \beta_6 = \frac{1}{12}. \qquad (2.11)$$

Using (2.11) in (2.5) yields a symmetric scheme

$$\begin{aligned}
y_{n+6} &= 6y_{n+5} + 15y_{n+4} - 20y_{n+3} + 15y_{n+2} + 6y_{n+1} - y_n \\
&\quad + \frac{h^2}{12}\{f_{n+6} + 6f_{n+5} - 33f_{n+4} + 52f_{n+3} - 33f_{n+2} + 6f_{n+1} - f_n\}. \qquad (2.12)
\end{aligned}$$

3. Predictors

In order to use equation (2.12), it is necessary to know the previous values y_{n+j} of y and f at x_{n+j}, $j = O(1)6$, $h > 0$. Applying the Taylor series expansion technique about x_n for y_{n+k} is applied.

$$\begin{aligned}
y_{n+k} =& \{y(x_n + kh) = y(x_n) + (kh)y^1(x_n) + \frac{(hk)^2}{2!}y^2(x_n) \\
&+ \frac{(kh)^3}{3!}y^3(x_n) + \cdots + \frac{(jk)^p}{p!}y^p(x_n) + O(h^{p+1}) \qquad (3.1)
\end{aligned}$$

and its first derivative

$$\begin{aligned}
y'_{n+k} =& \{y'(x_n + kh) = y^1(x_n) + (kh)y^2(x_n) + \frac{(hk)^2}{2!}y^3(x_n) \\
&+ \frac{(kh)^3}{3!}y^4(x_n) + \cdots + \frac{(jk)^p}{p!}y^{p+1}(x_n) + O(h^{p+2}). \qquad (3.2)
\end{aligned}$$

The procedure for development of predictors is the same as for the main method. The discrete schemes and their required first derivatives are determined by Taylor's method. Note that $f_{n+k} = f(x_{n+1}, y_{n+k}, y'_{k+n})$, $k = O(1)6$ also $y''_n = f_n$ (see [16, 17] and [3, 4] for details).

4. Basic Property of Equation (2.12)

In order to ascertain the accuracy and suitability of equation (2.12), it is important to carry out the analysis of its basic properties such as the order of accuracy and error constant, symmetry, consistency, convergence, zero stability, and region of absolute stability.

4.1. **Order of accuracy and error constant.** Substituting the results (2.9) and (2.11) into (2.8) yields

$$
\begin{aligned}
C_0 &= -1+6-15+20-15+6-1=0\\
C_1 &= 6-30+60-60+30-6=0\\
C_2 &= \frac{1}{2}(6-60+180-240+150-36)+\frac{1}{12}(1+6-33+52-33\\
&\quad +6+1)=0\\
&\ \vdots\\
C_9 &= \frac{1}{362880}(6-7680+393660-3932160+11718750-10077696)\\
&\quad +\frac{1}{60480}(6-4224+113724-540672+468750+279936)=0\\
C_{10} &= \frac{1}{3628800}(6-15360+1180980+58593750-60466176)\\
&\quad +\frac{1}{483840}(6-8444+341142-2162688+2343750-1679616)\\
&= -\frac{1}{240}.
\end{aligned}
\tag{4.1.1}
$$

Thus, $C_0 = C_1 = \cdots = C_9 = 0$, but $C_{10} = C_{p+2} \neq 0$, hence equation (2.12) is of order 8, since $p+2 = 10$ with truncation error $C_{p+2} = -\frac{1}{240}$.

4.2. **Symmetry.** Formula (2.12) is symmetry if $\alpha_j = \alpha_{k-j}$ and $\beta_j = \beta_{k-j}$ for $j = O(1)\frac{k}{2}$ and even k, that is

$$
\begin{array}{ll}
\alpha_0=\alpha_6=\ \ 1 & \beta_0=\beta_6=\ \ 1\\
\alpha_1=\alpha_5=-6 & \beta_1=\beta_5=\ \ 6\\
\alpha_2=\alpha_4=\ \ 15 & \beta_2=\beta_4=-33\\
\alpha_3=\alpha_3=-20 & \beta_3=\beta_3=\ \ 52\\
\alpha_4=\alpha_2=\ \ 15 & \beta_4=\beta_2=-33\\
\alpha_5=\alpha_1=-6 & \beta_5=\beta_1=\ \ 6\\
\alpha_6=\alpha_0=\ \ 1 & \beta_6=\beta_0=\ \ 1
\end{array}
$$

hence, Formula (2.12) is symmetric.

4.3. **Consistency.** Method (2.12) is consistent because it satisfies the following:

(i) it has order $p \geq 1$ since Formula (2.12) is order 8,

(ii) $\sum_{j}^{k} \alpha_j = O(1)6$ (see (2.9)),

(iii) $\rho(r) = \rho'(r) = 0$, $r = 1$,

(iv) $\rho''(r) = 2!\delta(r)$, $r = 0$ (see [17, 13] for details).

4.4. **Zero stability.** The stability polynomial of (2.12) is given by

$$\begin{aligned}\pi(r,h) &= r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1 \\ &\quad - \frac{h}{12}(r^6 + 6r^5 - 33r^4 + 52r^3 - 33r^2 + 6r + 1) = 0; \qquad (4.4.1)\end{aligned}$$

$\rho(r)$ in Section 4.3 is to have zero stability as demonstrated by

$$\begin{aligned}\rho(r) = r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1 &= 0 \\ (r-1)(r^5 - 5r^4 + 10r^3 - 10r^2 - 5r!) &= 0 \\ (r-1)^6 &= 0.\end{aligned}$$

Thus, equation (2.12) is zero stable since none of the roots of $\rho(r)$ has modulus greater than one $(|r|, 1)$.

4.5. **Convergence.** Since equation (2.12) has been shown to be consistent and zero stable, then it is convergent [1, 8, 13].

4.6. **Region of absolute stability.** To determine the region of absolute stability of Method (2.12) as discussed in [12] and [8], let

$$h(r) = \frac{\rho(r)}{\delta(r)}, \qquad (4.6.1)$$

where $\rho(r)$, $\delta(r)$ remain the same as defined earlier. Adopting the values of $\rho(r)$ and $\delta(r)$ as in Section 4.3, equation (4.6.1) becomes

$$h(r) = \frac{12(r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1)}{r^6 + 6r^5 - 33r^4 + 52r^3 - 33r^2 + 6r + 1}. \qquad (4.6.2)$$

Replacing r by $e^{i\theta}$, $0 \leq \theta \leq \pi$, (4.6.2) becomes

$$h(\theta) = \frac{P(\theta)}{Q(\theta)}, \qquad (4.6.3)$$

where $P(\theta) = 12(\cos 6\theta - 6\cos 5\theta + 15\cos 4\theta - 20\cos 3\theta + 15\cos 2\theta - 6\cos\theta + 1) + 12i(\sin 6\theta - 6\sin 5\theta + 15\sin 4\theta - 20\sin 3\theta + 15\sin 2\theta - 6\sin\theta)$ and $Q(\theta) =$

$(\cos 6\theta + 6\cos 5\theta - 33\cos 4\theta + 52\cos 3\theta - 33\cos 2\theta + 6\cos\theta + 1) + i(\sin 6\theta + 6\sin 5\theta - 33\sin 4\theta + 52\sin 3\theta - 33\sin 2\theta + 6\sin\theta)$.
Rationalizing and simplifying (4.6.3),

$$h(\theta) = x(\theta) + iy(\theta),$$

where $x(\theta) = \frac{12(-2\cos 5\theta - 4\cos 4\theta + 102\cos 3\theta - 432\cos 2\theta + 924\cos\theta - 588)}{-2\cos 5\theta - 28\cos 4\theta + 6\cos 3\theta + 624\cos 2\theta - 2052\cos\theta + 1452}$ and $y(\theta) = 0$.
Evaluation of $x(\theta)$, $0 \leq \theta \leq 180^\circ$ at intervals of 30° yields these results. Thus, the interval of absolute stability of (2.12) is $(-6, \infty)$.

θ	0	30	60	90	120	150	180
$x(\theta)$	∞	-0.274	-1.091	-2.400	-4.000	-5.417	-6.000

5. Numerical Experiments

In order to ascertain the applicability and accuracy of method 2.12, two numerical examples, linear and nonlinear for special and general case problems of second order initial value problems are solved to demonstrate the accuracy of the new method. The accuracy of the new method is determined via the size of the discretization error estimates ℓ_n obtained by subtracting the approximate solution from the corresponding exact solution of the problems. The results are compared with [16, 17] for $h = \frac{1}{40}$ and [5] for $h = \frac{1}{320}$.

Problem 1. $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$.
Exact solution: $y(x) = \sin x$.
Problem 2. $y'' = x(y')^2$, $y(0) = 1$, $y'(0) = \frac{1}{2}$.
Exact solution: $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$.

n	x_n	Owolobi & Ademiluyi (2010) error	Owolobi (2011) error	New method error
6	0.150	0.000311837	0.000000641	0.000000536
8	0.200	0.000872254	0.000002652	0.000001237
11	0.275	0.002564847	0.000012845	0.000000477
13	0.325	0.004438400	0.000029504	0.000009269
16	0.400	0.008664817	0.000083148	0.000054330
17	0.425	0.010509789	0.000112593	0.000084281
20	0.500	0.017553657	0.000431670	0.000251442

TABLE 1. Numerical sol. to Prob. 1 with step-size $h = \frac{1}{40}$

n	x_n	Owolobi & Ademiluyi (2010) error	Owolobi (2011) error	New method error
6	0.150	0.000137210	0.000001907	0.000000192
8	0.200	0.000220060	0.000006437	0.000000451
11	0.275	0.000653028	0.000033855	0.000007224
13	0.325	0.001138210	0.000047207	0.000016427
16	0.400	0.002250195	0.000113964	0.000098560
17	0.425	0.002742171	0.000148177	0.000051409
20	0.500	0.003938079	0.000241637	0.000012189

TABLE 2. Numerical sol. to Prob. 2 with step-size $h = \frac{1}{40}$

x_n	Exact solution	Computed with new method	Absolute error (method in [5])	Absolute error (New method)
0.1	1.050041729278491	1.050041695187980	5.891000e-06	3.409051e-008
0.2	1.100335347731076	1.100334606520453	8.239900e-05	7.412106e-007
0.3	1.151140435936467	1.15113589185372	3.464210e-04	4.544082e-006
0.4	1.202732554054082	1.202706513180778	7.521010e-04	2.604087e-005
0.5	1.255412811882995	1.255283475630027	1.380283e-03	1.293363e-004

TABLE 3. Comparison of errors arising from Method ([5]) and the new method for Problem 2 with $h = \frac{1}{320}$

6. Conclusion

General higher order k-step method for direct integration of second order initial value problems of ordinary differential equations has been proposed as an alternative to [16] and [17]. Two test problems on both general and special cases of nonlinear and linear second order ODE's were solved earlier by Owolabi and Ademiluyi [16] and by Owolabi [17]. The new method, whose results as shown in the tables above, is significantly better in accuracy than the results in [16, 17, 5] (see Table 3 for (2.1)). It was also revealed from the analysis of the basic properties that the new method is symmetric, consistent, convergent, zero stable with interval of absolute stability $(-6, \infty)$.

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