CATEGORIES WITH STRONG MONOMORPHIC STRONG COIMAGES

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ABSTRACT. Let $SE(\mathcal{C})$ (respectively, $SM(\mathcal{C})$) be the subcategory of a category $\mathcal{C}$ with the same objects and whose morphisms are strong epimorphisms (respectively, strong monomorphisms) of $\mathcal{C}$. In this paper, we give conditions in some categories $\mathcal{C}$ for an object X of pro-$\mathcal{C}$ to be isomorphic to an object of pro-$SE(\mathcal{C})$ (respectively, pro-$SM(\mathcal{C})$). As an application, we give conditions under which objects of pro-categories are stable.

1. Introduction

Let $\mathcal{C}$ be an arbitrary category. In [2], J. Dydak and F. R. Ruiz del Portal gave conditions in some categories for an object X of pro-$\mathcal{C}$ to be isomorphic to an object of pro-$E(\mathcal{C})$ or pro-$M(\mathcal{C})$, where $E(\mathcal{C})$ (respectively, $M(\mathcal{C})$) is the subcategory of $\mathcal{C}$ whose morphisms are epimorphisms (respectively, monomorphisms) of $\mathcal{C}$. In particular, the following results were obtained.

Proposition 1.1. *Let $\mathcal{C}$ be a balanced category with epimorphic images and let $f\colon X \to Y$ be an epimorphism (respectively, monomorphism) of pro-$\mathcal{C}$. If pro-$\mathcal{C}$ is balanced and X is isomorphic to an object of pro-$E(\mathcal{C})$ (respectively, Y is isomorphic to an object of pro-$M(\mathcal{C})$), then Y is isomorphic to an object of pro-$E(\mathcal{C})$ (respectively, X is isomorphic to an object of pro-$M(\mathcal{C})$).*

Proposition 1.2. *Let $\mathcal{C}$ be a balanced category with epimorphic images and let X be an object of pro-$\mathcal{C}$. If pro-$\mathcal{C}$ is balanced, then X is isomorphic to an object of pro-$M(\mathcal{C})$ if and only if there is a monomorphism $f\colon X \to P \in Ob(\mathcal{C})$.*

Let $SE(\mathcal{C})$ (respectively, $SM(\mathcal{C})$) be the subcategory of $\mathcal{C}$ with the same objects and whose morphisms are strong epimorphisms (respectively, strong monomorphisms) of $\mathcal{C}$. Given an object X of pro-$\mathcal{C}$, it is of interest to detect if X is isomorphic to an object of pro-$SE(\mathcal{C})$ or pro-$SM(\mathcal{C})$. Therefore, we will investigate the question: under what conditions is X isomorphic to an object of pro-$SE(\mathcal{C})$ or pro-$SM(\mathcal{C})$? Moreover, stability is an important property of objects of pro-$\mathcal{C}$ and hence we will present conditions which give stability of objects of pro-$\mathcal{C}$ (Corollary 3.11).

2. Preliminaries

First we recall some basic facts about pro-categories. The main reference is [1] and for more details see [3, 5].

Loosely speaking, the pro-category pro-$\mathcal{C}$ of $\mathcal{C}$ is the universal category with directed inverse limits containing $\mathcal{C}$ as a full subcategory. An object of pro-$\mathcal{C}$ is an inverse system in $\mathcal{C}$, denoted by $X = (X_\alpha,\ p_\alpha^\beta,\ A)$, consisting of a directed set A, called the *index set* (from now onward it will be denoted by $I(X)$), of $\mathcal{C}$ objects X_α for each $\alpha \in I(X)$, called the *terms* of X and of $\mathcal{C}$ morphisms $p_\alpha^\beta \colon X_\beta \to X_\alpha$ for each related pair $\alpha < \beta$, called the *bonding morphisms* of X. One requires that if $\alpha < \beta < \gamma$, then $p_\alpha^\gamma = p_\alpha^\beta \circ p_\beta^\gamma$. From now onward the bonding morphism p_α^β for each related pair $\alpha < \beta$ will be denoted by $p(X)_\alpha^\beta$.

If P is an object of $\mathcal{C}$ and X is an object of pro-$\mathcal{C}$, then a morphism $f \colon X \to P$ in pro-$\mathcal{C}$ is the direct limit of $\mathrm{Mor}(X_\alpha, P)$, $\alpha \in I(X)$ and so f can be represented by $g \colon X_\alpha \to P$. Note that the morphism from X to X_α represented by the identity $X_\alpha \longrightarrow X_\alpha$ is called the *projection morphism* and denoted by $p(X)_\alpha$.

If X and Y are two objects in pro-$\mathcal{C}$ with identical index sets, then a morphism $f \colon X \to Y$ is called a *level morphism* if for each $\alpha < \beta$, with f_α and f_β as representations, the following diagram

$$\begin{tikzcd} X_\beta \arrow[r, "f_\beta"] \arrow[d, "p(X)_\alpha^\beta"'] & Y_\beta \arrow[d, "p(Y)_\alpha^\beta"] \\ X_\alpha \arrow[r, "f_\alpha"] & Y_\alpha \end{tikzcd}$$

commutes.

The next result is well-known [3].

Theorem 2.1. *For any morphism $f \colon X \to Y$ of pro-$\mathcal{C}$ there exists a level morphism $f' \colon X' \to Y'$ and isomorphisms $i \colon X \to X'$, $j \colon Y' \to Y$ such that $f = j \circ f' \circ i$ and $I(X')$ is a cofinite directed set. Moreover, the bonding morphisms of X' (respectively, Y') are chosen from the set of bonding morphisms of X (respectively, Y).*

Recall that a morphism $f \colon X \to Y$ of $\mathcal{C}$ is called a *monomorphism* if $f \circ g = f \circ h$ implies $g = h$ for any two morphisms $g, h \colon Z \to X$. A morphism $f \colon X \to Y$ of $\mathcal{C}$ is called an *epimorphism* if $g \circ f = h \circ f$ implies $g = h$ for any two morphisms $g, h \colon Y \to Z$.

Also, recall that an object X in pro-$\mathcal{C}$ is called *stable* if there is an isomorphism $f\colon X \to P$ where P is an object of $\mathcal{C}$.

If f is a morphism of $\mathcal{C}$, then its domain will be denoted by $D(f)$ and its range will be denoted by $R(f)$. Hence, $f\colon D(f) \to R(f)$.

Next, we recall definitions of strong monomorphism and strong epimorphism [1].

Definition 2.2. A morphism $f\colon X \to Y$ in pro-$\mathcal{C}$ is called a *strong monomorphism* (*strong epimorphism*, respectively) if for every commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

with P, Q objects in $\mathcal{C}$, there is a morphism $h\colon Y \to P$ such that $h \circ f = a$ ($g \circ h = b$, respectively).

Note that if X and Y are objects of $\mathcal{C}$, then $f\colon X \to Y$ is a strong monomorphism (strong epimorphism, respectively) if and only if f has a left inverse (a right inverse, respectively).

The next lemma is proved in [1].

Lemma 2.3. *If $g \circ f$ is a strong monomorphism (strong epimorphism, respectively), then f is a strong monomorphism (g is a strong epimorphism, respectively).*

The following theorem is a characterization of isomorphisms in pro-$\mathcal{C}$ [1].

Theorem 2.4. *Let $f\colon X \to Y$ be a morphism in pro-$\mathcal{C}$. The following statements are equivalent.*

(i) *f is an isomorphism.*
(ii) *f is a strong monomorphism and an epimorphism.*

Proposition 2.5. *Suppose that*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{g} & T \end{array}$$

is a commutative diagram in a category $\mathcal{C}$. If f is an epimorphism and g is a strong monomorphism, then there is a unique morphism $h\colon Y \to Z$ such that $h \circ f = a$ and $g \circ h = b$.

Proof. Note that it suffices to prove the existence of h since f is an epimorphism. Also, it suffices to prove that $a = h \circ f$. Indeed, $(g \circ h) \circ f = g \circ (h \circ f) = g \circ a = b \circ f$, so that $g \circ h = b$ as f is an epimorphism.

Since g is a strong monomorphism, there is a morphism $u \colon T \to Z$ such that $u \circ g = \mathrm{id}_Z$. Now let $h = u \circ b$. Therefore, $h \circ f = (u \circ b) \circ f = u \circ (b \circ f) = u \circ (g \circ a) = (u \circ g) \circ a = \mathrm{id}_Z \circ a = a$. □

Corollary 2.6. *Suppose that*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ Z \arrow[r, "g"] & T \end{tikzcd}$$

is a commutative diagram in pro-$\mathcal{C}$. If f is an epimorphism and g is a strong monomorphism, then there is a unique morphism $h \colon Y \to Z$ such that $h \circ f = a$ and $g \circ h = b$.

3. Categories with Strong Monomorphic Strong Coimages

For the following definition, see [4].

Definition 3.1. $\mathcal{C}$ is a *category with coimages* if every morphism f of $\mathcal{C}$ factors as $f = u \circ g$ so that g is an epimorphism and this factorization is universal among such factorizations, that is, given another factorization $f = v \circ h$ with h being an epimorphism there is $t \colon D(v) \to D(u)$ such that $t \circ h = g$ and $u \circ t = v$.

$\mathcal{C}$ is a *category with monomorphic coimages* if it is a category with coimages and u in the universal factorization $f = u \circ g$ is a monomorphism.

Definition 3.2. $\mathcal{C}$ is a *category with strong coimages* if every morphism f of $\mathcal{C}$ factors as $f = u \circ g$ so that g is a strong epimorphism and this factorization is universal among such factorization, that is, given another factorization $f = v \circ h$ with h being a strong epimorphism there is $t \colon D(v) \to D(u)$ such that $t \circ h = g$ and $u \circ t = v$.

Definition 3.3. $\mathcal{C}$ is a *category with strong monomorphic strong coimages* if it is a category with strong coimages and u in the universal factorization $f = u \circ g$ is a strong monomorphism.

Lemma 3.4. *Let $\mathcal{C}$ be any category. Then the following conditions on $\mathcal{C}$ are equivalent:*

(i) *$\mathcal{C}$ is a category with strong monomorphic strong coimages.*
(ii) *Any morphism f factors as $f = u \circ g$ so that g is a strong epimorphism and u is a strong monomorphism. Given another factorization $f = v \circ h$ with h being a strong epimorphism and v a*

strong monomorphism, there is an isomorphism $t\colon D(v) \to D(u)$ *such that* $t \circ h = g$ *and* $u \circ t = v$.

Proof. (i) $\Rightarrow$ (ii) Any morphism f factors as $f = u \circ g$ such that g is a strong epimorphism and u is a strong monomorphism. Assume that f has another factorization $f = v \circ h$ with h being a strong epimorphism and v a strong monomorphism; there is $t\colon D(v) \to D(u)$ such that $t \circ h = g$ and $u \circ t = v$. Since g is a strong epimorphism, t is a strong epimorphism by Lemma 2.3. Now we show that t is an epimorphism. Suppose that $a, b\colon D(u) \to Z$ are two morphisms such that $a \circ t = b \circ t$. Since t is a strong epimorphism, there is a morphism $h\colon D(u) \to D(v)$ such that $t \circ h = \mathrm{id}_{D(u)}$. Therefore, $a = a \circ \mathrm{id}_{D(u)} = a \circ t \circ h = b \circ t \circ h = b \circ \mathrm{id}_{D(u)} = b$. Hence, t is an epimorphism. Since v is a strong monomorphism, t is a strong monomorphism by Lemma 2.3. Now applying Proposition 2.5 to the following diagram

$$\begin{tikzcd} D(v) \arrow[r, "t"] \arrow[d, "\mathrm{id}_{D(v)}"'] & D(u) \arrow[d, "\mathrm{id}_{D(u)}"] \\ D(v) \arrow[r, "t"] & D(u) \end{tikzcd}$$

shows that t is an isomorphism.

(ii) $\Rightarrow$ (i) Any morphism f factors as $f = u \circ g$ such that g is a strong epimorphism and u is a strong monomorphism. Now we show that the factorization is universal. Assume that $f = v \circ h$ is another factorization with h a strong epimorphism. Then v can be factored as $v = b \circ a$ where a is a strong epimorphism and b is a strong monomorphism; there is an isomorphism c such that $c \circ a \circ h = g$ and $u \circ c = b$. Let $t = c \circ a$. Hence, the result holds. □

Corollary 3.5. *Any morphism* f *of a category* $\mathcal{C}$ *with strong monomorphic strong coimages has a unique, up to isomorphism, factorization into a composition* $f = u \circ g$ *where* g *is a strong epimorphism and* u *is a strong monomorphism.*

We write this unique factorization as $f = SM(f) \circ SE(f)$.

The range of $SE(f)$ (which is the domain of $SM(f)$) will be called the *coimage* of f and denoted by $\mathrm{coim}(f)$.

Theorem 3.6. *Let* $\mathcal{C}$ *be a category with strong monomorphic strong coimages. Let* $f\colon X \to Y$ *be a level morphism in pro-*$\mathcal{C}$. *Then there exist level morphisms* $g\colon X \to Z$ *and* $h\colon Z \to Y$ *such that* $g_\alpha = SE(f_\alpha)$, $h_\alpha = SM(f_\alpha)$ *for each* $\alpha \in I(X)$ *and* $f = h \circ g$. *Moreover, if* f *is an isomorphism, then both* h *and* g *are isomorphisms.*

Proof. First note that we have $f_\alpha \circ p(X)^\beta_\alpha = p(Y)^\beta_\alpha \circ f_\beta$ for $\beta > \alpha$. Since $\mathcal{C}$ is a category with strong monomorphic strong coimages, we have $SM(f_\alpha) \circ SE(f_\alpha) \circ p(X)^\beta_\alpha = p(Y)^\beta_\alpha \circ SM(f_\beta) \circ SE(f_\beta)$. This implies that the following diagram

$$\begin{tikzcd}
X_\beta \arrow[r, "SE(f_\beta)"] \arrow[d, "SE(f_\alpha)\circ p(X)^\beta_\alpha"'] & \mathrm{coim}(f_\beta) \arrow[d, "p(Y)^\beta_\alpha \circ SM(f_\beta)"] \\
\mathrm{coim}(f_\alpha) \arrow[r, "SM(f_\alpha)"] & Y_\alpha
\end{tikzcd}$$

is commutative in pro-$\mathcal{C}$ with $SE(f_\beta)$ a strong epimorphism and $SM(f_\alpha)$ a strong monomorphism. Thus, there is a unique morphism $v\colon \mathrm{coim}(f_\beta) \to \mathrm{coim}(f_\alpha)$ by Corollary 2.6. Put $Z_\alpha = \mathrm{coim}(f_\alpha)$ and $p(Z)^\beta_\alpha = v$. Thus, Z is an object of pro-$\mathcal{C}$. Also, put $g_\alpha = SE(f_\alpha)$ and $h_\alpha = SM(f_\alpha)$ for each $\alpha \in I(X)$. Hence, $f = h \circ g$. Note that if f is an isomorphism, then g is a strong monomorphism and thus it is an isomorphism. Also, if f is an isomorphism, then h is a strong epimorphism and thus it is an isomorphism. Hence, the result holds. □

We denote g by $SE(f)$ and h by $SM(f)$. Therefore, we write f as $f = SM(f) \circ SE(f)$.

Theorem 3.7. *Let $\mathcal{C}$ be a category with strong monomorphic strong coimages. Then for any strong epimorphism (respectively, strong monomorphism) $f\colon X \to Y$ of pro-$\mathcal{C}$, there exists a level morphism $f'\colon X' \to Y'$ and isomorphisms $i\colon X \to X'$, $j\colon Y' \to Y$ such that $f = j \circ f' \circ i$, $I(X')$ is a cofinite directed set, and f'_α is a strong epimorphism (respectively, strong monomorphism) of $\mathcal{C}$ for each $\alpha \in I(Y')$. Moreover, the bonding morphisms of X' (respectively, Y') are chosen from the set of bonding morphisms of X (respectively, Y).*

Proof. By Theorem 2.1, we may consider f being a level morphism and $I(X)$ being cofinite. By Theorem 3.6, we can write f as $f = SM(f) \circ SE(f)$. If f is a strong epimorphism, then $SM(f)$ is a strong epimorphism and hence, it is isomorphism. If f is a strong monomorphism, then $SE(f)$ is a strong monomorphism and hence, it is isomorphism. If f is a strong epimorphism, we can put $f' = SE(f)$, $i = SM(f)$ and $j = id_Y$. Therefore, each f'_α is a strong epimorphism. If f is a strong monomorphism, we can put $f' = SM(f)$, $i = id_X$ and $j = SE(f)$. Therefore, each f'_α is a strong monomorphism. □

Theorem 3.8. *Let $\mathcal{C}$ be a category with strong monomorphic strong coimages. Then pro-$\mathcal{C}$ is a category with strong monomorphic strong coimages.*

Proof. Since every level morphism has the factorization $f = SM(f) \circ SE(f)$, we have that every morphism of pro-$\mathcal{C}$ factors as a composition of a strong epimorphism and a strong monomorphism. Now we show that this factorization is universal. This amounts to show that for any commutative diagram in pro-$\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ Z \arrow[r, "g"] & T \end{tikzcd}$$

in which f and a are strong epimorphism and g is a strong monomorphism, there is $t\colon Y \to Z$ such that the diagram commutes. And this is Corollary 2.6. Hence, the result holds. □

Corollary 3.9. *Let $\mathcal{C}$ be a category with strong monomorphic strong coimages. If $f\colon X \to Y$ is a strong epimorphism of pro-$\mathcal{C}$ and X is isomorphic to an object of pro-$SE(\mathcal{C})$, then Y is isomorphic to an object of pro-$SE(\mathcal{C})$.*

Proof. By Theorem 3.7, we have f_α is a strong epimorphism. But $p(Y)^\beta_\alpha \circ f_\beta = f_\alpha \circ p(X)^\beta_\alpha$ for all $\beta > \alpha$. Since f_α and $p(X)^\beta_\alpha$ are strong epimorphism, we have $p(Y)^\beta_\alpha$ is a strong epimorphism, that is, Y is isomorphic to an object of pro-$SE(\mathcal{C})$. □

The following result is a characterization of objects of pro-$\mathcal{C}$ which are isomorphic to objects of pro-$SM(\mathcal{C})$ for $\mathcal{C}$ a category with strong monomorphic strong coimages.

Corollary 3.10. *Let $\mathcal{C}$ be a category with strong monomorphic strong coimages. Then the following conditions on an object X of pro-$\mathcal{C}$ are equivalent.*

(i) *X is isomorphic to an object of pro-$SM(\mathcal{C})$.*
(ii) *There is a strong monomorphism $f\colon X \to P$ where P is an object of $\mathcal{C}$.*

Proof. (i) ⇒ (ii) Assume that $p(X)^\beta_\alpha$ is a strong monomorphism of $\mathcal{C}$ for each $\beta > \alpha$. Assume that the following diagram

$$\begin{tikzcd} X \arrow[r, "p(X)_\alpha"] \arrow[d, "a"'] & X_\alpha \arrow[d, "b"] \\ P \arrow[r, "g"] & Q \end{tikzcd}$$

is commutative in pro-$\mathcal{C}$ with P, Q objects in $\mathcal{C}$. We may find $\beta \in I(X)$, $\beta > \alpha$ and representative $a_\beta \colon X_\beta \to P$ of a such that the following diagram

$$\begin{tikzcd} X_\beta \arrow[r, "p(X)^\beta_\alpha"] \arrow[d, "a_\beta"'] & X_\alpha \arrow[d, "b"] \\ P \arrow[r, "g"] & Q \end{tikzcd}$$

is commutative. But $p(X)^\beta_\alpha$ is a strong monomorphism of $\mathcal{C}$. Thus, there is $h \colon X_\alpha \to P$ such that $h \circ p(X)^\beta_\alpha = a_\beta$. Therefore, $h \circ p(X)^\beta_\alpha \circ p(X)_\beta = a_\beta \circ p(X)_\beta$, that is, $h \circ p(X)_\alpha = a$. Hence, $p(X)_\alpha$ is a strong monomorphism of pro-$\mathcal{C}$ for each $\alpha \in I(X)$. Let $p(X)_\alpha = f$ and $X_\alpha = P$. Hence, the result holds.

(ii) $\Rightarrow$ (i) Suppose that $f \colon X \to P$ is a strong monomorphism. We may assume that f is a level morphism such that f_α $\alpha \in I(X)$, is a strong monomorphism of $\mathcal{C}$. But for all $\beta > \alpha$, $f_\alpha \circ p(X)^\beta_\alpha = f_\beta$. Hence, $p(X)^\beta_\alpha$ is a strong monomorphism. That is, X is isomorphic to an object of pro-$SM(\mathcal{C})$. $\square$

As an application, in the following corollary, we present conditions under which objects of pro-categories are stable.

Corollary 3.11. *Let $\mathcal{C}$ be a category with strong monomorphic strong coimages. Let P and Q be objects of $\mathcal{C}$ and X be an object of pro-$\mathcal{C}$. If $f \colon P \to X$ is a strong epimorphism of pro-$\mathcal{C}$ and $g \colon X \to Q$ is a strong monomorphism of pro-$\mathcal{C}$, then X is stable.*

Proof. Let $h = g \circ f$. h can be factored as $h = SM(h) \circ SE(h)$. Also, $h = g \circ f$ is another factorization of h. By Lemma 3.4, there is an isomorphism $v \colon X \to \mathrm{coim}(h)$ such that $v \circ f = SE(h)$ and $SM(h) \circ v = g$. Hence, X is stable. $\square$

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