THE EQUIVALENCE NUMBER OF A LINE GRAPH

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ABSTRACT. The chromatic index of a graph G is most often defined to be the minimum size of a partition of the edge set of G into matchings. An equivalent but different definition is the minimum size of a cover of the edge set of G by matchings. We consider the analogous problem of covering the edge set of G by subgraphs that are vertex-disjoint unions of cliques, known as equivalence graphs. The minimum size of such a cover is the equivalence number of G. We compute the equivalence number of the line graph of a clique on at most 12 vertices. We also construct a particular type of cover to show that, for all graphs G on at most n vertices, the equivalence number of the line graph of G has an upper bound on the order of log n. Finally, we show that if G is a clique on 13 vertices then the minimum size of this particular cover is 5.

1. INTRODUCTION

This work is based on part of my doctoral dissertation [12]. Furthermore, an unpublished version of this work [13] is cited by [8] which further develops the concepts we introduce here and improves upon the results. The equivalence number of a graph was first introduced by Duchet in [7] and further developed by Alon in [2]. A distinct and related parameter called the Prague dimension, or product dimension, of a graph was introduced earlier by Nešetřil and Pultr in [14] and further explored by Lovász, Nešetřil, and Pultr in [11] and Furedi in [9]. The equivalence number of a graph is the Prague dimension of the complement of the graph.

Throughout this article, G will denote a finite graph with no loops or multiple edges. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively, and we assume the vertex and edge sets to be disjoint. We denote the degree of a vertex x in G by $d_G(x)$. Given $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S. We denote the clique, or complete graph, on n vertices by K_n. The *line graph* of G, denoted by $L(G)$, is a graph whose vertex set is $E(G)$, and for which two vertices are adjacent if and only if they are adjacent as edges in G. We refer the reader to [6] for all other definitions of graph theoretic terms.
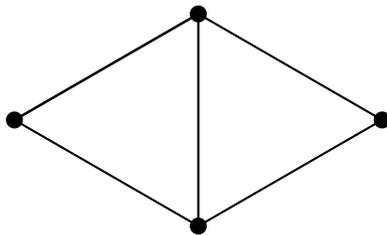
FIGURE 1. $eq(G) = 2$.

For a given graph G, an *equivalence graph* on the vertices of G is a subgraph whose connected components are cliques. Given a positive integer k, a proper *edge k-covering* of a graph G *by equivalence graphs* is a collection of equivalence graphs $G_1, G_2, \ldots, G_k$ whose union is G. The *equivalence number* of G, denoted by $eq(G)$, is the smallest positive integer k such that there exists a proper edge k-covering of G by equivalence graphs. By convention, if G is empty then $eq(G) = 0$. Whereas the *chromatic index* of a graph G is the minimum size of a cover of the edge set of G by matchings, the equivalence number of G is the minimum size of a cover of the edge set of G by equivalence graphs. A matching is simply an equivalence graph in which every clique has order at most two. Thus, if G is triangle-free then it is easy to see that the equivalence number of G is equal to the chromatic index of G.

At this point it may be important to emphasize again that we define the equivalence number using covers rather than partitions. This distinguishes the equivalence number from other parameters like the subchromatic number discussed in [1]. A *subcoloring* of G is a coloring of the vertices of G so that each color class induces an equivalence graph. The *subchromatic number* of G is the minimum number of colors required by any subcoloring. An analogous coloring of the edges yields a definition similar to that of the equivalence number, but using partitions rather than covers. For example, the graph in Figure 1 can be covered by two triangles and therefore has equivalence number 2. If we instead require that the equivalence graphs be edge disjoint, the graph would require an edge partition into three cliques (one triangle and two edges).

2. MAIN RESULTS

There is a natural lower bound for $eq(G)$. Let $d_G^K(x)$ be the minimum number of clique subgraphs of G needed to cover the edges of G incident

with x, and let

$$\Delta^K(G) = \max_{x \in V(G)} d_G^K(x).$$

If G is triangle free, then $d_G^K(x) = d_G(x)$ and $\Delta^K(G) = \Delta(G)$. Because any covering of G by equivalence graphs must in particular cover the edges of G incident with x, then $\Delta^K(G)$ is clearly a lower bound for $eq(G)$. It was conjectured in [4] that $eq(G) \leq \Delta^K(G) + 1$. We prove that $L(K_5)$ is a counterexample to this conjecture. Line graphs are worth examining because for every graph G, $\Delta^K(L(G)) \leq 2$. This follows from the fact that for every vertex x in G, the edges incident with x are pairwise adjacent vertices in $L(G)$. Because every edge e of G is incident with exactly two vertices of G, then every vertex of $L(G)$ that is adjacent to e belongs to one of two cliques, and so the edges of $L(G)$ that are incident with e are covered by two cliques in $L(G)$. The conjecture therefore implies that, for every graph G, $eq(L(G)) \leq 3$. Our first theorem demonstrates otherwise.

Theorem 2.1. $eq(L(K_5)) = 4$.

We next investigate the equivalence number for line graphs of higher order cliques by considering special covers whose equivalence graphs are generated greedily. Using these special covers, we define a function $c(n)$ such that $eq(L(K_n)) \leq c(n)$ for all integers n. By bounding $c(n)$ from above, we are able to bound $eq(L(K_n))$ from above, and we prove the following results.

Theorem 2.2. *If G is a graph on $n \leq 12$ vertices then $eq(L(G)) \leq 4$.*

Theorem 2.3. *If G is a graph on $n > 0$ vertices then $eq(L(G)) \leq 4 \left\lceil \dfrac{\ln n}{\ln 12} \right\rceil$.*

3. Optimal Coverings of $L(K_5)$

We first remark on the structure of the line graph of K_n. If $n \leq 2$ then K_n has at most one edge. It follows that $L(K_n)$ has at most one vertex and no edges, and so the equivalence number is 0. K_3 is a triangle, and so is $L(K_3)$. It follows that the equivalence number of $L(K_3)$ is 1. $L(K_4)$ is an octahedron, and the facial cycles (triangles) of three pairs of opposing faces cover the edges. It follows that the equivalence number of $L(K_4)$ is at most 3. It is easy to see that an equivalence graph on $L(K_4)$ has at most six edges, and that the only such maximum equivalence graphs are the aforementioned pairs of triangles. It follows that $L(K_4) = 2$. For every positive integer $n > 4$, $L(K_n)$ is the union of n cliques of order $n - 1$ that pairwise intersect in a single vertex. Each of these cliques is simply the subgraph of $L(G)$ induced by the edges incident with one of the n vertices in G. We now prove Theorem 2.1, i.e., $eq(L(K_5)) = 4$.
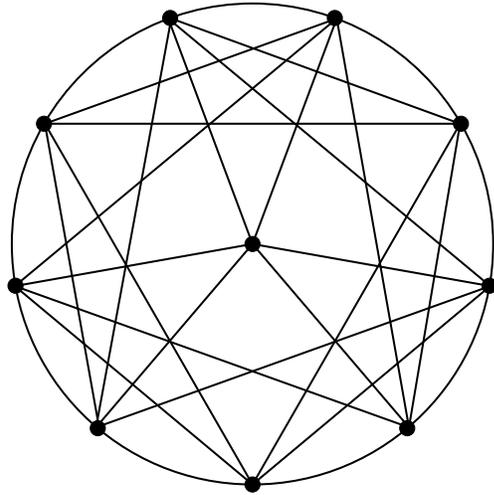
FIGURE 2. $L(K_5)$

Proof. The graph $L(K_5)$ has 10 vertices and 30 edges and is pictured in Figure 2. We first show that $eq(L(K_5)) > 3$. Suppose that $L(K_5)$ is the union of three graphs H_1, H_2, and H_3 such that the connected components of each of the graphs are cliques. The maximal order of such a clique is 4. First suppose that none of H_1, H_2, and H_3 have a clique component of order 4. Then the maximum size of a clique component is 3, and since $L(K_5)$ has 10 vertices, each subgraph H_i, $1 \leq i \leq 3$, has at most nine edges. Therefore, the union of the subgraphs has at most 27 edges and cannot possibly cover $L(K_5)$ which has 30 edges. So we may assume that at least one of the subgraphs H_i has a clique component of order 4. For any such subgraph, every other subgraph of $L(K_5)$ that is a 4-clique must intersect the 4-clique component of H_i at some vertex. Therefore, each H_i can have at most one clique component of order 4.

Next suppose that exactly one of the graphs, say H_1, has a 4-clique component. Then H_1 has at most 12 edges, and H_2 and H_3 each have at most 9 edges. Let x be a vertex in $L(K_5)$ that is covered by the 4-clique component of H_1, and notice that $d_{L(K_5)}(x) = 6$. Since the degree of any vertex in a 4-clique is 3, x is incident with 3 edges of $L(K_5)$ that are not covered by H_1. If these three edges are covered by triangles from H_2 and H_3, then $d_{H_1}(x) + d_{H_2}(x) + d_{H_3}(x) = 7$, and so one edge is covered twice. Therefore, the union of H_1, H_2, and H_3 has at most $(12+9+9)-1 = 29 < 30$ edges and cannot possibly cover $L(K_5)$. If these three edges are covered by
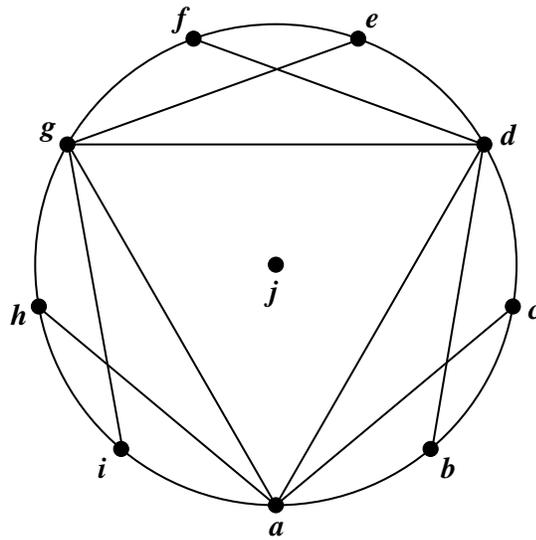
FIGURE 3. Three 4-cliques of $L(K_5)$

a triangle from one of H_1 and H_2, and a 2-clique from the other, then at least one of H_1 and H_2 has at most 8 edges. Therefore, the union of H_1, H_2, and H_3 has at most $12 + 9 + 8 = 29 < 30$ edges and cannot possibly cover $L(K_5)$.

Next suppose that exactly two of the graphs, say H_1 and H_2, has a 4-clique component. Let $V(L(K_5)) = \{a, b, c, d, e, f, g, h, i, j\}$, let the vertex set of the 4-clique component of H_1 be $\{b, e, h, j\}$, and let the vertex set of the 4-clique component of H_2 be $\{c, f, i, j\}$. In Figure 3 we have deleted from $L(K_5)$ the edges of these two 4-cliques and labeled the vertices to make the following argument more clear to the reader. We will also refer to edges by the vertex pairs. For example, the edge joining vertices a and b will be called ab.

Because H_1 has a 4-clique component on $\{b, e, h, j\}$ and H_2 has a 4-clique component on $\{c, f, i, j\}$, the edges bc, ef, and hi must all be covered by H_3. We now show that $de \notin E(H_3)$. Suppose $de \in E(H_3)$. Then because $ef \in E(H_3)$, the vertices d, e, and f are in the same clique component of H_3, and $df \in E(H_3)$. Since H_3 has no 4-clique components, none of the edges da, db, dc, or dg may be in $E(H_3)$. Vertex b is in a 4-clique in H_1, and so $db \notin E(H_1)$. Vertex c is in a 4-clique in H_1, and so $dc \notin E(H_2)$. Therefore, $db \in E(H_2)$ and $dc \in E(H_1)$. Then because g is not adjacent

to either b or c, the edge dg is not in $E(H_2)$ or $E(H_1)$. Since dg is not in $E(H_1)$, $E(H_2)$, or $E(H_3)$, the subgraphs H_1, H_2, and H_3 do not cover $L(K_5)$. Therefore, $de \notin E(H_3)$. By symmetry, none of the edges ab, cd, de, fg, gh, and ia are in $E(H_3)$. Any of these edges that are incident with a vertex in $\{b, e, h, j\}$ must not be in $E(H_1)$. Likewise, any of these edges that are incident with a vertex in $\{c, f, i, j\}$ must not be in $E(H_2)$. Therefore, the edges ab, de, and gh must be in $E(H_2)$, and the edges cd, fg, and ia must be in $E(H_1)$.

The edge df is not in $E(H_3)$ because otherwise, together with $ef \in E(H_3)$, that would imply $de \in E(H_3)$. By symmetry, none of the edges ac, bd, df, eg, gi, and ha are in $E(H_3)$. Any of these edges that are incident with a vertex in $\{b, e, h, j\}$ must not be in $E(H_1)$. Likewise, any of these edges that are incident with a vertex in $\{c, f, i, j\}$ must not be in $E(H_2)$. Therefore, the edges bd, eg, and ha must be in $E(H_2)$, and the edges ac, df, and gi must be in $E(H_1)$. Now we have that edges de, eg, and gh are all in $E(H_2)$, which implies that vertices d, e, g, and h are in the same clique component of H_2. This is a contradiction for two reasons: H_2 cannot have another 4-clique, and vertices d and h are not even adjacent in $L(K_5)$.

So now we prove the final case in which all three subgraphs H_1, H_2, and H_3 have a 4-clique component. We again refer to Figure 3, only this time we assume that the three 4-cliques that are pictured are actually the 4-cliques of H_1, H_2, and H_3, on vertex sets $\{a, b, c, d\}$, $\{d, e, f, g\}$, and $\{g, h, i, a\}$, respectively. Recall again that the 4-cliques in $L(K_5)$ that are missing from Figure 3 are on vertex sets $\{b, e, h, j\}$ and $\{c, f, i, j\}$.

Edges eh and fi must be in $E(H_1)$ because they each have endvertices in the 4-cliques of both H_2 and H_3. Likewise, edges bh and ci must be in $E(H_2)$ because they each have endvertices in the 4-cliques of both H_1 and H_3. Finally, edges be and cf must be in $E(H_3)$ because they each have endvertices in the 4-cliques of both H_1 and H_2. The only edges not yet accounted for are those incident with j. Notice that vertex j is not in the 4-clique of any H_i. Therefore, the six edges incident with j must be covered by three edge disjoint triangles, one from each H_i. Based upon how the edges have been allocated to covering subgraphs so far, there are only two such triangles that can possibly be in H_2: the triangle on $\{b, h, j\}$ or the triangle on $\{c, i, j\}$. Let us assume that the triangle on $\{b, h, j\}$ is in H_2; the other case is similar. Since hj has already been allocated to the triangle in H_2, the only possibility for the triangle from H_1 is on the vertex set $\{f, i, j\}$. That leaves only the edges cj and ej, but these two edges cannot be in the same triangle because c and e are not adjacent. Therefore, the union of H_1, H_2, and H_3 does not cover $L(K_5)$ in this case either, and we have finished the proof that $eq(L(K_5)) > 3$.

In order to prove that $eq(L(K_5)) \leq 4$, we need only exhibit a set of four equivalence graphs that covers $L(K_5)$. Using the labeling from 3 and $G = L(K_5)$, let

$$G_1 = G[\{a, b, c, d\}] \cup G[\{e, h, j\}],$$
$$G_2 = G[\{d, e, f, g\}] \cup G[\{b, h, j\}],$$
$$G_3 = G[\{g, h, i, a\}] \cup G[\{b, e, j\}],$$
$$G_4 = G[\{c, f, i, j\}] \cup G[\{b, e, h\}].$$

The set $\{G_1, G_2, G_3, G_4\}$ is an edge 4-covering of $L(K_5)$ by equivalence graphs. □

4. Greedy Equivalence Graphs

We first prove an easy lemma to justify our focus on line graphs of cliques.

Lemma 4.1. *For any nonnegative integer n and any graph G on at most n vertices, $eq(L(G)) \leq eq(L(K_n))$.*

Proof. Let G be a graph on at most n vertices. Then G is a subgraph of K_n. So $L(G)$ can be obtained from $L(K_n)$ simply by deleting those vertices which are edges in $E(K_n) \setminus E(G)$. If $\{H_1, H_2, \ldots H_k\}$ is a proper covering of $L(K_n)$ by subgraphs whose connected components are cliques, then after the deletion of vertices, the connected components of these subgraphs are still cliques, just possibly of smaller order. Therefore, the set of subgraphs obtained by deletion of $E(K_n) \setminus E(G)$ is a proper covering of $L(G)$. □

We now define a special type of equivalence graph of vertices in K_n that we call a greedy equivalence graph. Given a positive integer $n \geq 3$ and a set V with $|V| = n$, let S_V be the set of bijections from $[n]$ to V, where $[n] = \{1, 2, \ldots, n\}$. Suppose T is a subset of S_V with the following property: for any three distinct elements $a, b, c \in V$, there exists $\sigma \in T$ such that $\sigma^{-1}(a) < \min\{\sigma^{-1}(b), \sigma^{-1}(c)\}$. We call such a set a *nice* subset of S_V. If V is the vertex set of K_n, and for all $x \in V$, H_x is the $(n-1)$-clique in $L(K_n)$ corresponding to x, then for every $\sigma \in T$ we define the subgraph

$$C_\sigma = \bigcup_{j=1}^{n} \left[H_{\sigma(j)} \setminus \left(\bigcup_{i=1}^{j-1} H_{\sigma(i)} \right) \right].$$

The connected components of every subgraph in $\{C_\sigma : \sigma \in T\}$ are cliques. Moreover, if e and f are any two adjacent vertices in $L(K_n)$ with endvertices $\{a, b\}$ and $\{a, c\}$ in G, respectively, then there exists $\sigma \in T$ such that $\sigma^{-1}(a) < \min\{\sigma^{-1}(b), \sigma^{-1}(c)\}$, and consequently e and f are in a common connected component of C_σ. We call $\{C_\sigma : \sigma \in T\}$ an edge $|T|$-*covering of $L(K_n)$ by greedy equivalence graphs.* The labeling of the

x	1	2	3
$\sigma_1(x)$	1	2	3
$\sigma_2(x)$	2	3	1
$\sigma_3(x)$	3	1	2

x	1	2	3	4
$\tau_1(x)$	1	2	3	4
$\tau_2(x)$	3	2	1	4
$\tau_3(x)$	4	2	3	1

TABLE 1. $c(3) = 3$ and $c(4) = 3$

x	1	2	3	4	5	6	7	8	9	10	11	12
$\sigma_1(x)$	1	5	9	10	6	12	8	7	11	4	3	2
$\sigma_2(x)$	2	6	10	11	7	9	5	8	12	1	4	3
$\sigma_3(x)$	3	7	11	12	8	10	6	5	9	2	1	4
$\sigma_4(x)$	4	8	12	9	5	11	7	6	10	3	2	1

TABLE 2. Definition of $\sigma_i(x)\colon [12] \to [12]$, $i = 1, 2, 3, 4$

vertices is not important. For two sets V and V' of cardinality n, the set $S_{V'}$ can be obtained from S_V simply by composing all of the elements of S_V with a suitable bijection between V and V'. So it is only the cardinality of V that is important, and we define $c(n)$ to be the minimum size of such a cover, i.e. the minimum size of a nice subset of S_V. Likewise, in the proofs that follow, we will feel free to choose any V with appropriate cardinality with the understanding that once the result holds for V it holds for any set of the same cardinality.

We define $c(0) = 0$, $c(1) = 1$ and $c(2) = 2$. It is easy to see that $c(3) = 3$ and $c(4) = 3$. Clearly, $c(3) \geq 3$ and $c(4) \geq 3$. Inequalities in the other direction are established by the functions defined in Table 1.

The following lemma is obvious because every covering by greedy equivalence graphs is a covering by equivalence graphs.

Lemma 4.2. *For any nonnegative integer n, $eq(L(K_n)) \leq c(n)$.*

Combining Lemma 4.2 and Lemma 4.1, we have the following corollary.

Corollary 4.3. *For any graph on at most n vertices, $eq(L(G)) \leq c(n)$.*

We now complete the proof of Theorem 2.2 with the following lemma.

Lemma 4.4. $c(12) \leq 4$.

Proof. Let $\{1, 2, \ldots, 12\}$ be the vertex set of K_{12}, and for each vertex i, let H_i be the corresponding clique of order 11 in $L(K_{12})$. We define four bijections σ_1, σ_2, σ_3, and σ_4 from $\{1, 2, \ldots, 12\}$ to $\{1, 2, \ldots, 12\}$ by Table 2. As before, we define four greedy equivalence graphs C_1, C_2, C_3, and C_4 of $L(K_{12})$ as follows: for $k \in \{1, 2, 3, 4\}$,

a	{b,c}	σ	a	{b,c}	σ	a	{b,c}	σ	a	{b,c}	σ
5	{1,2}	σ_4	6	{2,1}	σ_3	7	{3,1}	σ_4	8	{4,1}	σ_3
5	{1,3}	σ_4	6	{2,3}	σ_1	7	{3,2}	σ_4	8	{4,2}	σ_3
5	{1,4}	σ_2	6	{2,4}	σ_1	7	{3,4}	σ_2	8	{4,3}	σ_2
5	{1,6}	σ_4	6	{2,5}	σ_3	7	{3,5}	σ_2	8	{4,5}	σ_3
5	{1,7}	σ_4	6	{2,7}	σ_1	7	{3,6}	σ_4	8	{4,6}	σ_3
5	{1,8}	σ_2	6	{2,8}	σ_1	7	{3,8}	σ_2	8	{4,7}	σ_1
5	{1,9}	σ_3	6	{2,9}	σ_3	7	{3,9}	σ_2	8	{4,9}	σ_3
5	{1,10}	σ_4	6	{2,10}	σ_4	7	{3,10}	σ_4	8	{4,10}	σ_3
5	{1,11}	σ_4	6	{2,11}	σ_1	7	{3,11}	σ_1	8	{4,11}	σ_1
5	{1,12}	σ_2	6	{2,12}	σ_1	7	{3,12}	σ_2	8	{4,12}	σ_2

TABLE 3. Case $a \in \{5,6,7,8\}$, $a - 4 \in \{b,c\}$

$$C_k = \bigcup_{j=1}^{n} \left[H_{\sigma_k(j)} \setminus \left(\bigcup_{i=1}^{j-1} H_{\sigma_k(i)} \right) \right].$$

We show that these subgraphs cover $L(K_{12})$ by showing that $T = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a nice subset of $S_{[12]}$. In other words, we need to show that given any $a, b, c \in \{1, 2, \ldots, 12\}$, a, b, and c distinct, there exists $k \in \{1, 2, 3, 4\}$ such that $\sigma_k^{-1}(a) < \sigma_k^{-1}(b)$ and $\sigma_k^{-1}(a) < \sigma_k^{-1}(c)$. If $a \in \{1, 2, 3, 4\}$ then σ_a suffices for all choices of b and c because in these four cases $\sigma_a^{-1}(a) = 1$. If $a \in \{5, 6, 7, 8\}$ and $\{b, c\} \cap \{a - 4\} = \emptyset$ then σ_{a-4} suffices because $\sigma_{a-4}^{-1}(a) = 2$ and $\sigma_{a-4}(1) = a - 4$. For the cases when $a \in \{5, 6, 7, 8\}$ and $\{b, c\} \cap \{a - 4\} \neq \emptyset$, we have listed solutions in Table 3. If $a \in \{9, 10, 11, 12\}$ and $\{b, c\} \cap \{a-4, a-8\} = \emptyset$ then σ_{a-8} suffices because $\sigma_{a-8}^{-1}(a) = 3$, $\sigma_{a-8}(1) = a - 8$, and $\sigma_{a-8}(2) = a - 4$. For the cases when $a \in \{9, 10, 11, 12\}$ and $\{b, c\} \cap \{a - 4, a - 8\} \neq \emptyset$, we have listed solutions in Table 4. $\square$

Together, the results in Theorem 2.1, Lemma 4.1, and Lemma 4.4 establish that

$$eq(L(K_n)) = c(n) = 4$$

for $n = 5, 6, \ldots, 12$. We will see that the greedy equivalence graphs cannot be used to establish that $eq(L(K_n)) = 4$ for $n \geq 13$, but first we will establish some upper bounds for $eq(L(K_n))$.

5. A LOGARITHMIC UPPER BOUND

Lemma 5.1. *For every nonnegative integer n, $c(n) \leq c(n+1)$.*

a	{b,c}	σ	a	{b,c}	σ	a	{b,c}	σ	a	{b,c}	σ
9	{1,2}	σ_4	10	{2,1}	σ_3	11	{3,1}	σ_2	12	{4,1}	σ_3
9	{1,3}	σ_4	10	{2,3}	σ_1	11	{3,2}	σ_4	12	{4,2}	σ_3
9	{1,4}	σ_2	10	{2,4}	σ_1	11	{3,4}	σ_2	12	{4,3}	σ_1
9	{1,5}	σ_4	10	{2,5}	σ_3	11	{3,5}	σ_2	12	{4,5}	σ_3
9	{1,6}	σ_4	10	{2,6}	σ_3	11	{3,6}	σ_4	12	{4,6}	σ_3
9	{1,7}	σ_4	10	{2,7}	σ_1	11	{3,7}	σ_2	12	{4,7}	σ_1
9	{1,8}	σ_2	10	{2,8}	σ_1	11	{3,8}	σ_2	12	{4,8}	σ_3
9	{1,10}	σ_4	10	{2,9}	σ_3	11	{3,9}	σ_2	12	{4,9}	σ_3
9	{1,11}	σ_4	10	{2,11}	σ_1	11	{3,10}	σ_4	12	{4,10}	σ_3
9	{1,12}	σ_2	10	{2,12}	σ_1	11	{3,12}	σ_2	12	{4,11}	σ_1
9	{5,2}	σ_4	10	{6,1}	σ_3	11	{7,1}	σ_2	12	{8,1}	σ_3
9	{5,3}	σ_4	10	{6,3}	σ_1	11	{7,2}	σ_4	12	{8,2}	σ_3
9	{5,4}	σ_2	10	{6,4}	σ_1	11	{7,4}	σ_2	12	{8,3}	σ_1
9	{5,6}	σ_4	10	{6,5}	σ_3	11	{7,5}	σ_2	12	{8,5}	σ_3
9	{5,7}	σ_4	10	{6,7}	σ_1	11	{7,6}	σ_4	12	{8,6}	σ_3
9	{5,8}	σ_2	10	{6,8}	σ_1	11	{7,8}	σ_2	12	{8,7}	σ_1
9	{5,10}	σ_4	10	{6,9}	σ_3	11	{7,9}	σ_2	12	{8,9}	σ_3
9	{5,11}	σ_4	10	{6,11}	σ_1	11	{7,10}	σ_4	12	{8,10}	σ_3
9	{5,12}	σ_2	10	{6,12}	σ_1	11	{7,12}	σ_2	12	{8,11}	σ_1

TABLE 4. Case $a \in \{9, 10, 11, 12\}$, $\{a-4\} \cap \{b, c\} \neq \emptyset$

Proof. The result is trivial for $n \in \{0, 1, 2\}$. Given a nice subset of $S_{[n+1]}$, we can find a nice subset of $S_{[n]}$ by simply restricting to $[n]$ the domain of each of the functions in $S_{[n+1]}$. $\square$

Corollary 5.2. *For all nonnegative integers m and n, if $m < n$ then $c(m) \leq c(n)$.*

Lemma 5.3. *For all nonnegative integers m and n, $c(mn) \leq c(m) + c(n)$.*

Proof. We may assume $m \leq n$. By Corollary 5.2, $c(m) \leq c(n)$. Let V be a set such that $|V| = mn$. Let $V_1, V_2, \ldots, V_m$ be a partition of V into m subsets, each of cardinality n. For each t, $1 \leq t \leq m$, let $f_t \colon [n] \to V_t$ be a bijection. Let P be a nice subset of $S_{[m]}$ with elements $\{\rho_1, \rho_2, \ldots, \rho_m\}$. Let R be a nice subset of $S_{[n]}$ with elements $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. For each k, $1 \leq k \leq mn$, we define a matrix M_k as follows: for $1 \leq i \leq m$ and $1 \leq j \leq n$, the (i, j) entry of M_k is

$$M_k(i,j) = \begin{cases} f_{\rho_k(i)}(\sigma_k(j)), & 1 \le k \le c(m), \\ f_i(\sigma_k(j)), & c(m)+1 \le k \le c(n), \\ f_{\rho_{k-c(n)}(i)}(\sigma_{k-c(n)}(n+1-j)), & c(n)+1 \le k \le c(n)+c(m). \end{cases}$$

Since for all t, $1 \le t \le m$, the function f_t is a bijection, and for all s, $1 \le s \le n$, the function σ_s is a bijection, then their composition is a bijection. Therefore, in any row of M_k, the entries are distinct. Since for all r, $1 \le r \le m$, the function ρ_r is a bijection and $V_1, V_2, \ldots, V_m$ is a partition of V, no two rows of M_k may contain common elements. Therefore, each element of V appears exactly once in M_k. For each k, let $\tau_k \colon [mn] \to V$ be the bijection which orders the elements of V according to their lexicographic order in M_k. So if x is the (i,j) entry of M_k and y is the (i',j') entry of M_k, then $\tau_k^{-1}(x) < \tau_k^{-1}(y)$ if and only if either $i < i'$, or $i = i'$ and $j < j'$. We now show that the set $T = \{\tau_k : 1 \le k \le c(n)+c(m)\}$ is a nice subset of S_V. Let $a, b, c \in V$ be distinct. We consider three cases.

First suppose that there exists t, $1 \le t \le m$, such that $a, b, c \in V_t$. Then for all k, $1 \le k \le c(n)+c(m)$, a, b, and c appear in the same row of M_k. Because R is a nice subset of $S_{[n]}$, there exists k, $1 \le k \le c(n)$, such that $\sigma_k^{-1}(f_t^{-1}(a)) < \min\{\sigma_k^{-1}(f_t^{-1}(b)), \sigma_k^{-1}(f_t^{-1}(c))\}$. Therefore, a, b, and c are ordered appropriately within the row, and we conclude that $\tau_k^{-1}(a) < \min\{\tau_k^{-1}(b), \tau_k^{-1}(c)\}$.

Next suppose that a appears in a subset V_{t_1} in the partition $V_1, V_2, \ldots, V_m$ of V, that is distinct from the subsets V_{t_2} and V_{t_3} containing b and c, respectively. Then for all k, $1 \le k \le c(n)+c(m)$, a appears in a row of M_k that is distinct from those containing b and c. Because P is a nice subset of $S_{[m]}$, there exists k, $1 \le k \le c(m)$, such that $\rho_k^{-1}(t_1) < \min\{\rho_k^{-1}(t_2), \rho_k^{-1}(t_3)\}$, and so a appears in a row of M_k higher than those of b and c, and we conclude that $\tau_k^{-1}(a) < \min\{\tau_k^{-1}(b), \tau_k^{-1}(c)\}$.

Finally suppose that a an b are contained in the same subset V_{t_1} that is distinct from V_{t_2} containing c. Just as in the previous case, there exists k, $1 \le k \le c(m)$, such that $\rho_k^{-1}(t_1) < \rho_k^{-1}(t_2)$ and so a and b appear in a row of M_k higher than that of c. If $\sigma_k^{-1}(f_t^{-1}(a)) < \sigma_k^{-1}(f_t^{-1}(b))$, then we conclude that $\tau_k^{-1}(a) < \min\{\tau_k^{-1}(b), \tau_k^{-1}(c)\}$. If instead $\sigma_k^{-1}(f_t^{-1}(a)) > \sigma_k^{-1}(f_t^{-1}(b))$, then $(n+1) - \sigma_k^{-1}(f_t^{-1}(a)) < (n+1) - \sigma_k^{-1}(f_t^{-1}(b))$, and we consider $M_{k+c(n)}$. The ordering of the rows in $M_{k+c(n)}$ is the same as that in M_k, but the columns are in reverse order. So we conclude that $\tau_{k+c(n)}^{-1}(a) < \min\{\tau_{k+c(n)}^{-1}(b), \tau_{k+c(n)}^{-1}(c)\}$.

The case in which a an c are contained in the same subset V_{t_1} is similar, and so we have shown that for any $a, b, c \in V$ there exists $\tau \in T \subseteq S_V$ such

that $\tau^{-1}(a) < \min\{\tau^{-1}(b), \tau_{-1}(c)\}$. Therefore, T is a nice subset of S_V. Since $|T| = c(m) + c(n)$, we have proven that $c(mn) \leq c(m) + c(n)$. □

Lemma 5.4. *For all nonnegative integers k, $c(12^k) \leq 4k$.*

Proof. We prove the statement by induction on k. For $k = 0$, the statement is trivial. For $k = 1$, the statement is simply Theorem 2.2. For $k > 1$, suppose $c(12^{k-1}) \leq 4(k-1)$. Then by Lemma 5.4, $c(12^k) \leq c(12) + c(12^{k-1}) \leq 4 + 4(k-1) = 4k$. □

We now prove Theorem 2.3.

Proof. Let $n = |G|$. For $n \leq 12$,

$$c(n) \leq 4 \leq 4 \left\lceil \frac{\ln n}{\ln 12} \right\rceil.$$

For $n \geq 12$, let

$$k(n) = \left\lceil \frac{\ln n}{\ln 12} \right\rceil$$

and notice that $n \leq 12^{k(n)}$. By Corollary 5.2 and Lemma 5.4,

$$c(n) \leq c(12^{k(n)}) \leq 4k(n) = 4 \left\lceil \frac{\ln n}{\ln 12} \right\rceil.$$

Therefore, by Corollary 4.3,

$$eq(L(G)) \leq c(n) \leq 4 \left\lceil \frac{\ln n}{\ln 12} \right\rceil = 4 \left\lceil \frac{\ln |G|}{\ln 12} \right\rceil.$$

□

It turns out that this bound is not sharp. In particular, the bound is improved by [8] in which they show that

$$eq(L(G)) \leq 2 \log_2 \log_2 \chi(G) + 2.$$

6. Covering $L(K_{13})$ by Greedy Equivalence Graphs

In this section we prove that $c(n)$ is not always bounded above by 4.

Theorem 6.1. $c(13) = 5$.

Proof. By Lemma 4.4 and Lemma 5.3, $c(13) \leq c(12) + c(1) \leq 4 + 1 = 5$. By Corollary 5.2 we know that $c(13) \geq c(12) = 4$. Now we show that $c(13) \neq 4$. Suppose that V is a set of 13 elements and $T = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a nice subset of S_V, i.e., for any three distinct elements $a, b, c \in V$, there exists $\sigma \in T$ satisfying $\sigma^{-1}(a) < \min\{\sigma^{-1}(b), \sigma^{-1}(c)\}$. Let $W = V \setminus \{\sigma_1(1), \sigma_2(1), \sigma_3(1), \sigma_4(1)\}$. We first prove a couple of useful lemmas.

Lemma 6.2. *For all $x, y \in W$, there exist distinct $\sigma, \tau \in T$ such that $\sigma^{-1}(x) < \sigma^{-1}(y)$ and $\tau^{-1}(x) < \tau^{-1}(y)$.*

Proof. Let $x, y \in W$. By the definition of T, we know that there exists at least one $\sigma \in T$ such that $\sigma^{-1}(x) < \sigma^{-1}(y)$. Since $x \neq \sigma(1)$, $\sigma^{-1}(\sigma(1)) < \sigma^{-1}(x) < \sigma^{-1}(y)$. By the definition of T, we know that there exists at least one $\tau \in T$ such that $\tau^{-1}(x) < \min\{\tau^{-1}(y), \tau^{-1}(\sigma(1))\}$. So $\sigma^{-1}(x) < \sigma^{-1}(y)$ and $\tau^{-1}(x) < \tau^{-1}(y)$ and we are done. $\square$

Lemma 6.3. *Let $a, b, c \in W$ and suppose $\sigma \in T$ such that $\sigma^{-1}(a) < \sigma^{-1}(b) < \sigma^{-1}(c)$. Then for all $\tau \in T \setminus \{\sigma\}$, either $\tau^{-1}(a) > \tau^{-1}(b)$ or $\tau^{-1}(b) > \tau^{-1}(c)$.*

Proof. Suppose $\sigma^{-1}(a) < \sigma^{-1}(b) < \sigma^{-1}(c)$ and $\tau^{-1}(a) < \tau^{-1}(b) < \tau^{-1}(c)$. Then by Lemma 6.2, $\pi^{-1}(c) < \pi^{-1}(b)$ for all $\pi \in T \setminus \{\sigma, \tau\}$. Consequently, there does not exist $\rho \in T$ such that $\rho^{-1}(a) < \min\{\rho^{-1}(b), \rho^{-1}(c)\}$, which contradicts the definition of T. $\square$

Recall that $T = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and let $b \in W$. Suppose $b \notin \{\sigma_1(2), \sigma_1(13)\}$. By Lemma 6.2, there exists $\sigma \in T \setminus \{\sigma_1\}$ such that $\sigma^{-1}(\sigma_1(2)) < \sigma^{-1}(b)$. We may assume, without loss of generality, that $\sigma = \sigma_2$. Also by Lemma 6.2, there exists $\tau \in T \setminus \{\sigma_1\}$ such that $\tau^{-1}(b) < \tau^{-1}(\sigma_1(13))$. If $\tau = \sigma_2$ then we have $\sigma_2^{-1}(\sigma_1(2)) < \sigma_2^{-1}(b) < \sigma_2^{-1}(\sigma_1(13))$ and $\sigma_1^{-1}(\sigma_1(2)) < \sigma_1^{-1}(b) < \sigma_1^{-1}(\sigma_1(13))$, which contradicts Lemma 6.3. So $\tau \in T \setminus \{\sigma_1, \sigma_2\}$, and we may assume without loss of generality that $\tau = \sigma_3$.

Define the set $A = \{a \in W : \sigma_1^{-1}(a) < \sigma_1^{-1}(b) \text{ and } \sigma_2^{-1}(b) < \sigma_2^{-1}(a)\}$. For all $a \in A$, $\sigma_3^{-1}(b) < \sigma_3^{-1}(a)$, otherwise for $i = 1, 3$, $\sigma_i^{-1}(a) < \sigma_i^{-1}(b) < \sigma_i^{-1}(\sigma_1(13))$, which contradicts Lemma 6.3. It follows that $\sigma_4^{-1}(a) < \sigma_4^{-1}(b)$ by Lemma 6.2. Define the set $C = \{c \in W : \sigma_1^{-1}(b) < \sigma_1^{-1}(c) \text{ and } \sigma_3^{-1}(c) < \sigma_3^{-1}(b)\}$. For all $c \in C$, $\sigma_2^{-1}(c) < \sigma_2^{-1}(b)$, otherwise for $i = 1, 2$, $\sigma_i^{-1}(\sigma_1(2)) < \sigma_i^{-1}(b) < \sigma_i^{-1}(c)$, which contradicts Lemma 6.3. It follows that $\sigma_4^{-1}(b) < \sigma_4^{-1}(c)$ by Lemma 6.2.

Now we see that for all $a \in A$ and $c \in C$, we have $\sigma_1^{-1}(a) < \sigma_1^{-1}(b) < \sigma_1^{-1}(c)$ and $\sigma_4^{-1}(a) < \sigma_4^{-1}(b) < \sigma_4^{-1}(c)$, which contradicts Lemma 6.3 if A and C are both nonempty. Therefore, we conclude that either $A = \emptyset$ or $C = \emptyset$. If $A = \emptyset$, then $\sigma_2^{-1}(b) > \sigma_2^{-1}(w)$ for all $w \in W \setminus \{b\}$ (because $\sigma_1^{-1}(\sigma_1(2)) < \sigma_1^{-1}(b)$). It follows that $\sigma_2^{-1}(b) = 13$. If $C = \emptyset$, then $\sigma_3^{-1}(b) < \sigma_3^{-1}(w)$ for all $w \in W \setminus \{b\}$ (because $\sigma_1^{-1}(\sigma_1(b)) < \sigma_1^{-1}(13)$). It follows that $\sigma_3^{-1}(b) = 2$. Our choice of $b \in W$ was arbitrary, and we showed that if there exists $\sigma \in T$ such that $\sigma^{-1}(b) \notin \{2, 13\}$, then there exists $\tau \in T$ such that $\sigma^{-1}(b) \in \{2, 13\}$. Therefore, we have shown that

$$W \subseteq \{\sigma_1(2), \sigma_2(2), \sigma_3(2), \sigma_4(2), \sigma_1(13), \sigma_2(13), \sigma_3(13), \sigma_4(13)\}.$$

C. MCCLAIN

So $|W| \leq 8$. Since $W = V \setminus \{\sigma_1(1), \sigma_2(1), \sigma_3(1), \sigma_4(1)\}$, $|V| \leq 12$. This contradicts our assumption that $|V| = 13$ and finishes the proof. $\square$

7. Conclusions and Future Work

As of the original writing of this work, it was still an open question whether $eq(L(G))$ could possibly have a constant upper bound. That question is answered negatively by [8] in which the authors prove that

$$\frac{1}{3} \log_2 \log_2 \chi(G) \leq eq(L(G)),$$

where $\chi(G)$ denotes the (vertex) chromatic number of G. In fact, their work also disproves a conjecture present in the original version of this paper.

Conjecture 7.1. *For any triangle-free graph G, $eq(L(G)) \leq 3$.*

Recall that our interest in line graphs is motivated by the fact that for line graphs the parameter Δ^K is at most 2. We can broaden our inquiry to include all graphs G such that $\Delta^K(G) \leq 2$, and it turns out that this collection of graphs is itself a subset of another interesting class of graphs. If $\Delta^K(G) \leq 2$ then for any vertex v in G and any neighbors a, b, and c of v, at least two of a, b, and c must be in a common clique and therefore adjacent. Graphs with this property are called *claw–free* and their structure has been characterized in [5]. Perhaps the main structure theorems in this series of papers may be used to understand the equivalence number of claw–free graphs, a natural generalization of the subject of this paper.

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