A NEW GENERALIZATION OF $\mathbf{T_D}$ SPACES

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Abstract. In this paper we introduce T_{ac} spaces as a generalization of T_D spaces and we present some properties of these spaces related to subspaces, quotients, continuous functions, and homeomorphisms. Also, we show that the property T_{ac} does not imply T_0 and T_0 does not imply T_{ac}.

1. Introduction. To each topological space X, we can naturally associate the subspaces $\gamma(X) = \{x \in X \mid \{x\} \text{ is closed}\}$ and $\varepsilon(X) = \{x \in X \mid \overline{\{x\}} \cap \gamma(X) = \emptyset\}$, known as the closed points and free points subspaces of X, respectively [1]. With them, one can obtain a sufficient condition to determine the compactness of X and they can be used to characterize the compactness of the T_0 spaces [1]. γ and ε can also be studied as operators of Top in Top or $Top(X)$ in $P(X)$. In this framework, we obtain that both operators are idempotent and mutually orthogonal [5], among other properties. In this work, we associate with X, the subspaces $\delta(X) = \{x \in X \mid \{x\}' \text{ is closed in } X\}$ and $\lambda(X) = \{x \in X \mid \{x\}' \cap \delta(X) = \emptyset\}$, called the ac-closed points and the ac-free points subspaces of X, respectively. The properties of the composition of these operators are presented and they are used to define T_{ac} spaces, which are a generalization of T_D spaces. These are different from T_{UD} spaces. Finally, some topological notions are studied, such as preservation under continuous functions, under homeomorphisms, and for subspaces.

2. Preliminaries. To facilitate the development, some definitions and propositions are presented. They will be used later. The proofs that are not given here can be consulted in [3, 4, 6].

<u>Definition 2.1</u>. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an accumulation point of A if for every open set V containing x, $(V - \{x\}) \cap A \neq \emptyset$. The set of accumulation points of A is the derived set of A (denoted A'). In some cases to avoid confusion, we will denote with $\{x\}'_M$ the set of accumulation points of $\{x\}$ in M.

<u>Proposition 2.2</u>. Let $(X, \mathcal{T})$ be a topological space. If Y is a subspace of X and $A \subseteq Y$, then $A'_Y = A'_X \cap Y$.

Definition 2.3. A topological space $(X, \mathcal{T})$ is called:

a) T_2 if for any $x, y \in X$ with $x \neq y$, there exists disjoint open sets U, V containing x and y, respectively.
b) T_1 if for any $x, y \in X$ with $x \neq y$, there are open sets U, V with $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$.
c) T_D if for any $x \in X$, $\{x\}'$ is closed.
d) T_{UD} if for any $x \in X$, $\{x\}'$ is the union of disjoint closed sets.
e) T_0 if for any $x, y \in X$ with $x \neq y$, there is an open set that contains one of them but it does not contain the other one.

It can be proved that $T_2 \Longrightarrow T_1 \Longrightarrow T_D \Longrightarrow T_{UD} \Longrightarrow T_0$. For this reason, the axioms T_D and T_{UD} are considered as separation axioms between T_0 and T_1; a detailed study of this topic is presented in [3].

Proposition 2.4. If $(X, \mathcal{T})$ is T_0, then the relation $\leq_{\mathcal{T}}$, defined by $x \leq_{\mathcal{T}} y$ if and only if $x \in \overline{\{y\}}_{\mathcal{T}}$, is an order.

Definition 2.5. A poset $(X, \leq)$ has enough minimals, if for any $x \in X$, there exists a y such that $y \leq x$ and there is no $z \in X$ such that $z < y$.

3. The Subspaces $\delta(\mathbf{X})$ and $\lambda(\mathbf{X})$. In this section the subspaces $\delta(X)$ and $\lambda(X)$ are introduced. Some of their properties are also shown. Since $\delta(X)$ is a T_D space, the operator δ is an idempotent and $\gamma(X) \subseteq \delta(X) \cap \lambda(X)$. Finally, a sufficient condition is presented so that $\delta(\lambda(X)) = \lambda(\delta(X)) = \lambda(X)$. This allows us to define T_{ac} spaces in the following section.

Definition 3.1. Let $(X, \mathcal{T})$ be a topological space. The subspaces $\delta(X)$ and $\lambda(X)$ are defined as:

$$\delta(X) = \{x \in X \mid \{x\}' \text{ is closed in } X\}$$
$$\lambda(X) = \{x \in X \mid \{x\}' \cap \delta(X) = \emptyset\}.$$

The elements of $\delta(X)$ are called ac-closed points and the elements of $\lambda(X)$ are called ac-free points.

In the following proposition, some properties of the operators δ and λ are proved.

Proposition 3.2. Let $(X, \mathcal{T})$ be a topological space. Then,

a) $\delta(\delta(X)) = \delta(X)$

b) $\delta(\lambda(X)) \subseteq \lambda(X)$

c) $\lambda(\delta(X)) = \delta(X) \cap \lambda(X)$

d) $\lambda(\delta(X)) \subseteq \lambda(X)$

e) $\lambda(\lambda(X)) \subseteq \lambda(X)$

Proof. a) The fact that $\delta(\delta(X)) = \delta(X)$ follows inmediately from Definition 3.1.

Suppose that $x \in \delta(X)$ and $x \notin \delta(\delta(X))$. If $x \in \delta(X)$, then $\{x\}'_X$ is closed in X. Therefore, $\{x\}'_X \cap \delta(X)$ is closed in $\delta(X)$. As $\{x\}'_{\delta(X)} = \{x\}'_X \cap \delta(X)$, one has that $\{x\}'_{\delta(X)}$ is closed in $\delta(X)$, which is false.

Statements b) and e) are evident from the definitions.

c) Let $x \in \lambda(\delta(X))$, then $x \in \delta\,(X)$ and $\{x\}'_{\delta(X)} \cap \delta\,(X) = \emptyset$. It is known that $\{x\}'_{\delta(X)} = \{x\}'_X \cap \delta\,(X)$, then $\{x\}'_{\delta(X)} \cap \delta\,(X) = (\{x\}'_X \cap \delta\,(X)) \cap \delta\,(X) = \{x\}'_X \cap \delta\,(X) = \emptyset$. Therefore, $x \in \lambda\,(X)$.

Let $x \in \delta(X) \cap \lambda(X)$, then $x \in \delta(X)$ and $x \in \lambda(X)$. Hence, $\{x\}'_X \cap \delta\,(X) = \emptyset$. Thus, $(\{x\}'_X \cap \delta\,(X)) \cap \delta\,(X) = \emptyset$ and $\{x\}'_{\delta(X)} \cap \delta\,(X) = \emptyset$.

d) This is a direct consequence of part c).

Corollary 3.3. For any topological space $(X, \mathcal{T})$, $\delta(X)$ is a T_D space.

Proof. This follows inmediately from the previous proposition, part a).

We notice that the operator δ is idempotent. In the case of λ, it is partially satisfied. We have not been able to provide either a counterexample or a complete proof for the remaining part. The strict implications in cases b) and d) will allow us to define the T_{ac} spaces.

The following example shows that the implications obtained in b) and d) of Proposition 3.2 are strict. For the other implication of e), we neither have a counterexample nor a proof.

Example 3.4. If $X = \{1, 2, 3, 4\}$, $\mathcal{T} = \{\emptyset, X, \{3\}, \{3, 4\}\}$, then $\delta\,(X) = \{3, 4\}$ and $\lambda\,(X) = \{1, 2, 4\}$, $\mathcal{T}_{\delta(X)} = \{\emptyset, \delta\,(X)\,, \{3\}\}$. Thus, $\{3\}'_{\delta(X)} = \{4\}$ and $\{4\}'_{\delta(X)} = \emptyset$. Since $\lambda\,(\delta\,(X)) = \{x \in \delta\,(X) \mid \{x\}'_{\delta(X)} \cap \delta\,(\delta\,(X)) = \emptyset\}$ we have that $\lambda\,(\delta\,(X)) = \{4\}$. Therefore, $\lambda\,(X) \not\subseteq \lambda\,(\delta\,(X))$.

Now $\mathcal{T}_{\lambda(X)} = \{\phi, \lambda\,(X)\,, \{4\}\}$ so $\{1\}'_{\lambda(X)} = \{2\}$ is not closed, $\{2\}'_{\lambda(X)} = \{1\}$ is not closed, and $\{4\}'_{\lambda(X)} = \{1, 2\}$ is closed in $\lambda\,(X)$.

Since $\delta(\lambda(X)) = \{x \in \lambda(X) \mid \{x\}'_{\lambda(X)}$ is closed in $\lambda(X)\}$, we have that $\delta(\lambda(X)) = \{4\}$. Consequently, $\lambda(X) \nsubseteq \delta(\lambda(X))$.

In the next proposition we give a sufficient condition to obtain the remaining implications in Proposition 3.2.

Proposition 3.5. Let $(X, \mathcal{T})$ be a topological space. If $\lambda(X) \subseteq \delta(X)$, then:

a) $\lambda(X) \subseteq \delta(\lambda(X))$
b) $\lambda(X) \subseteq \lambda(\delta(X))$
c) $\lambda(X) \subseteq \lambda(\lambda(X))$.

Proof. a) Let $x \in \lambda(X)$. Since $\lambda(X) \subseteq \delta(X)$, $\{x\}'_X$ is closed in X. From Proposition 2.2, we have $\{x\}'_{\lambda(X)} = \{x\}'_X \cap \lambda(X)$. Since $\{x\}'_X$ is closed in X, $\{x\}'_{\lambda(X)}$ is closed in $\lambda(X)$. Therefore, $x \in \delta(\lambda(X))$.

b) Let $x \in \lambda(X)$. Then $\{x\}'_X \cap \delta(X) = \emptyset$. Now, from Proposition 2.2, we have $\{x\}'_{\delta(X)} = \{x\}'_X \cap \delta(X)$. Then $\{x\}'_{\delta(X)} \cap \delta(X) = \emptyset$. Therefore, $x \in \lambda(\delta(X))$.

c) Let $x \in \lambda(X)$ and suppose $\{x\}'_{\lambda(X)} \cap \delta(\lambda(X)) \neq \emptyset$. Then from part a), we have $\{x\}'_{\lambda(X)} \cap \lambda(X) \neq \emptyset$. Now, using Proposition 2.2 we obtain $\{x\}'_X \cap \lambda(X) \neq \emptyset$ and since $\lambda(X) \subseteq \delta(X)$, we have $\{x\}'_X \cap \delta(X) \neq \emptyset$. Hence, $x \notin \lambda(X)$, which is false.

In the next proposition we show the relation between the set of closed points ($\gamma(X)$ according to [5]) and the subspaces $\delta(X)$ and $\lambda(X)$. Notice that if $\delta(X) = X$, then $\lambda(X) = \gamma(X)$.

Proposition 3.6. Any closed point is an ac-closed point and an ac-free point.

Proof. Let x be a closed point, then $\{x\}' = \emptyset$; therefore, x is an ac-closed point. Also, $\emptyset \cap \delta(X) = \emptyset$. Therefore, x is an ac-free point.

In the next example we show that the converse of the previous proposition is not certain. That is, there exist ac-closed points and ac-free points that are not closed points. Therefore, the set of closed points is strictly contained in the set of points that are ac-closed points and ac-free points.

Example 3.7. If $X = \{1, 2, 3, 4\}$ and $\mathcal{T} = \{\emptyset, X, \{3\}, \{3, 4\}\}$, then $\delta(X) = \{3, 4\}$ and $\lambda(X) = \{1, 2, 4\}$. Therefore, we have that 4 is an ac-closed point and an ac-free point, but it is not a closed point.

4. $\mathbf{T_{ac}}$ Spaces. In the previous section we showed that if $\lambda(X) \subseteq \delta(X)$, the other implications of Proposition 3.2 are satisfied. This condition gives birth to the T_{ac} spaces, which are defined in this paper. They will be the objects of study in this section.

Definition 4.1. A topological space $(X, \mathcal{T})$ is a T_{ac} space if $\lambda(X) \subseteq \delta(X)$.

Notice that any T_D space is T_{ac} and, in particular T_1 and T_2 spaces are T_{ac}. The following example shows a finite T_{ac} space.

Example 4.2. If $X = \{1,2,3,4,5,6\}$ and

$$\begin{aligned}\mathcal{T} = \{&\emptyset, X, \{1\}, \{1,2\}, \{1,3\}, \{4,5\}, \{1,2,3\}, \{1,4,5\}, \{4,5,6\}, \{1,2,4,5\},\\ &\{1,3,4,5\}, \{1,2,4,5,6\}, \{1,3,4,5,6\}, \{1,4,5,6\}, \{1,2,3,4,5\}\}\end{aligned}$$

then $\delta(X) = \{1,2,3,6\}$ and $\lambda(X) = \{2,3,6\}$.

From the previous example, note that if $A = \{3,4,5\}$ then $\mathcal{T}_A = \{\emptyset, A, \{3\}, \{4,5\}\}$. We have that $\delta(A) = \{3\}$ and $\lambda(A) = \{3,4,5\}$. Therefore, $(A, \mathcal{T}_A)$ is not a T_{ac} space. Therefore, not every subspace of a T_{ac} space is T_{ac} space. This property is inherited by closed subspaces as shown in Proposition 4.4.

Proposition 4.3. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. If A is a closed set, then $\lambda(A) \subseteq \lambda(X)$.

Proof. Suppose that $a \notin \lambda(X)$ and $a \in \lambda(A)$. Then $\{a\}'_X \cap \delta(X) \neq \emptyset$. Therefore, there exists a $b \in \{a\}'_X$ and $b \in \delta(X)$. If $b \notin A$ then $b \in X - A$. But it is an open set and $a \notin A^c$. Therefore, $b \notin \{a\}'_X$, which is false. Then $b \in A$ and this implies that $b \in \{a\}'_A$. Since $b \in \delta(X)$, $\{b\}'_X$ is closed and $\{b\}'_A = \{b\}'_X \cap A$ is closed in A. Therefore, $b \in \delta(A)$. Then $b \in \{a\}'_A \cap \delta(A)$ and this implies that $a \notin \lambda(A)$.

Proposition 4.4. Any closed subspace of a T_{ac} space is a T_{ac} space.

Proof. For any closed subset A in X, we will show that $\lambda(A) \subseteq \delta(A)$. Let $a \in \lambda(A)$. Since A is closed, from Proposition 4.3, we have that $\lambda(A) \subseteq \lambda(X)$. Then $a \in \lambda(X)$. Since X is a T_{ac} space, $\lambda(X) \subseteq \delta(X)$. Thus, $a \in \delta(X)$ and $\{a\}'_X$ is closed. Then $\{a\}'_A = \{a\}'_X \cap A$ is closed in A. Therefore, $a \in \delta(A)$.

There exist spaces satisfying the condition $\lambda(\lambda(X)) = \lambda(X)$, but they are not T_{ac} spaces. This is observed in the next example.

Example 4.5. Let X be the set of natural numbers. The open sets in X are $\emptyset$, X, and sets of form $\{n, n+1, \dots\}$ with $n \geq 3$. Thus,

$$\{1\}' = \{2\} \ \{2\}' = \{1\} \ \{3\}' = \{1,2\} \ \{4\}' = \{1,2,3\} \ \{5\}' = \{1,2,3,4\} \dots .$$

Then, $\delta(X) = \{3, 4, 5, \dots\}$ and $\lambda(X) = \{1, 2, 3\}$.

We can observe that this space is not a T_{ac} space.

The subspace topology for $\lambda(X)$ is: $\Omega_{\lambda(X)} = \{\emptyset, \lambda(X), \{3\}\}$. Thus, $\{1\}'_{\lambda(X)} = \{2\}$, $\{2\}'_{\lambda(X)} = \{1\}$, $\{3\}'_{\lambda(X)} = \{1,2\}$. Then, $\delta(\lambda(X)) = \{3\}$.

The subspace topology for $\delta(X)$ is: $\mathcal{T}_{\delta(X)} = \{\emptyset, \delta(X)\} \cup \{\{n, n+1, n+2, \dots\} \mid n > 3\}$. Thus, $\{3\}'_{\delta(X)} = \emptyset$, $\{4\}'_{\delta(X)} = \{3\}$, $\{5\}'_{\delta(X)} = \{3,4\}$. In general $\{m\}'_{\delta(X)} = \{3, 4, \dots, m-1\}$ with $m \geq 4$. Then, $\lambda(\delta(X)) = \{3\}$. Therefore, we have $\lambda(\lambda(X)) = \{1,2,3\} = \lambda(X)$.

In the following examples we show that the property T_{ac} is neither preserved under continuous image nor for quotients.

Example 4.6. The topological space $(\mathbb{Z}, \mathcal{T})$, where $\mathcal{T}$ is the right topology, is a T_{ac} space. The identity function $i \colon (\mathbb{Z}, \mathcal{T}) \to (\mathbb{Z}, \Omega)$, where Ω is the indiscrete topology, is a continuous function. But the image of i is not a T_{ac} space.

Example 4.7. For the ring $\mathbb{Z}$, the topological space $Spec(\mathbb{Z})$ with Zariski's topology is a T_{ac} space since $\delta(Spec(\mathbb{Z})) = Spec(\mathbb{Z}) - \{\{0\}\} = \lambda(Spec(\mathbb{Z}))$. Now the quotient topology for the quotient space $Spec(\mathbb{Z}) / \sim$, where $\sim$ is the relation defined by $x \sim y$ if and only if x and y are maximals, is the indiscrete topology. Thus, $Spec(\mathbb{Z}) / \sim$ is not a T_{ac} space.

Proposition 4.8. The property T_{ac} is a topological invariant.

Proof. Let X, Y be homeomorphic topological spaces and X be a T_{ac} space. If $y \in \lambda(Y)$, then $\{y\}' \cap \delta(Y) = \emptyset$. Now there exists $x \in X$ such that $f(x) = y$. Then $\{f(x)\}' = \{y\}'$. Therefore,

$$\begin{aligned}
\{f(x)\}' \cap \delta(Y) = \{y\}' \cap \delta(Y) = \emptyset \\
f^{-1}(\{f(x)\}' \cap \delta(Y)) = \emptyset \\
f^{-1}(\{f(x)\}') \cap f^{-1}(\delta(Y)) = \emptyset.
\end{aligned}$$

Since f is an homeomorphism we have that $\{f(x)\}' = f(\{x\}')$. The last equality becomes $\{x\}' \cap \delta(X) = \emptyset$. Thus, $x \in \lambda(X)$. Then, $x \in \delta(X)$ and $\{x\}'$ is a closed set. Therefore, $f(x) \in f(\delta(X))$, $f(x) \in \delta(Y)$, and $f(x) = y$ implies $y \in \delta(Y)$.

In [3], a diagram of implications is presented showing the separation axioms between T_0 and T_1. This diagram will be modified by inserting the T_{ac} spaces as in the following figure.

$$\begin{array}{l} T_D \Longrightarrow T_{ac} \\ \Downarrow \\ T_{UD} \Longrightarrow T_0 \end{array}$$

The next three examples show that $T_{ac} \nRightarrow T_{UD}$, $T_{ac} \nRightarrow T_0$, and $T_0 \nRightarrow T_{ac}$.

Example 4.9. For the topological spaces of Example 4.2, we have $\delta(X) = \{1, 2, 3, 6\}$ and $\lambda(X) = \{2, 3, 6\}$. Thus, this space is not a T_{ac} space. We observe that $\{4\}' = \{5, 6\}$ is not the union of disjoint closed sets. Thus, this space is neither a T_{UD} space nor a T_D space. Since 4 and 5 cannot be separated, this space is not a T_0 space. In conclusion, $T_{ac} \nRightarrow T_0$.

Example 4.10. Let $\mathbb{R}$ be the set of real numbers with the topology generated by the base $\beta = \{(-\infty, x) \mid x \in \mathbb{R}\}$. For $x < y$, we have that $(-\infty, y)$ is an open set that contains to x and not to y. For $x \in \mathbb{R}$ we have $\{x\}' = (x, \infty)$ which is not closed. Thus, $\delta(X) = \emptyset$ and $\lambda(X) = \mathbb{R}$. Then, this space is a T_0 space but it is not a T_{ac} space. Therefore, $T_0 \nRightarrow T_{ac}$.

A natural question which arises from the above is whether or not $T_D = T_{UD} \cap T_{ac}$. The answer is negative as we will show in the following example.

Example 4.11. As mentioned in Example 4.7, $Spec(\mathbb{Z})$ is a T_{ac} space, because $\delta(Spec(\mathbb{Z})) = \lambda(Spec(\mathbb{Z}))$. Also, $Spec(\mathbb{Z})$ is a T_{UD} space. However, it is not a T_D space.

Another interesting question is to determine when a T_0 space is a T_{ac} space or vice versa. The next result shows a partial answer to this question.

Proposition 4.12. Let $(X, \mathcal{T})$ be a T_0 space. If $(X, \leq_{\mathcal{T}})$ has enough minimals, then X is a T_{ac} space.

Proof. We see that $\lambda(X) \subseteq \delta(X)$. If $x \notin \delta(X)$, then $\{x\}'$ is not closed. Now there exists a minimal $y \in X$ and hence, a closed point such that $y \leq_{\mathcal{T}} x$. Thus, $y \in \{x\}'$ and $y \in \{x\}' \cap \delta(X)$. Therefore, $x \notin \lambda(X)$.

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References

1. L. Acosta and E. Lozano, "Compactificaciones por un Punto de un Espacio Topológico," *Memorias XVI Coloquio Distrital de Matemáticas y Estadística,* Bogotá, Colombia, 1999.
2. S. Andima and W. J. Thron, "Order-induced Topological Properties," *Pacific J. Math.,* 75 (1978), 297–318.
3. C. E. Aull and W. J. Thron, "Separation Axioms Between T_0 and T_1," *Indag. Math.,* 24 (1962), 26–37.
4. R. Engelking, *Outline of General Topology,* North Holland, Amsterdam, 1968.
5. P. Gaona, *El Operador de Puntos Cerrados en Topología General. Tesis Maestría,* Universidad Nacional de Colombia, Bogotá, Colombia, 2001.
6. J. R. Munkres, *Topology A First Course,* Prentice-Hall, New Jersey, 1975.

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