INVARIANT RELATIONS FOR THE DERIVATIVES OF TWO ARBITRARY POLYNOMIALS

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Abstract. Suppose $P(x)$ and $Q(x)$ are two arbitrary polynomials. In this paper we use the theory of resultants of two polynomials including Sylvester's Matrix to specify a large number of polynomial relations involving $P(x)$, $Q(x)$ and their derivatives. We also include a research problem for the reader to consider.

1. Introduction

Consider the quadratic $P(x) = Ax^2 + Bx + C = P''(0)x^2/2 + P'(0)x + P(0)$ whose discriminant is $B^2 - 4AC = (P'(0))^2 - 2P''(0)P(0)$. Surprisingly, for any number b, it is true that $B^2-4AC = (P'(0))^2-2P''(0)P(0) = (P'(b))^2-2P''(b)P(b)$. In this paper we prove the following much more general result, with the special case above, simply using a quadratic $P(x)$ and its derivative $P'(x)$ for the two polynomials $P(x)$, $Q(x)$ in the theorem.

Theorem 1. *Let $P(x)$ and $Q(x)$ be polynomials of degrees n and m, respectively. Let $P^{(i)}(x)$ and $Q^{(j)}(x)$ denote the ith and jth derivatives of $P(x)$ and $Q(x)$. Define the $(m+n)\times(m+n)$ matrix $M(x)$ as follows.*

(1) *Each row i, $1 \le i \le m$, of $M(x)$ is the following.*

$$\overbrace{0, 0, \ldots, 0}^{\leftarrow i-1 \rightarrow}, \frac{P^{(n)}(x)}{n!}, \frac{P^{(n-1)}(x)}{(n-1)!}, \ldots, \frac{P'(x)}{1!}, P(x), \overbrace{0, 0, \ldots, 0}^{\leftarrow m-i \rightarrow}.$$

(2) *Each row $m+i$, $1 \le i \le n$, of $M(x)$ is the following.*

$$\overbrace{0, 0, \ldots, 0}^{\leftarrow i-1 \rightarrow}, \frac{Q^{(m)}(x)}{m!}, \frac{Q^{(m-1)}(x)}{(m-1)!}, \ldots, \frac{Q'(x)}{1!}, Q(x), \overbrace{0, 0, \ldots, 0}^{\leftarrow n-i \rightarrow}.$$

Then the determinant $|M(x)|$ of $M(x)$ has a value that is independent of x.

Thus, we can call $|M(x)|$ an invariant. However, some readers may prefer to call $|M(x)|$ a constant. We will give the reader all of the necessary background material and this will make the paper accessible to almost any undergraduate mathematics student. Also near the end, we give some specific examples which involve derivatives and at the end, we outline a research problem.

2. The Resultant of Two Polynomials

The resultant $\rho\left(P\left(x\right),Q\left(x\right)\right)$ of two polynomials $P\left(x\right)$, $Q\left(x\right)$ is the standard determinant, given in Theorem 2, which gives by its zero or non-zero value the necessary and sufficient condition so that $P\left(x\right)$ and $Q\left(x\right)$ have no roots in common.

Also, if

$$P\left(x\right)=A_n\cdot\prod_{i=1}^{n}\left(x-r_i\right)\qquad\text{and}\qquad Q\left(x\right)=B_m\cdot\prod_{i=1}^{m}\left(x-s_i\right),$$

then

$$\rho\left(P\left(x\right),Q\left(x\right)\right)=A_n^mB_m^n\prod\left(r_i-s_j\right).$$

If this last property is taken as a definition, then Theorem 2 is a standard property of resultants that is proved in the theory of equations. See [3, pp. 99–104] and [2, page 21] for the details. Also, see [1] for many related problems. Of course, the reader will immediately see the similarity of Theorem 2 and the determinant $\left|M\left(x\right)\right|$ that was given in the Introduction.

In Theorems 2 and 3, we use the notation

$$P\left(x\right)=\sum_{i=0}^{n}A_ix^i=A_n\cdot\prod_{i=1}^{n}\left(x-r_i\right),A_n\neq 0,$$

and

$$Q\left(x\right)=\sum_{i=0}^{m}B_ix^i=B_m\cdot\prod_{i=1}^{m}\left(x-s_i\right),B_m\neq 0.$$

Theorem 2. *$\rho\left(P\left(x\right),Q\left(x\right)\right)$ equals the determinant of the $(m+n)\times(m+n)$ matrix M defined as follows.*

(1) *Each row i, $1\leq i\leq m$, of M is defined as follows.*

$$\overbrace{0,0,\ldots,0}^{\leftarrow i-1\rightarrow},A_n,A_{n-1},\ldots,A_1,A_0,\overbrace{0,0,\ldots,0}^{\leftarrow m-i\rightarrow}.$$

(2) *Each row $m+i$, $1\leq 1\leq n$, of M is defined as follows.*

$$\overbrace{0,0,\ldots,0}^{\leftarrow i-1\rightarrow},B_m,B_{m-1},\ldots,B_1,B_0,\overbrace{0,0,\ldots,0}^{\leftarrow n-i\rightarrow}.$$

The matrix used in Theorem 2 is usually called Sylvester's Matrix [2]. We now illustrate Theorem 2. First, suppose $P(x)=(x-a)(x-b)=$

$x^2-(a+b)x+ab$ and $Q(x)=x-c$. Then $\rho(P(x),Q(x))=(a-c)(b-c)$. Also, by Theorem 2,

$$\rho(P(x),Q(x))=\begin{vmatrix} 1 & -(a+b) & ab \\ 1 & -c & 0 \\ 0 & 1 & -c \end{vmatrix}=c^2-(a+b)c+ab=(a-c)(b-c).$$

Second, suppose $P(x)=(x-a)(x-b)$ and $Q(x)=(x-c)(x-d)$. Then $\rho(P(x),Q(x))=(a-c)(b-c)(a-d)(b-d)$. Also, by Theorem 2,

$$\rho(P(x),Q(x))=\begin{vmatrix} 1 & -(a+b) & ab & 0 \\ 0 & 1 & -(a+b) & ab \\ 1 & -(c+d) & cd & 0 \\ 0 & 1 & -(c+d) & cd \end{vmatrix}.$$

The expansion of this determinant is a little tedious and is left to the reader.

Theorem 3. *For all complex numbers b,*

$$\rho\left(P\left(x+b\right),Q\left(x+b\right)\right)=\rho\left(P\left(x\right),Q\left(x\right)\right).$$

Proof. Of course, $r_1,r_2,\ldots,r_n$ are the roots of $P(x)$ and $s_1,s_2,\ldots,s_m$ are the roots of $Q(x)$. Also, let $\overline{r}_1,\overline{r}_2,\ldots,\overline{r}_n$ be the roots of $P(x+b)$ and let $\overline{s}_1,\overline{s}_2,\ldots,\overline{s}_m$ be the roots of $Q(x+b)$. Now each $\overline{r}_i=r_i-b$ and each $\overline{s}_j=s_j-b$.

Also, $\rho(P,Q)=A_n^mB_m^n\prod(r_i-s_j)$. Therefore, $\rho(P(x+b),Q(x+b))=A_n^mB_m^n\prod(\overline{r}_i-\overline{s}_j)=A_n^mB_m^n\prod(r_i-s_j)=\rho(P,Q)$. $\square$

3. Proving the Theorem Given in the Introduction

As always, let $P(x)=\sum_{i=0}^n A_ix^i$, $A_n\neq 0$. Now

$$P(x)=\sum_{i=0}^n\frac{P^{(i)}(b)}{i!}(x-b)^i.$$

Therefore,

$$P(x+b)=\sum_{i=0}^n\frac{P^{(i)}(b)}{i!}x^i. \tag{1}$$

Likewise, if $Q(x)=\sum_{i=0}^m B_ix^i$, $B_m\neq 0$, then

$$Q(x+b)=\sum_{i=0}^m\frac{Q^{(i)}(b)}{i!}x^i. \tag{2}$$

From Theorem 3, $\rho(P(x+b),Q(x+b))=\rho(P(x),Q(x))$ which means that $\rho(P(x+b),Q(x+b))$ has a value that is independent of b.

If we now use Theorem 2 with (1) and (2) to evaluate $\rho\left(P\left(x+b\right),Q\left(x+b\right)\right)$, we immediately see that Theorem 1, given in the Introduction is true, where we are now using the variable b instead of the variable x.

4. Some Specific Examples

Suppose

$$P\left(x\right)=\sum_{i=0}^{2}A_ix^i=A_2x^2+A_1x+A_0$$

and

$$Q\left(x\right)=\sum_{i=0}^{2}B_ix^i=B_2x^2+B_1x+B_0.$$

Of course, $P\left(x\right)$ and $Q\left(x\right)$ are second degree polynomials and $P'\left(x\right)$, $Q'\left(x\right)$ are first degree polynomials. Therefore, by using the theorem with each pair (P,Q), (P',Q), (P,Q'), (P',Q'), we have the following four invariants (or constants) involving the derivatives of $P\left(x\right)$, $Q\left(x\right)$.

In 2), 3), and 4), we note that $(P')'=P''$, $(Q')'=Q''$.

$$1)\ \begin{vmatrix} \frac{P''(x)}{2} & P'\left(x\right) & P\left(x\right) & 0 \\ 0 & \frac{P''(x)}{2} & P'\left(x\right) & P\left(x\right) \\ \frac{Q''(x)}{2} & Q'\left(x\right) & Q\left(x\right) & 0 \\ 0 & \frac{Q''(x)}{2} & Q'\left(x\right) & Q\left(x\right) \end{vmatrix},$$

$$2)\ \begin{vmatrix} P''\left(x\right) & P'\left(x\right) & 0 \\ 0 & P''\left(x\right) & P'\left(x\right) \\ \frac{Q''(x)}{2} & Q'\left(x\right) & Q\left(x\right) \end{vmatrix},$$

$$3)\ \begin{vmatrix} \frac{P''(x)}{2} & P'\left(x\right) & P\left(x\right) \\ Q''\left(x\right) & Q'\left(x\right) & 0 \\ 0 & Q''\left(x\right) & Q'\left(x\right) \end{vmatrix},$$

$$4)\ \begin{vmatrix} P''\left(x\right) & P'\left(x\right) \\ Q''\left(x\right) & Q'\left(x\right) \end{vmatrix}.$$

We invite the reader to write the invariants when $P(x)$ is third degree and $Q(x)$ is second degree and also consider the case where $Q(x)=P^{(k)}(x)$ for some positive integer k. If we make $P\left(x\right)$ simple and let $Q\left(x\right)$ be arbitrary, then we can write down invariants that can easily be evaluated. For example, if $P\left(x\right)=x-b$ and $Q\left(x\right)$ is a cubic, then we have the following invariant:

$$\begin{vmatrix} 1 & x-b & 0 & 0 \\ 0 & 1 & x-b & 0 \\ 0 & 0 & 1 & x-b \\ \frac{Q'''(x)}{6} & \frac{Q''(x)}{2} & Q'(x) & Q(x) \end{vmatrix}$$
$$= Q(x) + Q'(x)(b-x) + \frac{Q''(x)}{2}(b-x)^2 + \frac{Q'''(x)}{3!}(b-x)^3 = Q(b).$$

This is the standard Taylor's series if we interchange x and b. Finally, suppose $\overline{P}(x)$ is an arbitrary polynomial of degree n. Define $P(x) = \overline{P}^{(i)}(x), Q(x) = \overline{P}^{(j)}(x)$. Then by varying $i, j \in \{0, 1, 2, \ldots, n-1\}$, $i < j$, we can create $\binom{n}{2}$ different invariants that involve the derivatives of $\overline{P}(x)$.

5. A Research Problem

Since we deal with $P(x)$ and its horizontal translation $P(x+b)$ in this paper, this motivated us to consider the following problem below.

Suppose
$$P(x) = \sum_{i=0}^{n} A_i x^i, \qquad A_n \neq 0,$$
and
$$\overline{P}(x) = \sum_{i=0}^{n} B_i x^i, \qquad B_n \neq 0,$$
are two nth degree polynomials. We say that P and $\overline{P}$ are *weakly congruent* (denoted $P(x) \cong \overline{P}(x)$) if $\overline{P}(x) = P(x+b)$ for some complex number b. It is easy to show that $\cong$ is reflexive, symmetric and transitive, and therefore an equivalence relation for the collection of all degree n polynomials. We can see quickly that $A_n = B_n$. Also, the values of A_{n-1}, B_{n-1}, $A_n = B_n$ allow us to compute the value of b, and this explains why we do not use $P_2(A_n, A_{n-1}) = P_2(B_n, B_{n-1})$ in the following list. We wish to find a collection of $n-1$ polynomials, $P_3(x_1, x_2, x_3)$, $P_4(x_1, x_2, x_3, x_4)$, $P_5(x_1, x_2, x_3, x_4, x_5)$, …, $P_{n+1}(x_1, x_2, \ldots, x_{n+1})$ such that $P(x) \cong \overline{P}(x)$ if and only if

1) $A_n = B_n$
2) $P_3(A_n, A_{n-1}, A_{n-2}) = P_3(B_n, B_{n-1}, B_{n-2})$
3) $P_4(A_n, A_{n-1}, A_{n-2}, A_{n-3}) = P_4(B_n, B_{n-1}, B_{n-2}, B_{n-3})$

$\vdots$

n) $P_{n+1}(A_n, A_{n-1}, \ldots, A_0) = P_{n+1}(B_n, B_{n-1}, \ldots, B_0)$.

The reader may wish to solve the quadratic version of this problem before going further. We can call P_3, P_4, …, P_{n+1} invariants under $\cong$ that classify the equivalence relation. We now start the reader off on the general solution.

If $P(x) \cong \overline{P}(x)$, then

$$\overline{P}(x) = \sum_{i=0}^{n} B_i x^i = P(x+b) = \sum_{i=0}^{n} \frac{P^{(i)}(b)}{i!} x^i.$$

Therefore, $B_i = \frac{P^{(i)}(b)}{i!}$. Also, $A_i = \frac{P^{(i)}(0)}{i!}$. It follows that $P^{(i)}(b) = i!B_i$, $P^{(i)}(0) = i!A_i$. For each $i, j \in \{0, 1, 2, \ldots, n-1\}$, $i < j$, if we call $x = b$, then we know that we can use each pair $(P^{(i)}, P^{(j)})$ to create an invariant (under $\cong$) that involve the coefficients $A_0, A_1, \ldots, A_n$, $B_0, B_1, \ldots, B_n$. These invariants are necessary conditions for $P(x) \cong \overline{P}(x)$. However, when $n \geq 3$, this collection is too large, and it overshoots the number of invariants required in the problem. So the problem is to cull out (with proof) a subcollection that solves the problem. As an example, let $P(x) = A_2x^2 + A_1x + A_0$, $\overline{P}(x) = B_2x^2 + B_1x + B_0$. Then $B_i = \frac{P^{(i)}(b)}{i!}$, $A_i = \frac{P^{(i)}(0)}{i!}$, $i = 0, 1, 2$. Using (P, P'), we have the invariant

$$\begin{vmatrix} \frac{P''(b)}{2} & P'(b) & P(b) \\ P''(b) & P'(b) & 0 \\ 0 & P''(b) & P'(b) \end{vmatrix} = \begin{vmatrix} \frac{P''(0)}{2} & P'(0) & P(0) \\ P''(0) & P'(0) & 0 \\ 0 & P''(0) & P'(0) \end{vmatrix}.$$

This gives

$$\begin{vmatrix} B_2 & B_1 & B_0 \\ 2B_2 & B_1 & 0 \\ 0 & 2B_2 & B_1 \end{vmatrix} = \begin{vmatrix} A_2 & A_1 & A_0 \\ 2A_2 & A_1 & 0 \\ 0 & 2A_2 & A_1 \end{vmatrix}$$

which gives $4B_0B_2^2 - B_1^2B_2 = 4A_0A_2^2 - A_1^2A_2$. Since $A_2 = B_2 \neq 0$, it follows that $4B_0B_2 - B_1^2 = 4A_0A_2 - A_1^2$. Of course this is the standard discriminant of a quadratic. Equating $\overline{P}(x) = B_2x^2 + B_1x + B_0 = P(x+b) = A_2x^2 + (2A_2b + A_1)x + A_2b^2 + A_1b + A_0$, we see that $A_2 = B_2$, $b = \frac{B_1 - A_1}{2A_2}$. Also, we require $A_2\left[\frac{B_1 - A_1}{2A_2}\right]^2 + A_1\left[\frac{B_1 - A_1}{2A_2}\right] + A_0 = B_0$. Since $A_2 \neq 0$, this is true if and only if $(B_1^2 - 2B_1A_1 + A_1^2) + 2A_1(B_1 - A_1) + 4A_0A_2 = 4A_2B_0$. Since $A_2 = B_2$, this is true if and only if $B_1^2 - 4B_0B_2 = A_1^2 - 4A_0A_2$, and these are the same two conditions as are given above.

As the reader explores higher degree polynomials, he will begin to appreciate the theory in this paper. Ed Barbeau suggested another way to attack this problem, but he did not solve the problem. Thus the reader might like to try solving this in a different way.

Let us now define another relation. We say $P(x), \overline{P}(x)$ are *strongly congruent* (denoted by $P(x) \bowtie \overline{P}(x)$) if $P(x) = P(x+b) + a$ for some complex numbers a, b. We invite the reader to very slightly modify his solution to the weak congruence problem above to define and solve an analogous problem for strong congruence. Finally, the reader might like to discuss the relationship between strong congruence and geometric congruence.

6. Discussion

Since in general the two polynomials $\overline{P}(x)$ of degree n and $\overline{Q}(x)$ of degree m need not be correlated in any way, it seems to be a remarkable fact that we can write down so many different invariants involving the derivatives of $\overline{P}(x), \overline{Q}(x)$. By defining $P(x) = \overline{P}^{(i)}(x), i \in \{0, 1, 2, \ldots, n-1\}$, and $Q(x) = \overline{Q}^{(j)}(x), j \in \{0, 1, 2, \ldots, m-1\}$, and then using each pair $\left(\overline{P}^{(i)}(x), \overline{Q}^{(j)}(x)\right)$ we can write down $n \cdot m$ different invariants involving the derivatives of $\overline{P}(x), \overline{Q}(x)$. Thus, if $n = m = 100$, we would have ten thousand different invariants involving two unrelated polynomials $\overline{P}(x), \overline{Q}(x)$. This seems almost unbelievable. Also, if we include $(P^{(i)}, P^{(j)})$ and $(Q^{(i)}, Q^{(j)})$, we can write down more invariants.

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