A REMARK ON CHARACTER DEGREES AND NILPOTENCE CLASS IN p-GROUPS

A. Iranmanesh and M. Viseh

Abstract. Let G be a finite metabelian p-group whose non-linear irreducible character degrees lie between p^a and p^b, where $1 < a \leq b$. In this paper it is shown that the nilpotence class of G is bounded by a function of p and $b - a$.

1. Introduction. Let G be a finite p-group and, as usual, write $cd(G)$ to denote the set of degrees of irreducible complex characters of G. It is an old result of Isaacs and Passman [3] that if $cd(G) = \{1, p^e\}$ with $e > 1$, then G has nilpotence class at most p. On the other hand if $cd(G) = \{1, p\}$, the class of G can be arbitrarily large. More generally, it is known that for some choices of S, where S is a finite set of powers of p, if $cd(G) = S$, then the nilpotence class of G is bounded by some integer $n(S)$ that depends on S [2, 5, 6]. It is also shown that $n(S)$ does not depend only on p [5]. Slattery in [6] showed that if G is a metabelian p-group, $p^a \leq \chi(1) \leq p^b$ for all non-linear χ in $\mathrm{Irr}(G)$ and $b \leq 2a - 2$, then $c(G)$, the nilpotence class of G, is bounded by a function of p and $b - a$. However, by Theorem A of [2], it is not really necessary and we only need $1 < a \leq 2b$.

In this paper we will prove the following Theorem.

<u>Main Theorem</u>. Given a prime p and integers a and b with $1 < a \leq b$, let G be a metabelian p-group such that $p^a \leq \chi(1) \leq p^b$ for all non-linear χ in $\mathrm{Irr}(G)$. Then the nilpotence class of G is bounded in terms of p and $b - a$.

Using a result in [1] and arguing as in the proof of the main result of [4], we conclude that if $cd(G)$ contains p, then $c(G)$ can be arbitrarily large. Thus, the hypothesis that $a > 1$ is really necessary.

Our result improves a theorem in [2], where Isaacs and Moretó proved that in the situation of Theorem 1.1, the nilpotence class of G is bounded in terms of p^b.

2. Proof of the Main Theorem. We will use the following two results.

<u>Lemma 2.1</u>. [2] Let G be a metabelian p-group and let p^e be the largest irreducible character degree of G. If $p \notin cd(G)$, then $c(G) \leq 2 + (e - 1)p^e$.

<u>Lemma 2.2</u>. Suppose that G is a p-group and that $p \notin cd(G)$. Let $1 < L \lhd G$ with G/L cyclic. Then $c(L) = c(G)$.

Proof. This is an immediate corollary of lemmas in [6] and [2].

To state the Theorem in a more precise way, we shall introduce some convenient notation. Given a finite p-group G, we define

$$a(G) = \begin{cases} \log_p(\min(cd(G)\backslash\{1\}, & \text{if } G \text{ is non-abelian} \\ 2, & \text{if } G \text{ is abelian} \end{cases}$$

and

$$\delta(G) = \log_p(b(G)) - a(G),$$

where $b(G)$ is the largest irreducible character degree of G.

We will also need the following key lemma.

Lemma 2.3. Let G be a finite p-group and suppose that A is an abelian normal subgroup of G with $|G : A| = b(G)$. Let H and K be subgroups of G, where $A \subset H \subset K$ and $|K : H| = p$. Then

(i) $a(K) \leq a(H) + 1$.
(ii) $\delta(H) \leq \delta(K)$.

Proof. Since $|G : H| < b(G)$, we deduce that H is non-abelian. Let $\varphi \in \mathrm{Irr}(H)$ with $\varphi(1) = p^{a(H)}$. If $a(K) > a(H) + 1$, then φ^K has a linear constituent λ. Hence, $\varphi(1) = \lambda_H(1) = 1$. This contradiction proves (i).
Using Theorem [1, 6.19], we conclude that $A \leq L \leq G$, then $b(L) = p^\beta$, where $p^\beta = b(G)/|G : L|$. In particular, it follows that $b(H) < b(K)$, and hence by (i), we have $\delta(H) \leq \delta(K)$, and the proof is complete.

Now, we are able to prove our main result.

Proof of the Main Theorem. Following the proof of Theorem 2.7 of [2], we proceed by induction on $|G|$ and we observe that the hypotheses on G are inherited by homomorphic image G/N, where $N \lhd G$. Since $\delta(G/N) \leq \delta(G)$ and the function $2 + (\delta(G) + 1)p^{\delta(G)+2}$ is monotonic in $\delta(G)$, it follows that $c(G/N) \leq 2 + (\delta(G) + 1)p^{\delta(G)+2}$ for every non-identity normal subgroup N. Therefore, we can assume that G has a unique minimal normal subgroup, and thus, $Z(G)$ is cyclic and G has a faithful irreducible character χ. Because G is metabelian, it follows that χ is induced from a linear character of a subgroup $A \supseteq G'$, and since $A \lhd G$, we see that all irreducible constituents of χ_A are linear. But χ is faithful and therefore, A is abelian and hence, no irreducible character of G has degree larger than $|G : A| = \chi(1)$. In particular, it implies that $|G : A| = b(G)$.

We may now assume

$$A = G_0 < G_1 < \cdots < G_{b(G)} = G.$$

Since $a(G_1) = 1$ [1] and $a(G) \geq 2$, Lemma 2.3(i) implies that $a(G_i) = 2$ for some i. Fix m with $a(G_m) = 2$ and observe that by the previous lemma that $p \notin cd(G_i)$ for $i = m, \ldots, b(G)$. Using Lemma 2.2, we deduce $c(G_m) = c(G)$, and it follows by Lemma 2.1 that

$$c(G_m) \leq 2 + (\delta(G_m) + 1)p^{\delta(G_m)+2}.$$

But Lemma 2.3(ii) implies that $\delta(G_m) \leq \delta(G)$. Hence,

$$c(G) \leq 2 + (\delta(G) + 1)p^{\delta(G)+2}.$$

That is, the nilpotence class of G is bounded in terms of p and $b - a$.

Acknowledgment. The authors are indebted to the referee for his/her valuable comments.

References

1. I. M. Isaacs, *Character Theory of Finite Groups,* Dover Publications, Inc., New York, NY, 1994.
2. I. M. Isaacs and A. Moretó, "The Character Degrees and Nilpotence Class of a p-Group," *J. Algebra,* 238 (2001), 827–842.
3. I. M. Isaacs and D. S. Passman, "A Characterization of Groups in Terms of the Degrees of Their Characters II," *Pacific J. Math.,* 24 (1968), 467–510.
4. I. M. Isaacs and M. C. Slattery, "Character Degree Sets That Do Not Bound the Class of a p-Group," *Proc. Amer. Math. Soc.,* 130 (2002), 2553–2558.
5. A. Jaikin-Zapirain and A. Moretó, "Character Degrees and Nilpotence Class of Finite p-Groups: An Approach Via Pro-p Groups," *Trans. Amer. Math. Soc.,* 345 (2002), 3907–3925.
6. M. C. Slattery, "Character Degrees and Nilpotence Class in p-Groups," *J. Austral Math. Soc. (Series A),* 57 (1994), 76–80.

Mathematics Subject Classification (2000): 20C15, 20D15

A. Iranmanesh
Department of Mathematics
Tarbiat Modarres University
P.O. Box 14115-137
Tehran, Iran
email: iranmana@modares.ac.ir

M. Viseh
Department of Mathematics
Tarbiat Modarres University
P.O. Box 14115-137
Tehran, Iran
email: m-v3@parsimail.com