A COLLECTION OF LOCI USING TWO FIXED POINTS

Dustin L. Jones

Abstract. The use of dynamic geometry software in constructing loci with various conditions placed on the distances from two fixed points is described, where either (1) the sum of the squares, (2) the difference of the squares, (3) the ratio, or (4) the product of the two distances is constant. The equation of each locus analytically is also derived. In the first three cases, the loci are circles or lines. In the fourth case, the locus is a spiric section described by a quartic equation.

In this article, I discuss the use of the *Construct Locus* tool of *The Geometer's Sketchpad* (GSP), a dynamic geometry software package. This tool is useful in constructing familiar and straightforward loci, such as an ellipse or the perpendicular bisector of a segment; it also allows the user to be creative in defining a locus. This article presents some creative (or at least, atypical) definitions, and the resulting loci that satisfy those definitions.

To begin, I will first describe the method that was used to construct an ellipse using GSP. Based on the definition of an ellipse as the locus of points in the plane such that the sum of the distances from any point on the ellipse to two fixed points is constant, I proceeded as follows. First, identify two fixed points P and Q to serve as foci. Given the constant sum k, construct line segment $\overline{AB}$ such that $AB = k$. Next, place point C on $\overline{AB}$. Due to the dynamic nature of GSP, one may move point C along $\overline{AB}$, but not beyond either endpoint of the segment. Note that $AC + CB = AB$, i.e., the sum of AC and AB equals k, regardless of the location of C. Construct circle centered at P with radius AC, and a circle centered at Q with radius CB. Label the points of intersection of these circles I and J (if they exist). If there are no points of intersection, no such ellipse exists. In this case, to make an ellipse, it is necessary to either select a larger value for k or lessen the distance from P to Q. I and J are both points on the ellipse, since $IP + IQ = JP + JQ = AC + CB = k$. By moving point C along $\overline{AB}$, it is possible to see I and J follow an elliptical path. (See Figure 1.) To construct the ellipse, select point C (the dynamic object) and point I (the object on the locus) and choose *Construct Locus* from the GSP menu. This will result in half an ellipse. To construct the other half, repeat this process by selecting points C and J. By constructing this ellipse with dynamic geometry software, we are now able to investigate this ellipse and make and test conjectures about what would happen to the ellipse if the foci were closer together or farther apart, and how varying the constant k affects the shape of the ellipse.

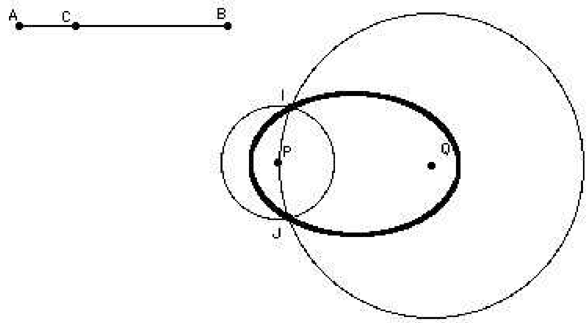


Figure 1. Ellipse construction

What would happen to the locus if the definition were changed slightly? For example, what if we were interested in the set of points in the plane such that the sum of the *squares* of the distances was constant? Or what if operations besides addition and subtraction are considered? In the following paragraphs, I will provide four examples of less typical definitions whose loci can be determined both geometrically and analytically. In general, I will present definitions of loci that are fairly simple to construct using either right triangles or similar triangles.

Proposition 1. The locus of points such that the sum of the squares of the distances to two distinct fixed points is (a nonnegative) constant is a circle.

Geometric construction: First, construct a segment $\overline{AB}$ such that the square of AB is the desired constant value. Construct a circle with diameter $\overline{AB}$, i.e., centered at the midpoint of $\overline{AB}$ and passing through point A. Place a point C on the circle, and construct $\triangle ABC$. Note that because $\overline{AB}$ is the diameter of the circle, ABC is a right triangle with hypotenuse $\overline{AB}$. Therefore, regardless of the location of point C on the circle, $AC^2 + BC^2 = AB^2$, the desired constant value. The lengths AC and BC will be used in the construction of this locus.

Given two fixed points P and Q, construct the circle centered at P with radius AC and the circle centered at Q with radius BC. Label the points of intersection (if they exist) as I and J. Note that the locations of I and J vary with the position of C on the circle with diameter $\overline{AB}$. Use the *Construct Locus* tool to construct the locus for points I and J as C

moves along its circle. Note that this locus, shown in Figure 2, appears to be a circle.

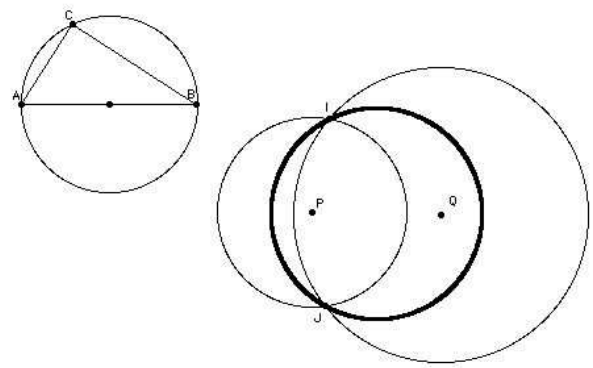


Figure 2. Locus where the sum of squares of the distances to two points is constant

Analytic equation: While this locus looks quite a bit like a circle, its mere appearance is not proof enough that it is indeed a circle. Below is an analytic derivation of the locus. For ease in calculations, a coordinate system will be used that places one of the fixed points P at the origin and the other fixed point Q at the point $(a, 0)$, with $a > 0$.

given that the sum of the squares of the distances from any point (x, y) to the two fixed points is a constant k, $k \geq 0$, we have

$$\left(\sqrt{x^2+y^2}\right)^2 + \left(\sqrt{(x-a)^2+y^2}\right)^2 = k \Rightarrow 2x^2 + 2y^2 - 2ax + a^2 = k$$

$$\Rightarrow x^2 - ax + y^2 = \frac{k-a^2}{2} \Rightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{2k-a^2}{4}.$$

This equation is easily recognized as that of a circle. Note that the center of this circle is the midpoint of PQ, $(a/2, 0)$. For this circle to exist, the square of the radius must be positive. That is to say, we must have $k \geq \frac{a^2}{2}$. When the circle exists, the radius of this circle is $\frac{1}{2}\sqrt{2k-a^2}$.

Proposition 2. The locus of points, such that the difference of the squares of the distances to two distinct fixed points is (a nonnegative) constant, is a line.

Geometric construction: Begin with $\overline{AB}$ such that the square of the distance AB is the desired constant value. Construct a line that is perpendicular to $\overline{AB}$ and that passes through point A. Place a point C on this line, and construct $\triangle ABC$. Note that ABC is a right triangle with hypotenuse $\overline{BC}$, regardless of the location of point C along this line perpendicular to $\overline{AB}$. Therefore, $BC^2 - AC^2 = AB^2$, the desired constant value.

Given two fixed points P and Q, construct the circle centered at P with radius BC and the circle centered at Q with radius AC. Label the points of intersection of these two circles as I and J, when they exist. By constructing the loci of points I and J as C moves along its linear path, we see a line which appears to be perpendicular to the line containing P and Q, as shown in Figure 3.

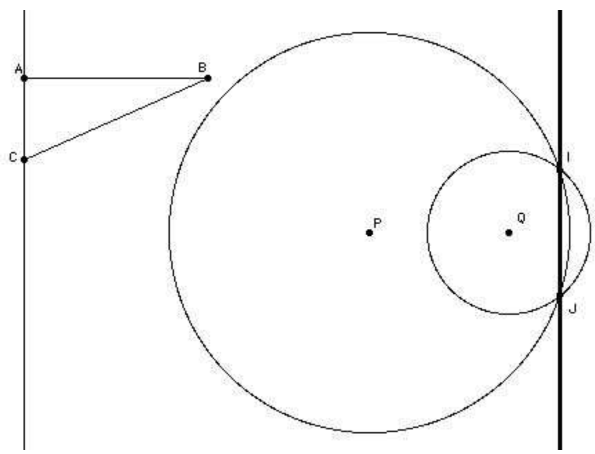


Figure 3. Locus where the difference of squares of the distances to two points is constant

Analytic equation: As in the previous proposition, we will utilize a coordinate system with $P(0,0)$ and $Q(a,0)$, $a > 0$ as fixed points. The locus of points (x,y) such that the difference of the squares of the distances from (x,y) to P and Q is a constant k, $k \geq 0$, is derived as follows:

$$\left(\sqrt{x^2+y^2}\right)^2 - \left(\sqrt{(x-a)^2+y^2}\right)^2 = k \Rightarrow x^2 - (x-a)^2 = k$$

$$\Rightarrow 2ax - a^2 = k \Rightarrow x = \frac{k+a^2}{2a}.$$

Given that our coordinate system places P and Q on the x-axis, we see that this locus is a line perpendicular to that axis. Note that this locus will always exist, as the value $\frac{k+a^2}{2a}$ is well-defined for all values of k and a that are considered. Furthermore, this perpendicular line will intersect line PQ at Q when $k = a^2$, and it will intersect $\overrightarrow{PQ} - \overline{PQ}$ when $k > a^2$. When $k < a^2$, this line will intersect the segment terminating at Q and the midpoint of $\overline{PQ}$. This is because $\frac{k+a^2}{2a} \geq \frac{a^2}{2a} = \frac{a}{2}$ and $\frac{k+a^2}{2a} < \frac{2a^2}{2a} = a$.

Proposition 3. The locus of points, such that the ratio of the distances to two distinct fixed points is (a nonnegative) constant, is a line when the constant $k = 1$, and a circle otherwise.

Case 1. The ratio of the distances to two fixed points is always equal to 1. In this case, the two distances must be equal. The locus of points equidistant from two fixed points is the familiar definition of the perpendicular bisector of a segment that terminates at the two fixed points.

Case 2. The ratio of the distances to two fixed points is a constant not equal to 1.

Geometric construction: Select three distinct points in the plane A, B, and C, so that $\angle BAC$ is acute. Construct $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Fix point D on $\overrightarrow{AB}$ and point E on $\overrightarrow{AC}$ so that the ratio AD/AE is the desired constant k. Construct $\overline{DE}$. Place a point F on $\overrightarrow{AB}$ and construct a line that is parallel to $\overline{DE}$ that passes through F. Label the intersection of this line with $\overrightarrow{AC}$ as point G. Note that $\triangle ADE$ and $\triangle AFG$ are similar, so therefore $\frac{AF}{AG} = \frac{AD}{AE} = k$ regardless of the location of point F on $\overrightarrow{AB}$.

Given two fixed points P and Q, construct the circle centered at P with radius AF and the circle centered at Q with radius AG. Label the points of intersection of these two circles I and J (if they exist). By constructing the loci of points I and J as F is allowed to vary along $\overrightarrow{AB}$, we see that a circle appears, as shown in Figure 4.

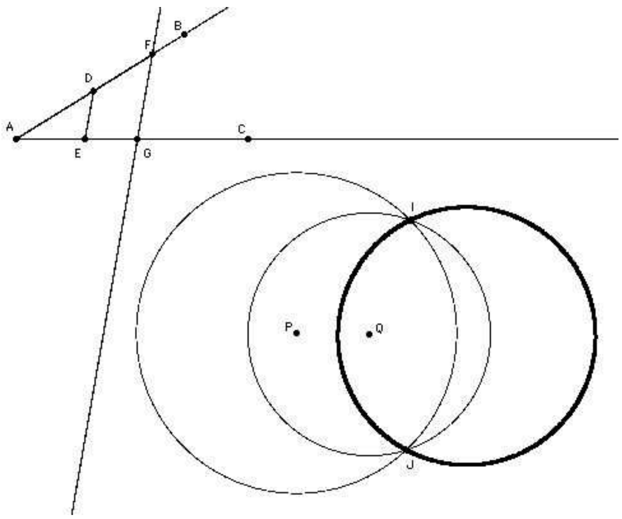


Figure 4. Locus where the ratio of distances to two points is constant

Analytic equation: As was done previously, utilize a coordinate system with fixed points $P(0,0)$ and $Q(a,0)$, $a > 0$. The locus of points (x,y), such that the ratio of distance to P to the distance to Q is a constant k ($k > 0$, $k \neq 1$), is determined as follows. Provided (x,y) is not equal to $(a,0)$,

$$\frac{\sqrt{x^2+y^2}}{\sqrt{(x-a)^2+y^2}} = k \Rightarrow x^2+y^2 = k^2\big((x-a)^2+y^2\big)$$

$$\Rightarrow (1-k^2)x^2 - 2ak^2x + (1-k^2)y^2 = a^2k^2$$

$$\Rightarrow x^2 + \frac{2ak^2}{1-k^2}x + y^2 = \frac{a^2k^2}{1-k^2} \Rightarrow \left(x + \frac{ak^2}{1-k^2}\right)^2 + y^2 = \frac{a^2k^2}{(1-k^2)^2}.$$

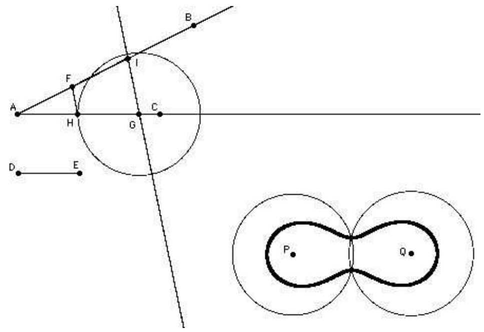
Note that this is a circle centered at $\left(\frac{-ak^2}{1-k^2}, 0\right)$, with radius $\left|\frac{ak}{1-k^2}\right|$. When $0 < k < 1$, we have $0 < k^2 < 1$ and $0 < 1-k^2 < 1$; therefore $\frac{k^2}{1-k^2} > 0$ and $\frac{-ak^2}{1-k^2} < 0$. Thus, the center of the circle is located on $\overrightarrow{QP} - \overline{PQ}$. If

$k > 1$, $k^2 > k^2 - 1 > 0$, then $\frac{k^2}{k^2-1} > 1$. Therefore, $\frac{-ak^2}{1-k^2} = \frac{ak^2}{k^2-1} > a$, and the center of the circle is located on $\overrightarrow{PQ} - \overline{PQ}$. Additionally, it can be shown that the entire locus is contained within the half-plane $\{(x, y)|x > \frac{a}{2}\}$ when $k > 1$ and in the half-plane $\{(x, y)|x < \frac{a}{2}\}$ when $0 < k < 1$.

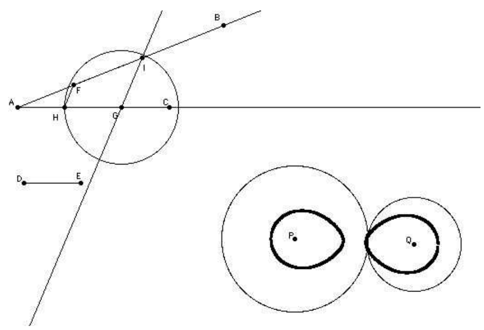
In the previous three propositions, these atypical definitions have yielded fairly well-known loci of circles and lines. In the following proposition, I will show an atypical locus definition that yields a locus that is not a conic section.

Proposition 4. The locus of points, such that the product of the distances to two distinct fixed points is (a nonnegative) constant, is either a closed curve or a pair of closed curves described by a quartic equation. This locus is known as a Cassini oval [1], a type of spiric section [2].

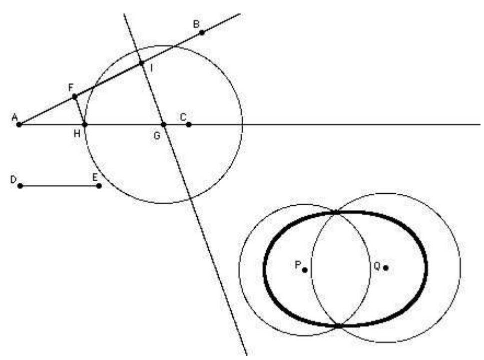
Geometric construction: Select three distinct points in the plane A, B, and C, so that $\angle BAC$ is acute. Construct $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Construct $\overline{DE}$ and locate point F on $\overrightarrow{AB}$ so that the product of DE and AF is the desired constant. Locate point G on $\overrightarrow{AC}$, and construct a circle centered at G with radius DE. This circle will intersect $\overrightarrow{AC}$ in at least one point; identify one of those points as H. Construct $\overline{FH}$ and a line parallel to $\overline{FH}$ that passes through point G. Identify the intersection of this line with $\overrightarrow{AB}$ as point I. Note that by definition, the product of AF and GH is the desired constant value. By the similarity of $\triangle AFH$ and $\triangle AIG$, $\frac{AF}{FI} = \frac{AH}{GH}$, which implies $AF \cdot GH = AH \cdot FI$. This is true regardless of the location of point G on $\overrightarrow{AC}$. Given two fixed points P and Q, construct the circle centered at P with radius AH and the circle centered at Q with radius FI. The locus of the points of intersection of these two circles, as G varies along $\overrightarrow{AC}$, takes on interesting shapes (see Figure 5), dependent upon the distance between P and Q and the desired constant product $AF \cdot DE$.



(a) Locus is the boundary of a nonconvex region



(b) Locus is the union of two disjoint curves



(c) Locus is the boundary of a convex region

Figure 5. Loci where the product of distances to two fixed points is constant

Analytic equation: In this case, we will utilize a coordinate system with fixed points $P(-a, 0)$ and $Q(a, 0)$, $a > 0$. Thus, the distance $PQ = 2a$. The locus of points (x, y) such that the product of the distances to P and Q is a constant k (with $k > 0$) is determined as follows:

$$\begin{aligned}
&\sqrt{(x+a)^2+y^2}\cdot\sqrt{(x-a)^2+y^2}=k\\
&\Rightarrow (x^2-a^2)^2+(x^2+2ax+a^2)y^2+(x^2-2ax+a^2)y^2+y^4=k^2\\
&\Rightarrow x^4-2a^2x^2+a^4+2x^2y^2+2a^2y^2+y^4=k^2.
\end{aligned}$$

In order to further analyze this locus, it is helpful to examine the explicit form of its equation, derived as follows:

$$\begin{aligned}
&y^4+2(x^2+a^2)y^2=k^2-a^4-x^2+2a^2x^2\\
&\Rightarrow (y^2+x^2+a^2)^2=k^2-a^4-x^2+2a^2x^2+(x^2+a^2)^2\\
&\Rightarrow (y^2+x^2+a^2)^2=k^2+4a^2x^2.
\end{aligned}$$

Since the quantity $y^2 + x^2 + a^2$ is always positive, we only need to consider the principal square root, and thus we have the following:

$$y^2+x^2+a^2=\sqrt{k^2+4a^2x^2}\Rightarrow y=\pm\sqrt{-(x^2+a^2)+\sqrt{k^2+4a^2x^2}}.$$

We may write this locus as the union of the graphs of the functions

$$f_1(x)=\sqrt{-(x^2+a^2)+\sqrt{k^2+4a^2x^2}}\text{ and }f_2(x)=-f_1(x).$$

It is readily apparent that this locus is symmetric about the x-axis. Additionally, since $f_1(-x) = f_1(x)$, the locus is also symmetric about the y-axis.

From the figures of different versions of this locus (see Figure 5), we can see that at times, the locus is composed by one closed curve, and at times by two closed curves. To determine the conditions that produce these different results, we can examine where the locus crosses the x-axis (i.e., where $f_1(x) = f_2(x) = 0$).

$$\begin{aligned}
f_1(x)=0\Rightarrow \sqrt{-(x^2+a^2)+\sqrt{k^2+4a^2x^2}}=0\Rightarrow \sqrt{k^2+4a^2x^2}=x^2+a^2\\
\Rightarrow k^2+4a^2x^2=x^4+2a^2x^2+a^4\Rightarrow k^2=(x^2-a^2)^2
\end{aligned}$$

$$\Rightarrow k = x^2 - a^2 \quad \text{or} \quad k = -x^2 + a^2$$

$$\Rightarrow x = \pm\sqrt{a^2 + k} \quad \text{or} \quad x = \pm\sqrt{a^2 - k}.$$

Since a and k are both positive, the roots $\pm\sqrt{a^2 + k}$ always exist. The other two roots $\pm\sqrt{a^2 - k}$ do not exist when $k > a^2$. This is the case in Figure 5a and 5c. Furthermore, these two roots have the same value when $k = a^2$. Figure 5b shows a case in which all four roots exist and are distinct.

These examples are only a few of the many loci that may be explored and examined using dynamic geometry software. Others that may be readily studied are those that involve square roots, such as the locus of points such that the sum of the square roots of the distances to two fixed points is constant.

In conclusion, it should be noted that the use of dynamic geometry software provided more than a means to investigate and discover these loci. Additionally, I was spurred on through my investigations and discoveries to pose new questions and attempt to answer them. Commensurate with my experiences reported in this article, I believe that this tool has the potential to allow and encourage students to conduct worthwhile, in-depth mathematical explorations.

References

1. E. W. Weisstein, "Cassini Ovals" From *Mathworld*–A Wolfram Web Resource. http://mathworld.wolfram.com/CassiniOvals.html.
2. E. W. Weisstein, "Spiric Section" From *Mathworld*–A Wolfram Web Resource. http://mathworld.wolfram.com/SpiricSection.html.

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Dustin L. Jones
Department of Mathematics and Statistics
Sam Houston State University
Huntsville, TX 77341
email: dustinjones@shsu.edu