ON u-ω-OPEN AND q-ω-OPEN SETS IN BITOPOLOGICAL SPACES

S. AL GHOUR AND S. ISSA

Abstract. ω-open sets are used to introduce two new classes of sets in bitopological spaces, namely, u-ω-open sets and q-ω-open sets. Several properties of these classes are given. The new classes of sets are used to introduce several types of continuity. Several results related to two known Lindelöfness bitopological concepts are introduced.

1. Introduction and preliminaries

Let (X, τ) be a topological space and let A be a subset of X. A point $x \in X$ is called a *condensation point of* A [8] if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. In 1982, Hdeib defined ω-closed sets and ω-open sets as follows. A is called ω-*closed* [12] if it contains all its condensation points. The complement of an ω-closed set is called ω-*open*. Many topological concepts and results related to ω-closed sets and ω-open sets have appeared in [1-5,8,11,16] and others. In 1963, Kelly [13] introduced the notion of *bitopological spaces* as an ordered triple (X, τ, σ) of a set X and two topologies τ and σ, (i.e., two bitopological spaces (X, τ, σ) and (X, τ', σ') are identical if and only if $\tau = \tau'$ and $\sigma = \sigma'$. After Kelly's initiation of the bitopological notion, many authors generalized many topological concepts to include bitopological spaces. Recently, the authors in [6] used ω-closed sets to introduce semi star generalized ω-closed sets as a class of sets in bitopological spaces. In the present work, ω-open sets will be used to obtain two new classes of sets in bitopological spaces, namely u-ω-open sets and q-ω-open sets. Several properties of these classes will be given. Several results related to two known Lindelöfness bitopological concepts will be introduced. The new classes of sets will be used to introduce several types of continuity and separation axioms in bitopological spaces.

Throughout this paper, we use $\mathbb{R}$ (resp. $\mathbb{Q}, \mathbb{Q}^c$) to denote the set of real numbers (resp. the set of rational numbers, the set of irrational numbers). For a subset A of a topological space (X, τ), we write $Cl_\tau(A)$ for the closure of A. Also, we write $\tau_{|A}$ to denote the relative topology on A when A is nonempty. For a nonempty set X, τ_{ind} will denote the indiscrete topology on X. We use τ_u (resp. τ_{lr}, τ_{rr}) to denote the usual (resp. left ray, right ray) topology on $\mathbb{R}$.

In this paper, the family of all ω-open sets of a topological space (X, τ) will be denoted by τ_ω.

At the end of this section, we recall basic definitions and propositions, which are used throughout this paper.

Proposition 1.1. [5] *Let (X, τ) be a topological space and $A \subseteq X$. Then*

(a) *A is ω-open of (X, τ) if and only if for each $x \in A$ there exists $U \in \tau$ and a countable set $C \subseteq X$ such that $x \in U - C \subseteq A$.*
(b) *τ_ω is a topology on X with $\tau \subseteq \tau_\omega$.*
(c) *If A is nonempty, then $(\tau |_A)_\omega = \tau_\omega |_A$.*

Definition 1.2. [11] *A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be ω-continuous at a point $x \in X$, if for every open set V containing $f(x)$ there is an ω-open set U containing x such that $f(U) \subseteq V$. If f is ω-continuous at each point of X then f is said to be ω-continuous on X.*

Proposition 1.3. [11] *For a function $f\colon (X, \tau) \to (Y, \sigma)$ the following are equivalent.*

(a) *f is ω-continuous.*
(b) *For each $U \in \sigma$, $f^{-1}(U) \in \tau_\omega$.*
(c) *$f\colon (X, \tau_\omega) \to (Y, \sigma)$ is continuous.*

Proposition 1.4. [11] *Every continuous function is ω-continuous, but not conversely.*

Recall that if τ and σ are two topologies on a set X, then the smallest topology on X which contains $\tau \cup \sigma$ is called the least upper bound topology on X.

Definition 1.5. [7] *A set $A \subseteq (X, \tau, \sigma)$ is said to be semi-open (briefly, s-open) if it is open in the least upper bound topology on X.*

If τ and σ are two topologies on a set X, then the least upper bound topology on X will be denoted by $\langle \tau, \sigma \rangle$.

The following useful result follows directly from the definition.

Proposition 1.6. *Let τ and σ be two topologies on a set X. Then $A \subseteq (X, \tau, \sigma)$ is s-open if and only if for each $x \in A$ there exists $U \in \tau$, and $V \in \sigma$ such that $x \in U \cap V \subseteq A$.*

Definition 1.7. [7] *A set $A \subseteq (X, \tau, \sigma)$ is said to be quasi-open (briefly, q-open) if for every $x \in A$ there exists $U_x \in \tau$ such that $x \in U_x \subseteq A$ or $V_x \in \sigma$ such that $x \in V_x \subseteq A$. Equivalently, a set $A \subseteq (X, \tau, \sigma)$ is q-open if and only if $A = B \cup C$, where $B \in \tau$ and $C \in \sigma$. A set $A \subseteq (X, \tau, \sigma)$ is said to be q-closed if $X - A$ is q-open.*

The family of all q-open sets in (X, τ, σ) is denoted by $q(\tau, \sigma)$.

Definition 1.8. [14] *A set $A \subseteq (X, \tau, \sigma)$ is said to be u-open if $A \in \tau \cup \sigma$.*

The family of all u-open sets in (X, τ, σ) is denoted by $u(\tau, \sigma)$. The authors in [14] called u-open set in Definition 3.1 (i) as p_1-open set.

Proposition 1.9. [7] *For a topological space (X, τ, σ), we have the following:*

(a) *$u(\tau, \sigma) \subseteq q(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle$, $\tau \cup \sigma \neq q(\tau, \sigma)$ in general, and $q(\tau, \sigma) \neq \langle \tau, \sigma \rangle$ in general.*
(b) *$q(\tau, \sigma)$ is closed under arbitrary union but $q(\tau, \sigma)$ is not a topology on X, in general.*
(c) *Arbitrary intersection of q-closed sets is q-closed.*

Definition 1.10. *A cover $\mathcal{U}$ of the bitopological space (X, τ, σ) is called:*

(a) *[17] $\tau\sigma$-open if $\mathcal{U} \subseteq u(\tau, \sigma)$.*
(b) *[9] p-open if it is $\tau\sigma$-open, and $\mathcal{U}$ contains at least one nonempty member of τ and at least one nonempty member of σ.*

Definition 1.11. [10] *A bitopological space (X, τ, σ) is called:*

(a) *s-Lindelöf if every $\tau\sigma$-open cover of (X, τ, σ) has a countable subcover.*
(b) *p-Lindelöf if every p-open cover of (X, τ, σ) has a countable subcover.*

Definition 1.12. *Let $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function.*

(a) *[15] f is said to be p-continuous if the functions $f\colon (X, \tau_1) \to (Y, \sigma_1)$ and $f\colon (X, \tau_2) \to (Y, \sigma_2)$ are continuous.*
(b) *[18] f is said to be u-continuous if for each $A \in u(\sigma_1, \sigma_2)$, $f^{-1}(A) \in u(\tau_1, \tau_2)$.*
(c) *[7] f is said to be q-continuous if for each $A \in q(\sigma_1, \sigma_2)$, $f^{-1}(A) \in q(\tau_1, \tau_2)$.*

The authors in [18] called u-continuous functions in Definition 5.1, p-continuous functions. Also the author in [7] called q-continuous functions in Definition 2.6 quasi-continuous.

The following result is well-known.

Proposition 1.13. *Every p-continuous function from a bitopological space (X, τ_1, τ_2) to a bitopological space (Y, σ_1, σ_2) is u-continuous, but not conversely.*

Proposition 1.14. [11] *If $f\colon (X, \tau) \to (Y, \sigma)$ is ω-continuous and $g\colon (Y, \sigma) \to (Z, \rho)$ is continuous, then $g \circ f\colon (X, \tau) \to (Z, \rho)$ is ω-continuous.*

Definition 1.15. [7] *The q-closure of A in (X, τ, σ) is denoted by q-$Cl_{(\tau,\sigma)}(A)$ and defined as follows:*

$$q\text{-}Cl_{(\tau,\sigma)}(A) = Cl_{\tau}(A) \cap Cl_{\sigma}(A).$$

The author in [7] called a q-closure of A a quasi-closure of A and denoted it by $\overline{A}$.

Proposition 1.16. [7] *If $A \subseteq (X, \tau, \sigma)$, then $q\text{-}Cl_{(\tau,\sigma)}(A)$ is the smallest q-closed set containing A.*

Definition 1.17. *Let (X, τ, σ) be a bitopological space. A subset M of X is called an s-Lindelöf subset of (X, τ, σ) if for each $\mathcal{A} \subseteq u(\tau, \sigma)$ with $M \subseteq \bigcup \mathcal{A}$, there exists a countable set $\mathcal{A}' \subseteq \mathcal{A}$ such that $M \subseteq \bigcup \mathcal{A}'$. Equivalently: A subset M of X is an s-Lindelöf subset of (X, τ, σ) if and only if M is empty or $((M, \tau_{|M}, (\sigma)_{|M})$ is s-Lindelöf.*

2. u-ω-open Sets and q-ω-open Sets

Definition 2.1. *Let (X, τ, σ) be a bitopological space and let $A \subseteq X$. Then*

(a) *A is said to be u-ω-open in (X, τ, σ) if $A \in \tau_{\omega} \cup \sigma_{\omega}$. Equivalently: $A \subseteq (X, \tau, \sigma)$ is u-ω-open if and only if $A \in u(\tau_{\omega}, \sigma_{\omega})$.*
(b) *A is said to be u-ω-closed in (X, τ, σ) if $X - A$ is u-ω-open in (X, τ, σ).*
(c) *A is said to be s-ω-open in (X, τ, σ) if it is open in the least upper bound topology on X, of τ_{ω} and σ_{ω}.*

For a bitopological space (X, τ, σ), the family of all u-ω-open sets in (X, τ, σ) is denoted by $u\omega(\tau, \sigma)$, and the family of all ω-open sets in the topological space $(X, \langle \tau, \sigma \rangle)$ is denoted by $\langle \tau, \sigma \rangle_{\omega}$.

Theorem 2.2. *Let (X, τ, σ) be a bitopological space. Then*

(a) $\langle \tau, \sigma \rangle_{\omega} = \langle \tau_{\omega}, \sigma_{\omega} \rangle$.
(b) $u(\tau, \sigma) \subseteq u\omega(\tau, \sigma)$.
(c) $u\omega(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle_{\omega}$.

Proof. (a) Let $A \in \langle \tau, \sigma \rangle_{\omega}$ and let $x \in A$. Then by Proposition 1.1 (a), there exists $H \in \langle \tau, \sigma \rangle$ and $C \subseteq X$ countable set such that $x \in H - C \subseteq A$. Since $x \in H \in \langle \tau, \sigma \rangle$ then by Proposition 1.6, there exists $U \in \tau$, and $V \in \sigma$ such that $x \in U \cap V \subseteq H$. Note that $U - C \in \tau_{\omega}$, $V - C \in \sigma_{\omega}$, and $x \in (U - C) \cap (V - C) \subseteq (U \cap V) - C \subseteq H - C \subseteq A$. Thus, again by Proposition 1.6, $A \in \langle \tau_{\omega}, \sigma_{\omega} \rangle$. Conversely, let $A \in \langle \tau_{\omega}, \sigma_{\omega} \rangle$ and let $x \in A$. Then there exist $W_1 \in \tau_{\omega}$ and $W_2 \in \sigma_{\omega}$ such that $x \in W_1 \cap W_2 \subseteq A$. Since $x \in W_1 \cap W_2$ then there exist $U \in \tau$, $V \in \sigma$ and $C_1, C_2 \subseteq X$ countable sets such that $x \in U - C_1 \subseteq W_1$ and $x \in V - C_2 \subseteq W_2$. Note that $U \cap V \in \langle \tau, \sigma \rangle$

and $C_1 \cap C_2$ countable set. Also, $x \in (U \cap V) - (C_1 \cap C_2) \subseteq W_1 \cap W_2 \subseteq A$. Thus, $A \in \langle \tau, \sigma \rangle_\omega$.

(b) By Proposition 1.1 (b), we have $\tau \cup \sigma \subseteq \tau_\omega \cup \sigma_\omega$, and consequently, $A \in \tau_\omega \cup \sigma_\omega = u\omega(\tau, \sigma)$.

(c) Note that $u\omega(\tau, \sigma) = \tau_\omega \cup \sigma_\omega = u(\tau_\omega, \sigma_\omega)$ and by Proposition 1.9 (a), $u(\tau_\omega, \sigma_\omega) \subseteq \langle \tau_\omega, \sigma_\omega \rangle$. Therefore, $u\omega(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle_\omega$. □

The following two examples will show, respectively, that the inclusion in each of the parts (b) and (c) of Theorem 2.2 cannot be replaced by equality, in general.

Example 2.3. *Let $X = \mathbb{R}$, $\tau = \tau_{ind}$. Then $\mathbb{Q}^c \in u\omega(\tau, \tau) - u(\tau, \tau)$.*

Example 2.4. *Let $X = \mathbb{R}$, $\tau = \tau_{lr}$, and $\sigma = \tau_{rr}$. Then $(1, 2) \in \langle \tau, \sigma \rangle \subseteq \langle \tau, \sigma \rangle_\omega$, while $(1, 2) \notin \tau_\omega \cup \sigma_\omega = u\omega(\tau, \sigma)$.*

Definition 2.5. *A set $A \subseteq (X, \tau, \sigma)$ is said to be q-ω-open if for every $x \in A$ there exists $U_x \in \tau_\omega$ such that $x \in U_x \subseteq A$ or $V_x \in \sigma_\omega$ such that $x \in V_x \subseteq A$. Equivalently, $A \subseteq (X, \tau, \sigma)$ is q-ω-open if and only if $A \in q(\tau_\omega, \sigma_\omega)$. A set $A \subseteq (X, \tau, \sigma)$ is said to be q-ω-closed, if $X - A$ is q-ω-open.*

The family of all q-ω-open sets in (X, τ, σ) is denoted by $q\omega(\tau, \sigma)$.

Theorem 2.6. *Let (X, τ, σ) be a bitopological space and $A \subseteq X$. Then the following are equivalent:*

(a) *A is q-ω-open.*
(b) *For each $x \in A$ there exists $B \in u(\tau, \sigma)$ and is a countable set $C \subseteq X$ such that $x \in U - C \subseteq A$.*

Proof. (a) → (b) Suppose that A is q-ω-open and let $x \in A$. Since A is q-ω-open, $A = B \cup C$, where $B \in \tau_\omega$ and $C \in \sigma_\omega$. Without loss of generality we may assume that $x \in B$. Take $U_x \in \tau \subseteq u(\tau, \sigma)$ and a countable set $C_x \subseteq X$ such that $x \in U_x - C_x \subseteq B \subseteq A$. This ends the proof.

(b) → (a) By assumption, for each $x \in A$, there exists $U_x \in u(\tau, \sigma)$ and a countable set C_x such that $x \in U_x - C_x \subseteq A$. Put $B = \bigcup \{U_x - C_x : U_x \in \tau\}$ and $C = \bigcup \{U_x - C_x : U_x \in \sigma\}$. Then $B \in \tau_\omega, C \in \sigma_\omega$, and $A = B \cup C$. Hence, A is q-ω-open. □

Theorem 2.7. *Let (X, τ, σ) be a bitopological space. Then*

(a) *$u\omega(\tau, \sigma) \subseteq q\omega(\tau, \sigma)$.*
(b) *$q(\tau, \sigma) \subseteq q\omega(\tau, \sigma)$.*
(c) *$q\omega(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle_\omega$.*
(d) *$\{\emptyset, X\} \subseteq q\omega(\tau, \sigma)$.*
(e) *The family $q\omega(\tau, \sigma)$ is closed under arbitrary union.*

(f) *The family of all q-ω-closed sets in (X, τ, σ) is closed under arbitrary intersection.*

Proof. (a) As $u\omega(\tau, \sigma) = u(\tau_\omega, \sigma_\omega)$ and by Proposition 1.9 (a), $u(\tau_\omega, \sigma_\omega) \subseteq q(\tau_\omega, \sigma_\omega) = q\omega(\tau, \sigma)$, we have $u\omega(\tau, \sigma) \subseteq q\omega(\tau, \sigma)$.

(b) Let $A \in q(\tau, \sigma)$. Then $A = B \cup C$, where $B \in \tau$ and $C \in \sigma$. By Proposition 1.1 (b), $B \in \tau_\omega$ and $C \in \sigma_\omega$ and so $A \in q(\tau_\omega, \sigma_\omega) = q\omega(\tau, \sigma)$.

(c) By Proposition 1.9 (a), we have $q\omega(\tau, \sigma) = q(\tau_\omega, \sigma_\omega) \subseteq \langle \tau_\omega, \sigma_\omega \rangle$. Thus, by Theorem 2.2 (a), it follows that $q\omega(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle_\omega$.

(d) As $\{\emptyset, X\} \subseteq u\omega(\tau, \sigma)$, then by part (a) we have $\{\emptyset, X\} \subseteq q\omega(\tau, \sigma)$.

(e) Since $q\omega(\tau, \sigma) = q(\tau_\omega, \sigma_\omega)$ and by Proposition 1.9 (b), $q(\tau_\omega, \sigma_\omega)$ is closed under arbitrary union, we get the result.

(f) Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of q-ω-closed sets of (X, τ, σ). Then $\bigcap\limits_{\alpha\in\Delta} A_\alpha = (\bigcup\limits_{\alpha\in\Delta} A_\alpha^c)^c$. Since A_α^c is q-ω-open for every $\alpha \in \Delta$, then by part (f), $\bigcup\limits_{\alpha\in\Delta} A_\alpha^c \in q\omega(\tau, \sigma)$. Therefore, $\bigcap\limits_{\alpha\in\Delta} A_\alpha$ is q-ω-closed. □

The following example shows that the inclusion in Theorem 2.7 (a) cannot be replaced by equality, in general.

Example 2.8. *Consider the bitopological space $(\mathbb{R}, \tau_{lr}, \tau_{rr})$. Then $(-\infty, 0) \cup (1, \infty) \in q\omega(\tau_{lr}, \tau_{rr}) - u\omega(\tau_{lr}, \tau_{rr})$.*

The following example shows that the inclusion in Theorem 2.7 (b) cannot be replaced by equality, in general.

Example 2.9. *Consider the bitopological space $(\mathbb{R}, \tau_u, \tau_u)$. Then $\mathbb{Q}^c \in q\omega(\tau_u, \tau_u) = q((\tau_u)_\omega, (\tau_u)_\omega) = (\tau_u)_\omega$, while as $q(\tau_u, \tau_u) = \tau_u$, then $\mathbb{Q}^c \notin q(\tau_u, \tau_u)$.*

The following example shows that the inclusion in Theorem 2.7 (c) cannot be replaced by equality, in general.

Example 2.10. *Consider the bitopological space $(\mathbb{R}, \tau_{lr}, \tau_{rr})$. Then $(0, 1) \in \langle \tau_{lr}, \tau_{rr} \rangle \subseteq \langle \tau_{lr}, \tau_{rr} \rangle_\omega$, while $(0, 1) \notin q\omega(\tau_{lr}, \tau_{rr})$.*

The next example shows that the intersection of two q-ω-open sets is not q-ω-open in general. Therefore, the family of all q-ω-open sets of a bitopological space (X, τ, σ) does not form a topology on X, in general.

Example 2.11. *Let $X = \mathbb{R}$, $\tau = \tau_{lr}$, $\sigma = \tau_{rr}$, $A_1 = (-\infty, 1)$, and $A_2 = (0, \infty)$. Then A_1 and A_2 are q-ω-open sets in (X, τ, σ) but $A_1 \cap A_2 = (0, 1)$ is not q-ω-open.*

For a bitopological space (X, τ, σ), Proposition 1.9 (b) says that $q(\tau, \sigma)$ does not form a topology on X, in general. However, we have the following result.

Proposition 2.12. *Let (X, τ, σ) be a bitopological space. Then $q(\tau, \sigma)$ is a topology on X if and only if $q(\tau, \sigma) = \langle \tau, \sigma \rangle$.*

Proof. Necessity. Suppose $q(\tau, \sigma)$ is a topology on X. By Proposition 1.9 (a), we have $q(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle$. Also, since $\tau \cup \sigma \subseteq q(\tau, \sigma)$, then by the definition of $\langle \tau, \sigma \rangle$, we have $\langle \tau, \sigma \rangle \subseteq q(\tau, \sigma)$.

Sufficiency. Suppose that $q(\tau, \sigma) = \langle \tau, \sigma \rangle$. As $\langle \tau, \sigma \rangle$ is a topology on X, then $q(\tau, \sigma)$ is a topology on X. □

Corollary 2.13. *Let (X, τ, σ) be a bitopological space. Then $q\omega(\tau, \sigma)$ is a topology on X if and only if $q\omega(\tau, \sigma) = \langle \tau, \sigma \rangle_\omega$.*

Proof. By Theorem 2.2 (a), $\langle \tau, \sigma \rangle_\omega = \langle \tau_\omega, \sigma_\omega \rangle$, also $q\omega(\tau, \sigma) = q(\tau_\omega, \sigma_\omega)$. Therefore, applying Proposition 2.12 on $(X, \tau_\omega, \sigma_\omega)$ we get the result. □

The following lemma will be used in the proof of the next main result.

Lemma 2.14. *Let (X, τ, σ) be a bitopological space and let $\mathcal{A} = \{W - C : W \in u(\tau, \sigma)$ and $C \subseteq X$ is countable $\}$. Then $(X, \tau_\omega, \sigma_\omega)$ is s-Lindelöf if and only if every cover of X, consisting of elements of $\mathcal{A}$, has a countable subcover.*

Proof. Necessity. Suppose $(X, \tau_\omega, \sigma_\omega)$ is s-Lindelöf and let $\mathcal{W}$ be a cover of X with $\mathcal{W} \subseteq \mathcal{A}$. Since $\mathcal{W} \subseteq \mathcal{A} \subseteq \tau_\omega \cup \sigma_\omega$, then $\mathcal{W}$ is $\tau_\omega\sigma_\omega$-open cover of $(X, \tau_\omega, \sigma_\omega)$, and thus, there exists a countable family of elements of $\mathcal{W}$ covers X.

Sufficiency. Let $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ be a $\tau_\omega\sigma_\omega$-open cover of $(X, \tau_\omega, \sigma_\omega)$. For each $\alpha \in \Delta$, there exists an indexed set Ω_α such that

$$W_\alpha = \bigcup_{\beta \in \Omega_\alpha} (V_\beta - C_\beta)$$

where $V_\beta \in \tau \cup \sigma$ and $C_\beta \subseteq X$ is countable set for every $\beta \in \Omega_\alpha$. Thus, $\{V_\beta - C_\beta : \beta \in \bigcup_{\alpha \in \Delta} \Omega_\alpha\}$ is a cover of X consists of elements of $\mathcal{A}$, and by assumption it has a countable subcover. This implies that $\mathcal{W}$ also has a countable subcover. □

Theorem 2.15. *For a bitopological space (X, τ, σ), the following are equivalent.*

(a) *(X, τ, σ) is s-Lindelöf,*
(b) *$(X, \tau_\omega, \sigma_\omega)$ is s-Lindelöf,*
(c) *every cover of X, consisting of elements of $q\omega(\tau, \sigma)$, has a countable subcover, and*
(d) *every cover of X, consisting of elements of $q(\tau, \sigma)$, has a countable subcover.*

Proof. (a) → (b) Suppose that (X,τ,σ) is s-Lindelöf. We apply Lemma 2.14. Let $\mathcal{W}=\{W_\alpha - C_\alpha : \alpha\in\Delta$, where $W_\alpha\in u(\tau,\sigma)$ and $C_\alpha\subseteq X$ is a countable set$\}$ be a cover of X. Since $\bigcup\limits_{\alpha\in\Delta} W_\alpha = X$, then by (a), there exists a countable set $\Delta'\subseteq\Delta$ such that $\{W_\alpha:\alpha\in\Delta'\}$ covers X. Put $G=\bigcap\limits_{\alpha\in\Delta'} C_\alpha$ and for each $x\in G$, choose $\alpha_x\in\Delta$ such that $x\in(W_{\alpha_x}-C_{\alpha_x})$. Thus, $\{W_\alpha-C_\alpha:\alpha\in\Delta'\}\cup\{W_{\alpha_x}-C_{\alpha_x}:x\in G\}$ is a countable subcover of $\mathcal{W}$.

(b) → (c) Suppose that $(X,\tau_\omega,\sigma_\omega)$ is s-Lindelöf. Let $\mathcal{W}=\{W_\alpha:\alpha\in\Delta\}$ be a cover of X consists of elements of $q\omega(\tau,\sigma)$. For each $\alpha\in\Delta$, there exist $A_\alpha\in\tau_\omega$ and $B_\alpha\in\sigma_\omega$ such that $W_\alpha=A_\alpha\cup B_\alpha$. Since $\{A_\alpha,B_\alpha:\alpha\in\Delta\}$ covers X and $\{A_\alpha,B_\alpha:\alpha\in\Delta\}\subseteq u(\tau_\omega,\sigma_\omega)$, then by (b), there exists a countable set $\Delta'\subseteq\Delta$ such that $\{A_\alpha,B_\alpha:\alpha\in\Delta'\}$ covers X. It follows that $\{W_\alpha:\alpha\in\Delta'\}$ is a countable subcover of $\mathcal{W}$.

(c) → (d) Let $\mathcal{U}$ be a cover of X with $\mathcal{U}\subseteq q(\tau,\sigma)$. Then by Theorem 2.7 (b), $\mathcal{U}\subseteq q\omega(\tau,\sigma)$, and so $\mathcal{U}$ has a countable subcover.

(d) → (a) Follows because $u(\tau,\sigma)\subseteq q(\tau,\sigma)$. □

Theorem 2.16. *For a bitopological space (X,τ,σ), the following are equivalent.*

(a) (X,τ,σ) *is p-Lindelöf,*
(b) $(X,\tau_\omega,\sigma_\omega)$ *is p-Lindelöf.*

Proof. (a) → (b) Suppose that (X,τ,σ) is p-Lindelöf. Let $\mathcal{W}=\{W_\alpha:\alpha\in\Delta\}$ be a p-open cover of $(X,\tau_\omega,\sigma_\omega)$. Take $\alpha_1,\alpha_2\in\Delta$ such that $W_{\alpha_1}\in\tau_\omega, W_{\alpha_2}\in\sigma_\omega$, and $W_{\alpha_i}\neq\emptyset$ for every $i=1,2$. For each $\alpha\in\Delta$, there exists an indexed set Ω_α such that $W_\alpha=\bigcup\limits_{\beta\in\Omega_\alpha}(V_\beta-C_\beta)$ where $\{C_\beta:\beta\in\Omega_\alpha\}$ is a family of countable subsets of X and

$$(\{V_\beta:\beta\in\Omega_\alpha\}\subseteq\tau \text{ or } \{V_\beta:\beta\in\Omega_\alpha\}\subseteq\sigma).$$

For every $i=1,2$, choose $\beta_i\in\Omega_{\alpha_i}$ such that $V_{\beta_1}\in\tau, V_{\beta_2}\in\sigma$, and $V_{\beta_i}\neq\emptyset$. Therefore, $\{V_\beta:\beta\in\bigcup\limits_{\alpha\in\Delta}\Omega_\alpha\}$ is a p-open cover of (X,τ,σ). Since (X,τ,σ) is p-Lindelöf, then there exists a countable set $\Delta'\subseteq\Delta$ such that for every $\alpha\in\Delta'$, there exists a countable set $\Gamma_\alpha\subseteq\Omega_\alpha$ such that

$$\left\{V_\beta:\beta\in\bigcup_{\alpha\in\Delta'}\Gamma_\alpha\right\}$$

covers X. Put

$$G=\bigcap\left\{C_\beta:\beta\in\bigcup_{\alpha\in\Delta'}\Gamma_\alpha\right\}.$$

Then G is countable and

$$\left\{V_\beta - C_\beta : \beta \in \bigcup_{\alpha\in\Delta'} \Gamma_\alpha\right\}$$

is a cover of $X - G$. For each $x \in G$, choose $\alpha_x \in \Delta$ such that $x \in W_{\alpha_x}$. Thus, $\{W_\alpha : \alpha \in \Delta'\} \cup \{W_{\alpha_x} : x \in G\}$ is a countable subcover of $\mathcal{W}$.

(b) → (a) Suppose that $(X, \tau_\omega, \sigma_\omega)$ is p-Lindelöf. Let $\mathcal{U}$ be a p-open cover of (X, τ, σ). Then $\mathcal{U}$ is a p-open cover of $(X, \tau_\omega, \sigma_\omega)$ and hence is has a countable subcover. □

Theorem 2.17. *Let (X, τ, σ) be an s-Lindelöf bitopological space and A be a q-ω-closed subset in (X, τ, σ). Then A is an s-Lindelöf subset of (X, τ, σ).*

Proof. Suppose that (X, τ, σ) is s-Lindelöf. Let $\mathcal{U}$ be a $\tau\sigma$-open cover of A. Then $\mathcal{U} \subseteq q\omega(\tau, \sigma)$ and $X - A \in q\omega(\tau, \sigma)$, and thus $\mathcal{U} \cup \{X - A\}$ is a cover of X and consists of elements of $q\omega(\tau, \sigma)$. Therefore, by Theorem 2.15, $\mathcal{U} \cup \{X - A\}$ has a countable subcover $\mathcal{A}$. Put $\mathcal{U}' = \mathcal{A} - \{X - A\}$. Then $\mathcal{U}'$ is countable, $\mathcal{U}' \subseteq \mathcal{U}$, and $\mathcal{U}'$ covers A. Therefore, A is an s-Lindelöf subset of (X, τ, σ). □

Corollary 2.18. *Let (X, τ, σ) be an s-Lindelöf space and $A \subseteq X$. If A is a u-ω-closed subset in (X, τ, σ), then A is an s-Lindelöf subset of (X, τ, σ).*

Proof. Since u-ω-closed sets are q-ω-closed in (X, τ, σ), by Theorem 2.17, we get the result. □

Corollary 2.19. *Let (X, τ, σ) be an s-Lindelöf space and $A \subseteq X$. If A is a u-closed subset in (X, τ, σ), then A is an s-Lindelöf subset of (X, τ, σ).*

Proof. Since u-ω-closed sets are q-ω-closed in (X, τ, σ), by Theorem 2.17, we get the result. □

Theorem 2.20. *Let (X, τ, σ) be a p-Lindelöf bitopological space and A be a q-ω-closed nonempty proper subset in (X, τ, σ). Then A is a Lindelöf subset of (X, τ) or A is a Lindelöf subset of (X, σ).*

Proof. Since A is q-ω-closed, $X - A = W \cup M$ where $W \in \tau_\omega$ and $M \in \sigma_\omega$. Since A is a proper subset of X, $W \neq \emptyset$ or $M \neq \emptyset$.

Case 1. $W \neq \emptyset$. We show that A is a Lindelöf subset of (X, σ). Let $\mathcal{U}$ be a cover of A with $\mathcal{U} \subseteq \sigma$. Since A is nonempty, then there exists $U_0 \in \mathcal{U}$ such that $U_0 \neq \emptyset$. Thus, $\mathcal{U} \cup \{W, M\}$ is a p-open cover of $(X, \tau_\omega, \sigma_\omega)$, and by Theorem 2.16, it follows that there exists a countable family $\mathcal{A} \subseteq \mathcal{U} \cup \{W, M\}$ which covers X. Let $\mathcal{U}' = \mathcal{A} - \{W, M\}$. Then $\mathcal{U}'$ is countable, $\mathcal{U}' \subseteq \mathcal{U}$, and $\mathcal{U}'$ covers A. It follows that A is a Lindelöf subset of (X, σ).

Case 2. $M \neq \emptyset$. Similar to that used in Case 1, we can show that A is a Lindelöf subset of (X, τ). □

Corollary 2.21. *Let (X, τ, σ) be a p-Lindelöf bitopological space and A be a q-closed proper subset in (X, τ, σ). Then A is a Lindelöf subset of (X, τ) or A is a Lindelöf subset of (X, σ).*

Proof. Since every q-closed is q-ω-closed, then by Theorem 2.20, we get the result. □

Corollary 2.22. *Let (X, τ, σ) be a p-Lindelöf bitopological space and A be a u-closed proper subset in (X, τ, σ). Then A is a Lindelöf subset of (X, τ) or A is a Lindelöf subset of (X, σ).*

Proof. Since every u-closed is q-ω-closed, then by Theorem 2.20, we get the result. □

3. u-ω-continuous and q-ω-continuous functions

Definition 3.1. *Let $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then*

(a) *f is said to be p-ω-continuous if both $f\colon (X, \tau_1) \to (Y, \sigma_1)$ and $f\colon (X, \tau_2) \to (Y, \sigma_2)$ are ω-continuous.*
(b) *f is said to be u-ω-continuous if for each $A \in u(\sigma_1, \sigma_2)$, $f^{-1}(A) \in u\omega(\tau_1, \tau_2)$.*
(c) *f is said to be q-ω-continuous if for each $A \in q(\sigma_1, \sigma_2)$, $f^{-1}(A) \in q\omega(\tau_1, \tau_2)$.*

Theorem 3.2. *Let $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then*

(a) *f is p-ω-continuous if and only if $f\colon (X, (\tau_1)_\omega, (\tau_2)_\omega) \to (Y, \sigma_1, \sigma_2)$ is p-continuous.*
(b) *f is u-ω-continuous if and only if $f\colon (X, (\tau_1)_\omega, (\tau_2)_\omega) \to (Y, \sigma_1, \sigma_2)$ is u-continuous.*
(c) *f is q-ω-continuous if and only if $f\colon (X, (\tau_1)_\omega, (\tau_2)_\omega) \to (Y, \sigma_1, \sigma_2)$ is q-continuous.*

Proof. (a) follows from the definitions and Proposition 1.3.
(b) and (c) follow directly from the definitions. □

Proposition 3.3. *Let $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then the following conditions are equivalent:*

(a) *The function f is q-continuous.*
(b) *The inverse image of every u-open set is q-open.*
(c) *The inverse image of every u-closed set is q-closed.*
(d) *For each $x \in X$ and each u-open set $V \subseteq Y$ containing $f(x)$, there is a q-open set $U \subseteq X$ containing x such that $f(U) \subseteq V$.*
(e) *The inverse image of every q-closed set is q-closed.*
(f) *For every $A \subseteq X$, $f(q\text{-}Cl_{(\tau_1,\tau_2)}(A)) \subseteq q\text{-}Cl_{(\sigma_1,\sigma_2)}(f(A))$.*
(g) *For every $B \subseteq Y$, $q\text{-}Cl_{(\tau_1,\tau_2)}(f^{-1}(B)) \subseteq f^{-1}(q\text{-}Cl_{(\sigma_1,\sigma_2)}(B))$.*

Proof. (a) $\rightarrow$ (b) Let $A \in u(\sigma_1, \sigma_2)$. Then $A \in q(\sigma_1, \sigma_2)$ and by (a), $f^{-1}(A)$ is q-open.

(b) $\rightarrow$ (c) Let H be be a u-closed set in Y. Then by (b), $f^{-1}(Y - H) = X - f^{-1}(H)$ is q-open and hence $f^{-1}(H)$ is q-closed.

(c) $\rightarrow$ (d) Let $x \in X$ and $V \subseteq Y$ be a u-open set with $f(x) \in V$. Then V is q-open and $Y - V$ is q-closed. By (c), we have $X - f^{-1}(V)$ is q-closed and $f^{-1}(V)$ is q-open. Put $U = f^{-1}(V)$. Then, U is q-open, $x \in U$, and $f(U) \subseteq V$.

(d) $\rightarrow$ (e) Let $C \subseteq Y$ be q-closed. We are going to see that $X - f^{-1}(C)$ is q-open. As $Y - C$ is q-open, there exist $A \in \sigma_1$and $B \in \sigma_2$ such that $Y - C = A \cup B$. Thus for each $x \in X - f^{-1}(C) = f^{-1}(Y - C)$, $f(x) \in A$ (which is u-open) or $f(x) \in B$ (which is u-open) and by (d), there exists a q-open set U_x containing x such that $f(U_x) \subseteq A \subseteq Y - C$ or $f(U_x) \subseteq B \subseteq Y - C$. Since arbitrary union of q-open sets is q-open, it follows that, $X - f^{-1}(C) = \bigcup \left\{U_x : x \in X - f^{-1}(C)\right\}$ is q-open.

(e) $\rightarrow$ (f) Let $A \subseteq X$. Since $q\text{-}Cl_{(\sigma_1,\sigma_2)}(f(A))$ is q-closed in Y, then by (e), $f^{-1}(q\text{-}Cl_{(\sigma_1,\sigma_2)}(f(A)))$ is q-closed in X. Since

$$A \subseteq f^{-1}(q\text{-}Cl_{(\sigma_1,\sigma_2)}(f(A))),$$

then by Proposition 1.16, it follows that

$$q\text{-}Cl_{(\tau_1,\tau_2)}(A) \subseteq f^{-1}(q\text{-}Cl_{(\sigma_1,\sigma_2)}(f(A))),$$

and hence,

$$f(q\text{-}Cl_{(\tau_1,\tau_2)}(A)) \subseteq q\text{-}Cl_{(\sigma_1,\sigma_2)}(f(A)).$$

(f) $\rightarrow$ (g) Let $B \subseteq Y$. Then by (f),

$$f(q\text{-}Cl_{(\tau_1,\tau_2)}(f^{-1}(B))) \subseteq q\text{-}Cl_{(\sigma_1,\sigma_2)}(f(f^{-1}(B))).$$

Consequently, we have

$$q\text{-}Cl_{(\tau_1,\tau_2)}(f^{-1}(B)) \subseteq f^{-1}(q\text{-}Cl_{(\sigma_1,\sigma_2)}(B)).$$

(g) $\rightarrow$ (a) Let U be q-open in Y. We show that

$$X - f^{-1}(U) = f^{-1}(Y - U)$$

is q-closed in X. By (g), it follows that

$$q\text{-}Cl_{(\tau_1,\tau_2)}(f^{-1}(Y - U)) \subseteq f^{-1}(q\text{-}Cl_{(\sigma_1,\sigma_2)}(Y - U)).$$

Since U is q-open in Y, then $Y - U$ is q-closed in X and $q\text{-}Cl_{(\sigma_1,\sigma_2)}(Y - U) = Y - U$. Therefore, $q\text{-}Cl_{(\tau_1,\tau_2)}(f^{-1}(Y - U)) \subseteq f^{-1}(Y - U)$ and hence $f^{-1}(Y - U)$ is q-closed in X. □

Corollary 3.4. *Let $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then the following conditions are equivalent.*

(a) *The function f is q-ω-continuous.*

(b) *The inverse image of every u-open set is q-ω.*
(c) *The inverse image of every u-closed set is q-ω-closed.*
(d) *For each $x \in X$ and each u-open set $V \subseteq Y$ containing $f(x)$, there is an q-ω-open set $U \subseteq X$ containing x such that $f(U) \subseteq V$.*
(e) *The inverse image of every q-closed is q-ω-closed.*
(f) *For every $A \subseteq X$,*

$$f(q\text{-}Cl_{(\tau_1)_\omega,(\tau_2)_\omega)}(A)) \subseteq q\text{-}Cl_{((\sigma_1)_\omega,(\sigma_2)_\omega)}(f(A)).$$

(g) *For every $B \subseteq Y$,*

$$q\text{-}Cl_{(\tau_1)_\omega,(\tau_2)_\omega)}(f^{-1}(B)) \subseteq f^{-1}(q\text{-}Cl_{((\sigma_1)_\omega,(\sigma_2)_\omega)}(B)).$$

Proof. By Theorem 3.2 (c), $f\colon (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is q-ω-continuous if and only if $f\colon (X,(\tau_1)_\omega,(\tau_2)_\omega) \to (Y,\sigma_1,\sigma_2)$ is q-continuous. Thus, applying Proposition 3.3 on $f\colon (X,(\tau_1)_\omega,(\tau_2)_\omega) \longrightarrow (Y,\sigma_1,\sigma_2)$, we get the result. □

Theorem 3.5. *Let $f\colon (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a function.*

(a) *If f is p-continuous, then it is p-ω-continuous.*
(b) *If f is p-ω-continuous, it is u-ω-continuous.*
(c) *If f is u-continuous, it is u-ω-continuous.*
(d) *If f is u-ω-continuous, it is q-ω-continuous.*
(e) *If f is u-continuous, it is q-continuous.*
(f) *If f is q-continuous, it is q-ω-continuous.*

Proof. (a) follows from the definitions and Proposition 1.4.

(b) Suppose that f is p-ω-continuous. Let $U \in u(\sigma_1,\sigma_2)$. If $U \in \sigma_i, i = 1,2$, then $f^{-1}(U) \in (\tau_i)_\omega \subseteq u\omega(\tau_1,\tau_2)$.

(c) Suppose that is u-continuous. Let $U \in u(\sigma_1,\sigma_2)$. Then $f^{-1}(U) \in u(\tau_1,\tau_2)$. Since $u(\tau_1,\tau_2) \subseteq u\omega(\tau_1,\tau_2)$, then $f^{-1}(U) \in u\omega(\tau_1,\tau_2)$.

(d) Suppose that f is u-ω-continuous. Let $U \in u(\sigma_1,\sigma_2)$. Then $f^{-1}(U) \in u\omega(\tau_1,\tau_2)$. Since $u\omega(\tau_1,\tau_2) \subseteq q\omega(\tau_1,\tau_2)$, then $f^{-1}(U) \in q\omega(\tau_1,\tau_2)$.

(e) Suppose that f is u-continuous. Let $U \in u(\sigma_1,\sigma_2)$. Since f is u-continuous, then $f^{-1}(U) \in u(\tau_1,\tau_2)$. $u(\tau_1,\tau_2) \subseteq q(\tau_1,\tau_2)$, then $f^{-1}(U) \in q(\tau_1,\tau_2)$.

(f) Suppose that $f\colon (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is q-continuous. Let $U \in u(\sigma_1,\sigma_2)$. Then $f^{-1}(U) \in q(\tau_1,\tau_2)$. Since $q(\tau_1,\tau_2) \subseteq q\omega(\tau_1,\tau_2)$, then $f^{-1}(U) \in q\omega(\tau_1,\tau_2)$. □

Each of the implications in (a), (c), and (f) in Theorem 3.5 is not reversible, as the following example shows.

Example 3.6. *The function $f\colon (\mathbb{R},\tau_{ind},\tau_{ind}) \to (\mathbb{R},\tau_{cof},\tau_{cof})$, where $f(x) = x$ for all $x \in \mathbb{R}$ is p-ω-continuous, u-ω-continuous, q-ω-continuous, not p-continuous, not u-continuous, and not q-continuous.*

Implication (b) of Theorem 3.5 is not reversible, as the following example shows.

Example 3.7. *The function $f\colon (\mathbb{R}, \tau_{lr}, \tau_{rr}) \to (\mathbb{R}, \tau_{rr}, \tau_{lr})$, where $f(x) = x$ for all $x \in \mathbb{R}$ is u-continuous and hence it is u-ω-continuous. However,it is not p-ω-continuous.*

Implications (d) and (e) of Theorem 3.5 is not reversible, as the following example shows.

Example 3.8. *Consider the function $f\colon (\mathbb{R}, \tau_{lr}, \tau_{rr}) \to (\mathbb{R}, \tau_{rr}, \tau_{rr})$, where $f(x) = x^2$. Let $U \in u(\tau_{rr}, \tau_{rr}) = \tau_{rr}$. Then $f^{-1}(U) = (-\infty, -\sqrt{a}) \cup (\sqrt{a}, \infty)$ if $U = (a, \infty)$ with $a \geq 0$, $f^{-1}(U) = \mathbb{R}$ if $U = (a, \infty)$ with $a < 0$, and $f^{-1}(U) = \emptyset$ if $U = \emptyset$. In all cases, $f^{-1}(U) \in q(\tau_{lr}, \tau_{rr})$. Therefore, f is q-continuous and hence it is q-ω-continuous. On the other hand, since $(1, \infty) \in u(\tau_{rr}, \tau_{rr})$, while $f^{-1}((1, \infty)) = (-\infty, -1) \cup (1, \infty) \notin u\omega(\tau_{lr}, \tau_{rr})$, it follows that f is not u-ω-continuous and hence it is not u-continuous.*

The following diagram summarizes the implications among the known and introduced continuity concepts.

p-continuous	$\longrightarrow$	u-continuous	$\longrightarrow$	q-continuous
$\downarrow$		$\downarrow$		$\downarrow$
p-ω-continuous	$\longrightarrow$	u-ω-continuous	$\longrightarrow$	q-ω-continuous

It is known that the composition of two u-continuous functions is u-continuous. The following result says that the composition of two q-continuous functions is q-continuous.

Theorem 3.9. *Consider two functions $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g\colon (Y, \sigma_1, \sigma_2) \to (Z, \rho_1, \rho_2)$. If f and g are q-continuous, then $g \circ f$ is q-continuous.*

Proof. Let $U \in q(\rho_1, \rho_2)$. Since g is q-continuous, $g^{-1}(U) \in q(\sigma_1, \sigma_2)$. Since f is q-continuous, $f^{-1}\left(g^{-1}(U)\right) \in q(\tau_1, \tau_2)$. Thus, $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right) \in q(\tau_1, \tau_2)$. □

Theorem 3.10. *Let (X, τ), (Y, σ) be two topological spaces and let $f\colon X \to Y$ be a function. Then the following are equivalent:*

(a) *$f\colon (X, \tau, \tau) \to (Y, \sigma, \sigma)$ is p-ω-continuous.*
(b) *$f\colon (X, \tau, \tau) \to (Y, \sigma, \sigma)$ is u-ω-continuous.*
(c) *$f\colon (X, \tau, \tau) \to (Y, \sigma, \sigma)$ is q-ω-continuous.*
(d) *$f\colon (X, \tau) \to (Y, \sigma)$ is ω-continuous.*

Proof. (a) $\to$ (b) Theorem 3.5 (b).

(b) $\to$ (c) Theorem 3.5 (d).

(c) $\to$ (d) Let $U \in \sigma = \sigma \cup \sigma = u(\sigma, \sigma)$. Then by (c), $f^{-1}(U) \in q\omega(\tau, \tau) = \tau_\omega$. Hence, $f\colon (X, \tau) \to (Y, \sigma)$ is ω-continuous.

(d) $\rightarrow$ (a) Follows directly from the definition of $f\colon (X,\tau,\tau) \rightarrow (Y,\sigma,\sigma)$ is ω-continuous.

The following example shows that the composition of two ω-continuous functions need not to be ω-continuous, in general. Thus, by Theorem 3.10, it follows that the composition of two p-ω-continuous (resp. u-ω-continuous, q-ω-continuous) functions need not be p-ω-continuous (resp. u-ω-continuous, q-ω-continuous), in general. $\square$

Example 3.11. *Let $f\colon (\mathbb{R},\tau_u) \rightarrow (\mathbb{R},\tau_{ind})$ be a function defined by*

$$f(x) = \begin{cases} \sqrt{2}, & \text{if } x \text{ is rational;} \\ 2, & \text{if } x \text{ is irrational.} \end{cases}$$

and $g\colon (\mathbb{R},\tau_{ind}) \rightarrow (\mathbb{R},\tau)$ be a function defined by

$$g(x) = \begin{cases} 2, & \text{if } x \text{ is rational;} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

where $\tau = \{\mathbb{R},\phi,\{1\}\}$. Then f is continuous (and so ω-continuous) and g is ω-continuous. On the other hand, since $f^{-1}(g^{-1}(\{1\})) = f^{-1}(\mathbb{Q}^c) = \mathbb{Q}$ and $\mathbb{Q}$ is not ω-open set in $(\mathbb{R},\tau_u)$, it follows that $g \circ f$ is not ω-continuous.

Theorem 3.12. *Consider two functions $f\colon (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ and $g\colon (Y,\sigma_1,\sigma_2) \rightarrow (Z,\rho_1,\rho_2)$.*

(a) *If f is p-ω-continuous and g is p-continuous, then $g \circ f$ is p-ω-continuous.*
(b) *If f is u-ω-continuous and g is u-continuous, then $g \circ f$ is u-ω-continuous.*
(c) *If f is q-ω-continuous and g is q-continuous, then $g \circ f$ is q-ω-continuous.*

Proof. (a) follows from the definitions and Proposition 1.14.

(b) Let $U \in u(\rho_1,\rho_2)$. Since g is u-continuous, $g^{-1}(U) \in u(\sigma_1,\sigma_2)$. Since f is u-ω-continuous, $f^{-1}\left(g^{-1}(U)\right) \in u\omega(\tau_1,\tau_2)$. Thus, $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right) \in u\omega(\tau_1,\tau_2)$.

(c) Let $U \in u(\rho_1,\rho_2)$. Since g is q-continuous, $g^{-1}(U) \in q(\sigma_1,\sigma_2)$. Thus, $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right) \in q\omega(\tau_1,\tau_2)$. $\square$

Corollary 3.13. *Let $f\colon (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ and $g\colon (Y,\sigma_1,\sigma_2) \rightarrow (Z,\rho_1,\rho_2)$ be two functions. If f is q-ω-continuous and g is u-continuous, then $g \circ f$ is q-ω-continuous.*

Proof. Follows from Theorem 3.5 (e) and Theorem 3.12 (c). $\square$

Theorem 3.14. *Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be two bitopological spaces, A a nonempty subset of X, and $f\colon (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be a function.*

Let $g\colon (A, (\tau_1)_{|A}, (\tau_2)_{|A}) \to (Y, \sigma_1, \sigma_2)$ *be the restriction of* f *to* A *defined by* $g(x) = f(x)$ *for every* $x \in A$. *Then*

(a) *If* f *is* p-ω-*continuous, then* g *is* p-ω-*continuous.*
(b) *If* f *is* u-ω-*continuous, then* g *is* u-ω-*continuous.*
(c) *If* f *is* q-ω-*continuous, then* g *is* q-ω-*continuous.*

Proof. (a) Let $i \in \{1, 2\}$ and $U \in \sigma_i$. Since f is p-ω-continuous, then $f^{-1}(U) \in (\tau_i)_\omega$. So, $f^{-1}(U) \cap A \in ((\tau_i)_\omega)_{|A}$. Therefore, by Proposition 1.1 (c), $g^{-1}(U) \in \left((\tau_i)_{|A}\right)_\omega$. Hence, g is p-ω-continuous.

(b) Let $U \in u(\sigma_1, \sigma_2)$. Since f is u-ω-continuous, then $f^{-1}(U) \in (\tau_1)_\omega \cup (\tau_2)_\omega$. So, $f^{-1}(U) \cap A \in ((\tau_1)_\omega)_{|A} \cup ((\tau_1)_\omega)_{|A}$. Therefore, by Proposition 1.1 (c), $g^{-1}(U) \in \left((\tau_1)_{|A}\right)_\omega \cup \left((\tau_2)_{|A}\right)_\omega$. Hence, g is u-ω-continuous.

(c) Let $U \in u(\sigma_1, \sigma_2)$. Since f is q-ω-continuous, then

$$f^{-1}(U) \in q\omega(\tau_1, \tau_2).$$

So, there exist $B \in (\tau_1)_\omega$ and $C \in (\tau_2)_\omega$ such that $f^{-1}(U) = B \cup C$. Then $g^{-1}(U) = (B \cap A) \cup (C \cap A)$ with $B \cap A \in ((\tau_1)_\omega)_{|A}$ and $C \cap A \in ((\tau_2)_\omega)_{|A}$. Therefore, by Proposition 1.1 (c), it follows that $g^{-1}(U) \in q\omega((\tau_1)_{|A}, (\tau_1)_{|A})$. This shows that g is q-ω-continuous. □

Theorem 3.15. *Let* $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be* q-ω-*continuous and surjective. If* (X, τ_1, τ_2) *is* s-*Lindelöf, then* (Y, σ_1, σ_2) *is* s-*Lindelöf.*

Proof. Suppose that (X, τ_1, τ_2) is s-Lindelöf. Let $\{W_\alpha : \alpha \in \Delta\}$ be a $\sigma_1\sigma_2$-open cover of (Y, σ_1, σ_2). Since f is q-ω-continuous, then for each $\alpha \in \Delta$, $f^{-1}(W_\alpha) \in q\omega(\tau_1, \tau_2)$. Since,

$$X = f^{-1}(Y) = f^{-1}\left(\bigcup_{\alpha\in\Delta} W_\alpha\right) = \bigcup_{\alpha\in\Delta} f^{-1}(W_\alpha)$$

then, $\{f^{-1}(W_\alpha) : \alpha \in \Delta\}$ is a cover of X consisting of elements of $q\omega(\tau_1, \tau_2)$. Thus, by Theorem 2.15, there exists a countable set $\Delta' \subseteq \Delta$ such that $\{f^{-1}(W_\alpha) : \alpha \in \Delta'\}$ covers X. Since f is surjective, then $Y = f(X)$. Accordingly,

$$Y = f(X) = f\left(\bigcup_{\alpha\in\Delta'} f^{-1}(W_\alpha)\right) = \bigcup_{\alpha\in\Delta'} f(f^{-1}(W_\alpha)) \subseteq \bigcup_{\alpha\in\Delta'} W_\alpha.$$

Thus we obtain $\{W_\alpha : \alpha \in \Delta'\}$ as a countable subcover of $\{W_\alpha : \alpha \in \Delta\}$. This shows that (Y, σ_1, σ_2) is s-Lindelöf. □

Corollary 3.16. *Let* $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a surjective function. If* f *is* p-*continuous (resp.* u-*continuous,* q-*continuous,* p-ω-*continuous,* u-ω-*continuous) and* (X, τ_1, τ_2) *is* s-*Lindelöf, then* (Y, σ_1, σ_2) *is* s-*Lindelöf.*

Proof. By Theorem 3.5, every p-continuous (resp. u-continuous, q-continuous, p-ω-continuous, u-ω-continuous) is q-ω-continuous. Then by Theorem 3.15, (Y, σ_1, σ_2) is s-Lindelöf. □

Theorem 3.17. *Let $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be p-ω-continuous and surjective. If (X, τ_1, τ_2) is p-Lindelöf, then (Y, σ_1, σ_2) is p-Lindelöf.*

Proof. Suppose that (X, τ_1, τ_2) is p-Lindelöf. Let $\{U_\alpha : \alpha \in \Delta\}$ be a p-open cover of (Y, σ_1, σ_2). Since f is p-ω-continuous and

$$X = f^{-1}(Y) = f^{-1}(\bigcup_{\alpha\in\Delta} W_\alpha) = \bigcup_{\alpha\in\Delta} f^{-1}(W_\alpha)$$

then $\{f^{-1}(W_\alpha) : \alpha \in \Delta\}$ is a $(\tau_1)_\omega\,(\tau_2)_\omega$-open cover of $(X, (\tau_1)_\omega, (\tau_2)_\omega)$. Take $\alpha_1, \alpha_2 \in \Delta$ such that $W_{\alpha_i} \in \sigma_i - \{\emptyset\}$ for $i = 1, 2$. Since f is p-ω-continuous, it follows that $f^{-1}(W_{\alpha_i}) \in (\tau_i)_\omega$ for $i = 1, 2$. Also, since f is surjective, $f^{-1}(W_{\alpha_i}) \neq \emptyset$ for $i = 1, 2$. Therefore, $\{f^{-1}(W_\alpha) : \alpha \in \Delta\}$ is a p-open cover of $(X, (\tau_1)_\omega, (\tau_2)_\omega)$. Since (X, τ_1, τ_2) is p-Lindelöf, then by Theorem 2.16, there exists a countable set $\Delta' \subseteq \Delta$ such that $\{f^{-1}(W_\alpha) : \alpha \in \Delta'\}$ covers X. Again since f is surjective, then $Y = f(X)$. Accordingly,

$$Y = f(X) = f\left(\bigcup_{\alpha\in\Delta'} f^{-1}(W_\alpha)\right) = \bigcup_{\alpha\in\Delta'} f(f^{-1}(W_\alpha)) \subseteq \bigcup_{\alpha\in\Delta'} W_\alpha.$$

Thus we obtain $\{W_\alpha : \alpha \in \Delta'\}$ as a countable subcover of $\{W_\alpha : \alpha \in \Delta\}$. This shows that (Y, σ_1, σ_2) is p-Lindelöf. □

Corollary 3.18. [10] *Let $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be p-continuous and surjective. If (X, τ_1, τ_2) is p-Lindelöf, then (Y, σ_1, σ_2) is p-Lindelöf.*

Proof. This follows from Theorem 3.5 (a) and Theorem 3.17. □

References

[1] S. Al Ghour, *Some generalizations of paracompactness*, Missouri Journal of Mathematical Sciences, **18** (2006), 64–77.

[2] A. Al-Omari and M. Noorani, *Regular generalized ω-closed sets*, International Journal of Mathematics and Mathematical Sciences, 2007, Art. ID 16292:11 pp.

[3] A. Al-Omari and M. Noorani, *Contra-ω-continuous and almost contra-ω-continuous*, International Journal of Mathematics and Mathematical Sciences, 2007, Art. ID 40469:13 pp.

[4] K. Al-Zoubi, *On generalized ω-closed sets*, International Journal of Mathematics and Mathematical Sciences, **13** (2005), 2011–2021.

[5] K. Al-Zoubi and B. Al-Nashef, *The topology of ω-open subsets*, Al-Manarah Journal, **9** (2003), 169–179.

[6] K. Rao and D. Narasimhan, *Semi star generalized ω-closed sets in bitopological spaces*, Int. J. Contemp. Math. Sci., **4** (2009), 587–595.

[7] M. Datta, *Projective bitopological spaces*, J. Austral. Math. Soc., **13** (1972), 327–334.

[8] R. Engelking, *General Topology*, Hilderman, Berlin, (1989).
[9] P. Fletcher, B. Hoyle, and C. Patty, *The comparison of topologies*, Duke Math. J., **36** (1969), 325–331.
[10] A. Fora and H. Hdeib, *On pairwise Lindelöf spaces*, Rev. Colombiana Mat., **17** (1983), 37–57.
[11] H. Hdeib, *ω-continuous functions*, Dirasat Journal, **16** (1989), 136–153.
[12] H. Hdeib, *ω-closed mappings*, Revista Colombiana de Mathematicas, **16** (1982), 65–78.
[13] J. Kelly, *Bitopological spaces*, Proc. London Math. Soc., **13** (1963), 71–89.
[14] A. Kılıçman and Z. Salleh, *A note on pairwise continuous mappings and bitopological spaces*, Eur. J. Pure Appl. Math., **2** (2009), 325–337.
[15] M. Saegrove, *Pairwise complete regularity and compactification in bitopological spaces*, J. London Math. Soc., **7** (1973), 286–290.
[16] M. Sarsak, *ω-almost Lindelöf spaces*, Questions Answers General Topology, **21** (2003), 27–35.
[17] J. Swart, *Total disconnectedness in bitopological spaces and product bitopological spaces*, Nederl. Akad. Wetensch. Proc. Ser. A 74, Indag. Math., **33** (1971), 135–145.
[18] A. Tallafha, A. Al-Bsoul, and A. Fora, *Countable dense homogeneous bitopological spaces*, Tr. J. Math., **23** (1999), 233–242.

MSC2010: 54A05, 54A10, 54D10

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
E-mail address: algore@just.edu.jo

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
E-mail address: si.alzyoud87@yahoo.com