NETWORKS OF MORPHISMS

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Abstract. This paper deals with some classes of morphisms in a category which we call networks of morphisms. These networks are linked with full reflective subcategories. The notion of saturated subcategory is introduced and studied. Each reflective subcategory is shown to be saturated.

1. Introduction and Terminologies. It is well-known that some elementary types of morphisms in Mathematics play a crucial role. Among these morphisms one can mention injections, surjections, and bijections in the category **SET** of sets and more generally monics, epics, and isomorphisms in an arbitrary category.

The following properties are well-known.

(1) Each isomorphism is both monic and epic.
(2) The composite of two monics (resp. epics, resp. isomorphisms) is a monic (resp. an epic, resp. an isomorphism).
(3) If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ and gf is monic (resp. g is epic), then f is monic (resp. epic).
(4) Let $A \xrightarrow{f_1} B$ and $B \xrightarrow{f_2} C$ be two morphisms and $f_3 = f_2 f_1$. If two among the three morphisms f_1, f_2, f_3 are isomorphisms, then so is the third one.

It is known that several other types of morphisms satisfy at least one of the previous properties.

These observations motivate us to introduce a new concept in order to obtain information concerning reflective subcategories.

Notice that all undefined terms are standard as in [7].

<u>Definition 1.1</u>. Let **C** be a category. Suppose that for each object A, B of **C** a subset $\Gamma(A, B)$ of $hom_{\mathbf{C}}(A, B)$ is given. We consider the following axioms.

(Net_1) For each object A, B of **C** the set of isomorphisms $Isom_{\mathbf{C}}(A, B)$ from A to B is contained in $\Gamma(A, B)$.

(Net_2) If $\theta_1 \in \Gamma(A_1, A_2)$ and $\theta_2 \in \Gamma(A_2, A_3)$, then $\theta_3 = \theta_2\theta_1 \in \Gamma(A_1, A_3)$.

(Net_3) If $\theta_2 \in \Gamma(A_2, A_3)$ and $\theta_3 = \theta_2\theta_1 \in \Gamma(A_1, A_3)$, then $\theta_1 \in \Gamma(A_1, A_2)$.

(Net_4) If $\theta_1 \in \Gamma(A_1, A_2)$ and $\theta_3 = \theta_2\theta_1 \in \Gamma(A_1, A_3)$, then $\theta_2 \in \Gamma(A_2, A_3)$.

(1) We will say that Γ is a *left network of morphisms* (LNM, for short) of $\mathbf{C}$ if it satisfies axioms (Net_i), for each $i \in \{1, 2, 3\}$.

(2) Γ is said to be a *right network of morphisms* (RNM, for short) of $\mathbf{C}$ if it satisfies axioms (Net_i), for each $i \in \{1, 2, 4\}$.

(3) Γ is said to be a *network of morphisms* (NM, for short) of $\mathbf{C}$ if it satisfies axioms (Net_i), for each $i \in \{1, 2, 3, 4\}$.

A morphism $f\colon A \longrightarrow B$ and an object X in a category $\mathbf{C}$ are called *orthogonal* [5], if the mapping $hom_{\mathbf{C}}(f, X)\colon hom_{\mathbf{C}}(B, X) \longrightarrow hom_{\mathbf{C}}(A, X)$ which takes g to gf is bijective. For a class of morphisms Σ (resp. a class of objects $\mathbf{D}$), we denote by $\Sigma^{\perp}$ the class of objects orthogonal to every f in Σ (resp. by $\mathbf{D}^{\perp}$ the class of morphisms orthogonal to all X in $\mathbf{D}$). The internal saturation of Σ is $\Sigma^{\perp\perp}$ and say that Σ is *internally saturated* if $\Sigma^{\perp\perp} = \Sigma$ [1].

As a fundamental example of networks of morphisms (see Example 2.10), we mention $\mathbf{D}^{\perp}$; and consequently each internally saturated class of morphisms defines a network of morphisms.

There is another concept of saturation of a class of morphisms introduced in [6] and [2]; the saturation of a class of morphisms Σ in a category $\mathbf{C}$ consists of the morphisms rendered invertible by the canonical functor from $\mathbf{C}$ to the category of fractions $\mathbf{C}[\Sigma^{-1}]$. This saturation is called *the external saturation* $\widehat{\Sigma}$ of Σ [1]. The class Σ is said to be *externally saturated* if $\widehat{\Sigma} = \Sigma$ [1].

Casacuberta and Frei [1] have remarked that a class of morphisms in a category is externally saturated if and only if it is rendered invertible by some functor, which was already pointed out in [2]. This makes clear the fact that any externally saturated class of morphisms determines a network of morphisms. Note also that if a class of morphisms in a category is internally saturated, then it is externally saturated [1]. The converse does not hold [1].

All subcategories $\mathbf{D}$ of $\mathbf{C}$ considered in this paper are assumed to be full and closed under isomorphisms (i.e., if $A \to B$ is an isomorphism of objects in $\mathbf{C}$, then A is in $\mathbf{D}$ if and only if so is B).

A subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is called *reflective* in $\mathbf{C}$ when the inclusion functor $I: \mathbf{D} \longrightarrow \mathbf{C}$ has a left adjoint $F: \mathbf{C} \longrightarrow \mathbf{D}$. It is well-known that $F: \mathbf{C} \longrightarrow \mathbf{D}$ is a left adjoint functor of I if and only if there is a natural transformation $\mu: 1_{\mathbf{C}} \longrightarrow IF$ such that for each object A of $\mathbf{C}$ the couple $(F(A), \mu_A)$ is a universal to I from A.

If $\mathbf{D}$ is a reflective subcategory of $\mathbf{C}$, we are aiming to give information about the network of morphisms defined by $\mathbf{D}^{\perp}$. This is the main goal of Section 2.

Let $\mathbf{D}$ be a class of objects of a category $\mathbf{C}$. The *saturation* of $\mathbf{D}$ is $\mathbf{D}^{\perp\perp}$. We will say that $\mathbf{D}$ is *saturated* if $\mathbf{D}^{\perp\perp} = \mathbf{D}$. Section 3 is devoted to the study of saturated classes of objects. It is proved that each reflective subcategory of a given category is saturated and the converse does not hold.

In Section 1 many examples of networks of morphisms are provided to illustrate the theory.

2. Fundamental Constructions of Networks of Morphisms. This section is devoted to some fundamental examples of networks of morphisms.

Example 2.1. Let $\mathbf{C}$ be a category.

(1) There are two trivial networks of morphisms of $\mathbf{C}$, namely $Isom_{\mathbf{C}}(-,-)$ and $hom_{\mathbf{C}}(-,-)$.
(2) By letting $Monic(A,B)$ be the set of monics of $\mathbf{C}$ from A to B, we define a left network of morphisms $Monic(-,-)$.
(3) By letting $Epic(A,B)$ be the set of epics of $\mathbf{C}$ from A to B, we define a right network of morphisms $Epic(-,-)$.
(4) - $Monic$ is an LNM of **SET** but not an RNM.
- $Epic$ is an RNM of **SET** but not an LNM.
- Both $Epic$ and $Monic$ are not NM.

Definition 2.2. Let Γ, Γ' be two NM's (resp. LNM's, resp. RNM's) of a category $\mathbf{C}$.

(1) We will say that Γ' is a sub-network (resp. sub-left network, resp. sub-right network) of Γ, if $\Gamma'(A,B) \subseteq \Gamma(A,B)$, for each object A, B of $\mathbf{C}$. We write $\Gamma \subseteq \Gamma'$.

(2) We will say that Γ is equal to Γ', if $\Gamma \subseteq \Gamma'$ and $\Gamma' \subseteq \Gamma$.

Proposition 2.3. Let $\mathbf{C}$ be a category and $(\Gamma_i, i \in I)$ a class of NM's (resp. LNM's, resp. RNM's) of $\mathbf{C}$. We define the intersection Γ of all Γ_i by letting $\Gamma(A,B)$ be the $\cap_{i\in I}\Gamma_i(A,B)$. Then Γ is an NM (resp. LNM, resp. RNM) of $\mathbf{C}$.

The previous proposition leads to the following natural definition.

Definition 2.4. Let $\mathbf{C}$ be a category and Σ a class of morphisms of $\mathbf{C}$. The NM (resp. LNM, resp. RNM) generated by Σ is the intersection of all NM's (resp. LNM's, resp. RNM's) of $\mathbf{C}$ containing Σ.

Example 2.5. In the category **SET** of sets, the network of morphisms $hom_{\mathbf{C}}(-,-)$ is generated by the class of all one-to-one maps and onto maps.

The following two propositions give more examples of networks of morphisms.

Proposition 2.6. Let $\mathbf{C}$, $\mathbf{D}$ be two categories and $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ a covariant functor. Let Γ' be an NM (resp. LNM, resp. RNM) of $\mathbf{D}$. We define $\Gamma = F^{-1}(\Gamma')$ by letting $\Gamma(A,B)$ be the set $\{f \in hom_{\mathbf{C}}(A,B) \mid F(f) \in \Gamma'(F(A), F(B))\}$. Then Γ is an NM (resp. LNM, resp. RNM) of $\mathbf{C}$ (called the inverse image of Γ' by F).

Proposition 2.7. Let $\mathbf{C}$, $\mathbf{D}$ be two categories and $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ a contravariant functor. Let Γ' be an NM (resp. LNM, resp. RNM) of $\mathbf{D}$. We define $\Gamma = F^{-1}(\Gamma')$ by letting $\Gamma(A,B)$ be the set

$$\{f \in hom_{\mathbf{C}}(A,B) \mid F(f) \in \Gamma'(F(B), F(A))\}.$$

(1) If Γ' is an RNM of $\mathbf{D}$, then Γ is an LNM of $\mathbf{C}$.

(2) If Γ' is an LNM of $\mathbf{D}$, then Γ is an RNM of $\mathbf{C}$.

(3) If Γ' is an NM of $\mathbf{D}$, then Γ is an NM of $\mathbf{C}$.

Example 2.8. Let $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ be a functor. By letting $\Gamma(F)(A,B)$ be the set of all arrows $f\colon A \longrightarrow B$ rendered invertible by F, we define a network of morphisms of $\mathbf{C}$, called the network of morphisms associated to F. In fact, $\Gamma(F) = F^{-1}(Isom(-,-))$.

Example 2.8 and [1] yield immediately the following.

Corollary 2.9. Any externally saturated class of morphisms of a category $\mathbf{C}$ is a network of morphisms.

Example 2.10 (A Fundamental Example). Let $\mathbf{C}$ be a category and $\mathbf{D}$ a subcategory of $\mathbf{C}$. Then $\mathbf{D}^{\perp}$ determines a network of morphisms of $\mathbf{C}$.

Proof. The proof is straightforward, but I would like to write it.

Let $\theta_1\colon A \longrightarrow B$, $\theta_2\colon B \longrightarrow C$ be two morphisms of $\mathbf{C}$ and $\theta_3 = \theta_2\theta_1$. Since $hom_{\mathbf{C}}(-, X)$ is a contravariant functor, then

$$hom_{\mathbf{C}}(\theta_3, X) = hom_{\mathbf{C}}(\theta_1, X)hom_{\mathbf{C}}(\theta_2, X).$$

Thus, if two of the three maps $hom_{\mathbf{C}}(\theta_1, X)$, $hom_{\mathbf{C}}(\theta_2, X)$, $hom_{\mathbf{C}}(\theta_3, X)$ are bijective, then so is the third one. Therefore, $\mathbf{D}^{\perp}$ is a network of morphisms of $\mathbf{C}$. In fact, this is an elementary proof. Note also that this may be derived immediately combining the following facts; $\mathbf{D}^{\perp}$ is internally saturated; each internally saturated class of morphisms is externally saturated [1]; and then apply Corollary 2.9.

Question 2.11. Is there a network of morphisms which is not externally saturated?

Particular networks of morphisms are introduced in the following.

Definition 2.12. Let Γ be an NM (resp. LNM, resp. RNM) of a category $\mathbf{C}$ and $F\colon \mathbf{C} \longrightarrow \mathbf{C}$ a functor. Γ is said to be compatible (resp. strongly compatible) with F if $\Gamma \subseteq F^{-1}(\Gamma)$ (resp. $\Gamma = F^{-1}(\Gamma)$).

The following proposition gives some examples of networks of morphisms strongly compatible with a functor.

Proposition 2.13. Let $\mathbf{C}$ be a category and $F\colon \mathbf{C} \longrightarrow \mathbf{C}$ a covariant functor. Suppose that there exists a natural transformation $\mu\colon 1_{\mathbf{C}} \longrightarrow F$. Let Γ be a network of morphisms of $\mathbf{C}$ such that $\mu_A \in \Gamma(A, F(A))$, for each object A of $\mathbf{C}$. Then Γ is strongly compatible with F.

Proof. We have to prove that for each $u \in hom_{\mathbf{C}}(A,B)$, the following equivalence holds:

$$u \in \Gamma(A,B) \Longleftrightarrow F(u) \in \Gamma(F(A),F(B)).$$

Let us first remark that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \mu_A \downarrow & & \downarrow \mu_B \\ F(A) & \xrightarrow{F(u)} & F(B) \end{array}$$

[$\Longrightarrow$]. If we suppose that $u \in \Gamma(A,B)$, then $F(u)\mu_A = \mu_B u \in \Gamma(A,F(B))$. Thus, $F(u) \in \Gamma(F(A),F(B))$, since Γ is an RNM.
[$\Longleftarrow$]. Conversely, suppose that $F(u) \in \Gamma(F(A),F(B))$. Then $F(u)\mu_A \in \Gamma(A,F(B))$; hence, $\mu_B u \in \Gamma(A,F(B))$; so that $u \in \Gamma(A,B)$, since Γ is an LNM.

Remark 2.14. One may easily see that a compatible NM with a functor F is not necessarily strongly compatible. It suffices to remark that the network $Isom_{\mathbf{C}}(-,-)$ is compatible with any functor $F\colon \mathbf{C} \longrightarrow \mathbf{C}$; however the equality $Isom_{\mathbf{C}}(-,-) = F^{-1}(Isom_{\mathbf{C}}(-,-))$ does not hold in general.

3. Reflective Sub-Categories. This section is devoted to shed some light on the network of morphisms associated to a reflective subcategory $\mathbf{D}$ of $\mathbf{C}$.

Notation 3.1. Let $\mathbf{D}$ be a reflective subcategory of $\mathbf{C}$. We denote by $\mathbf{D}^{\perp}(-,-)$ the network of morphisms associated to $\mathbf{D}$ (see Example 2.10).

The following result illustrates some type of "minimality" of the network $\mathbf{D}^{\perp}(-,-)$.

Theorem 3.2. Let $\mathbf{D}$ be a reflective subcategory of $\mathbf{C}$, F a left adjoint functor of the inclusion functor $I\colon \mathbf{D} \longrightarrow \mathbf{C}$ and μ the unit of the adjunction. Let Γ be an LNM of $\mathbf{C}$ such that the following properties hold:

(i) $\Gamma(A,B) \subseteq \mathbf{D}^{\perp}(A,B)$ for each object A, B of $\mathbf{C}$;
(ii) $\mu_A \in \Gamma(A,F(A))$ for each object A of $\mathbf{C}$.

Then $\Gamma = \mathbf{D}^{\perp} = \mathcal{S}(F)$, where $\mathcal{S}(F)$ is the class of morphisms of $\mathbf{C}$ rendered invertible by the functor F.

Since the LNM $\mathbf{D}^{\perp}(-,-)$ of $\mathbf{C}$ satisfies properties (i), (ii) of Theorem 3.2, it will be sufficient to show that $\Gamma = \mathcal{S}(F)$. Thus, it suffices to prove the following lemma.

Lemma 3.3. Under the same assumptions of Theorem 3.2, the following statements are equivalent:

(i) $f \in \Gamma(A, B)$;
(ii) $F(f)$ is an isomorphism;
(iii) $F(f) \in \Gamma(F(A), F(B))$.

Proof.

[(i) $\Longleftrightarrow$ (iii)]. This follows immediately from Proposition 2.13.

[(i) $\Longrightarrow$ (ii)]. Since $\mu_B f \in \Gamma(A, F(B)) \subseteq \mathbf{D}^{\perp}(A, F(B))$, there exists a unique morphism $g\colon F(B) \longrightarrow F(A)$ such that $g(\mu_B f) = \mu_A$. Hence, $(gF(f))\mu_A = \mu_A$. Thus, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu_A} & F(A) \\ & \mu_A \searrow & \\ & & F(A) \end{array}$$

is rendered commutative by the two morphisms $1_{F(A)}$ and $gF(f)$. As $\mu_A \in \mathbf{D}^{\perp}(A, F(A))$, we get $gF(f) = 1_{F(A)}$.

Now, $\mu_B f \in \mathbf{D}^{\perp}(A, F(B))$ and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu_B f} & F(B) \\ & \mu_B f \searrow & \\ & & F(B) \end{array}$$

is rendered commutative by the two morphisms $1_{F(B)}$ and $F(f)g$. This yields immediately $F(f)g = 1_{F(B)}$. Therefore, $F(f)$ is an isomorphism.

[(ii) $\Longrightarrow$ (iii)]. Clearly $Isom(F(A), F(B)) \subseteq \Gamma(F(A), F(B))$.

Next, we give further information about reflective subcategories.

Proposition 3.4. Let $\mathbf{D}$ be a reflective subcategory of $\mathbf{C}$, F a left adjoint functor of the inclusion functor $I\colon \mathbf{D} \longrightarrow \mathbf{C}$ and μ the unit of the adjunction (F, I). Then the following properties hold.

(1) If A is in $\mathbf{D}$, then μ_A is an isomorphism.
(2) For each A, B in $\mathbf{D}$, $\mathbf{D}^{\perp}(A, B) = Isom(A, B)$.
(3) $F(\mu_A) = \mu_{F(A)}$ for each A in $\mathbf{C}$.

Proof.

(1) There exists a unique morphism $g\colon F(A) \longrightarrow A$ such that $g\mu_A = 1_A$. Hence, $(\mu_A g)\mu_A = \mu_A$. It follows that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu_A} & F(A) \\ & \mu_A \searrow & \\ & & F(A) \end{array}$$

is rendered commutative by the two morphisms $1_{F(A)}$ and $\mu_A g$; so that $\mu_A g = 1_{F(A)}$. Therefore, μ_A is an isomorphism.

(2) Let A, B in $\mathbf{D}$ and $f \in \mathbf{D}^{\perp}(A, B)$. By (1), μ_A and μ_B are isomorphisms. Hence, $f = (\mu_B)^{-1} F(f) \mu_C$. On the other hand, $F(f)$ is an isomorphism, by Theorem 3.2. Now clearly, f is an isomorphism.

(3) By letting $f = F(\mu_A)\mu_A = \mu_{F(A)}\mu_A$, the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu_A} & F(A) \\ & f \searrow & \\ & & F(F(A)) \end{array}$$

is rendered commutative by the two morphisms $\mu_{F(A)}$ and $F(\mu_A)$. Since, in addition, $\mu_A \in \mathbf{D}^{\perp}(A, F(A))$, we easily obtain $F(\mu_A) = \mu_{F(A)}$.

Next, we give an interesting result about reflective subcategories.

Theorem 3.5. Let $\mathbf{D}$ be a subcategory of a category $\mathbf{C}$ and $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ a covariant functor. Then the following statements are equivalent.

(1) F is a left adjoint functor of the inclusion functor $I\colon \mathbf{D} \longrightarrow \mathbf{C}$.
(2) There exists a natural transformation $\mu\colon 1_{\mathbf{C}} \longrightarrow IF$ such that the following properties hold:

(i) for each object A in $\mathbf{C}$, $F(\mu_A)$ is an isomorphism.
(ii) for each object A in $\mathbf{D}$, μ_A is an isomorphism.

Proof.

[(1) $\Longrightarrow$ (2)]. Here $\mathbf{D}$ is a reflective subcategory of $\mathbf{C}$. Now, since $\mu_A \in \mathbf{D}^{\perp}(A, F(A))$, we conclude that μ_A is an isomorphism, by Theorem 3.2.

If A is in $\mathbf{D}$, then, according to Proposition 3.4, μ_A is an isomorphism.

[(2) $\Longrightarrow$ (1)]. We are aiming to prove that $(F(A), \mu_A)$ is universal to the inclusion functor $I\colon \mathbf{D} \longrightarrow \mathbf{C}$ from A.

Let C be an object of $\mathbf{D}$ and $f\colon A \longrightarrow C$ a morphism in $\mathbf{C}$. We must prove that there is a unique morphism $\widetilde{f}\colon F(A) \longrightarrow C$ such that $\widetilde{f}\mu_A = f$. Suppose that such a morphism $\widetilde{f}$ exists. Then we have $F(\widetilde{f})F(\mu_A) = F(f)$. Thus, $F(\widetilde{f}) = F(f)(F(\mu_A))^{-1}$.

On the other hand, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\widetilde{f}} & C \\ \mu_{F(A)} \downarrow & & \downarrow \mu_C \\ F(F(A)) & \xrightarrow{F(\widetilde{f})} & F(C) \end{array}$$

commutes. Consequently,

$$\widetilde{f} = (\mu_C)^{-1} F(\widetilde{f}) \circ \mu_{F(A)} = (\mu_C)^{-1} F(f)(F(\mu_A))^{-1} \mu_{F(A)}.$$

This implies the uniqueness of $\widetilde{f}$, if it exists. Now, it suffices to verify that $\widetilde{f} = (\mu_C)^{-1} F(f)(F(\mu_A))^{-1} \mu_{F(A)}$ does the job. Indeed, the following diagrams are commutative.

$$\begin{array}{ccccc} C & \xleftarrow{f} & A & \xrightarrow{\mu_A} & F(A) \\ \mu_C \downarrow & & \mu_A \downarrow & & \downarrow \mu_{F(A)} \\ F(C) & \xleftarrow{F(f)} & F(A) & \xrightarrow{F(\mu_A)} & F(F(A)) \end{array}$$

Hence,

$$\begin{aligned} \widetilde{f}\mu_A &= (\mu_C)^{-1} F(f)(F(\mu_A))^{-1} \mu_{F(A)} \mu_A \\ &= (\mu_C)^{-1} F(f)(F(\mu_A))^{-1} F(\mu_A) \mu_A \\ &= (\mu_C)^{-1} F(f) \mu_A \\ &= (\mu_C)^{-1} \mu_C f \\ &= f. \end{aligned}$$

4. Saturated Subcategories. An interesting example of saturated subcategories is given in the following.

Theorem 4.1. If $\mathbf{D}$ is a reflective subcategory of $\mathbf{C}$, then $\mathbf{D}$ is saturated (i.e., $\mathbf{D}^{\perp\perp} = \mathbf{D}$).

Proof. Let $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ be a left adjoint functor of the canonical inclusion functor $I\colon \mathbf{D} \longrightarrow \mathbf{C}$. Let μ be a natural transformation from the functor $1_{\mathbf{C}}$ to the functor IF such that $(F(A), \mu_A)$ is a universal of A to I, for each object A of $\mathbf{C}$.

According to Theorem 3.2, it suffices to show that $\mathcal{S}(F) = \mathbf{D}$.

(•) $X \in ob(\mathbf{D}) \Longrightarrow X \in \mathcal{S}(F)^{\perp}$.

Let $X \in ob(\mathbf{D})$, $q\colon A \longrightarrow B$ in $\mathcal{S}(F)$ and $f\colon A \longrightarrow X$. We look for a unique arrow $\widetilde{f}\colon B \longrightarrow X$ such that $\widetilde{f}q = f$. Suppose that such an arrow exists. Then we have $F(\widetilde{f})F(q) = F(f)$. In this way, $F(\widetilde{f}) = F(f)(F(q))^{-1}$. On the other hand, the diagram

$$\begin{array}{ccc} B & \xrightarrow{\widetilde{f}} & X \\ \mu_B \downarrow & & \downarrow \mu_X \\ F(B) & \xrightarrow{F(\widetilde{f})} & F(X) \end{array}$$

is commutative. Hence, $\mu_X \widetilde{f} = F(\widetilde{f})\mu_B$. But μ_X is an isomorphism, by Proposition 3.4 (1). Thus,

$$\widetilde{f} = (\mu_X)^{-1} F(\widetilde{f})\mu_B = (\mu_X)^{-1} F(f)(F(q))^{-1}\mu_B.$$

This shows the uniqueness of $\widetilde{f}$. Now, it suffices to check that $\widetilde{f} = (\mu_X)^{-1} F(f)(F(q))^{-1}\mu_B$ does the job. Indeed,

$$\begin{aligned} \widetilde{f}q &= (\mu_X)^{-1} F(f)(F(q))^{-1}\mu_B q \\ &= (\mu_X)^{-1} F(f)(F(q))^{-1} F(q)\mu_A \\ &= (\mu_X)^{-1} F(f)\mu_A \\ &= (\mu_X)^{-1}\mu_X f \\ &= f. \end{aligned}$$

(•) $X \in \mathcal{S}(F)^{\perp} \Longrightarrow X \in ob(\mathbf{D})$.

As $\mathbf{D}$ is closed under isomorphisms, it suffices to show that μ_X is an isomorphism.

It is known that $\mu_X \in \mathbf{D}^{\perp}$. Hence, $\mu_X \in \mathcal{S}(F)$, by Theorem 3.2. Thus, $X \perp \mu_X$; consequently, there is a unique arrow $g\colon F(X) \longrightarrow X$ such that $g\mu_X = 1_X$. This leads to $\mu_X g \mu_X = \mu_X$. As $F(X) \perp \mu_X$, we easily obtain $\mu_X g = 1_{F(X)}$, completing the proof.

The next example shows that the converse of Theorem 4.1 does not hold.

<u>Example 4.2</u>. Let $\mathbf{C}$ be the category whose objects are W, X, Y, Z and with arrows

$$1_W,\ 1_X,\ 1_Y,\ 1_Z,\ f_1,\ f_1,\ f_2,\ f_3,\ f_4$$

such that the following diagram is commutative.

$$\begin{array}{ccccc} Z & \overset{f_4}{\longleftarrow} & W & \overset{f_1}{\longrightarrow} & X \\ & & f_2 \downarrow & \swarrow f_3 & \\ & & Y & & \end{array}$$

Let $\mathbf{D} = \{Y\}$. Then

$$\mathbf{D}^{\perp} = \{1_W,\ 1_X,\ 1_Y,\ 1_Z,\ f_1,\ f_1,\ f_2,\ f_3\}.$$

and $\mathbf{D}^{\perp\perp} = \mathbf{D}^{\perp}$. Thus, $\mathbf{D}$ is saturated, however $\mathbf{D}$ is not reflective in $\mathbf{C}$.

Particular saturated subcategories are provided by the following definition.

<u>Definition 4.3</u>. Let $\mathbf{C}$ be a category. An object A of $\mathbf{C}$ is said to be saturated if $\{A\}^{\perp\perp} = \langle A \rangle$, where $\langle A \rangle$ is the class of objects of $\mathbf{C}$ which are isomorphic to A.

The following proposition, which has a straightforward proof, links saturated objects with terminal ones.

Proposition 4.4. Let $\mathbf{C}$ be a category. Then the following properties hold.

(1) If T is a terminal object of $\mathbf{C}$, then $\{T\}^{\perp} = hom_{\mathbf{C}}(-,-)$.
(2) If $A, B \in hom_{\mathbf{C}}(-,-)^{\perp}$ and there is an arrow $\theta\colon A \longrightarrow B$, then θ is an isomorphism.
(3) If T is a terminal object of $\mathbf{C}$, then it is the unique saturated object (up to an isomorphism).

Example 4.5. A saturated object which is not terminal.

Let $\mathbf{C}$ be a multiplicative monoid $(M, \times)$ such that M is of cardinality ≥ 2. Hence, $\mathbf{C}$ may be regarded as a category with single object A, $ob(\mathbf{C}) = \{A\}$ with $hom_{\mathbf{C}}(A, A) = M$. Thus, A is a saturated object; however, A is not terminal, since the cardinality of M is ≥ 2.

Example 4.6. A category without saturated objects.

Let $\mathbf{C}$ be the full subcategory of **SET** whose objects are sets of cardinality ≥ 2. If X is an object of $\mathbf{C}$, then $\{X\}^{\perp} = Isom_{\mathbf{C}}(-,-)$. Hence, $\{X\}^{\perp\perp} = \mathbf{C}$.

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