A NEW CHARACTERIZATION OF THE NAGEL POINT

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Abstract. In this note we provide a new method of constructing the Nagel Point.

1. Introduction. Let the incircle of some arbitrary $\triangle ABC$ be tangent to the sides of the triangle at points D, E, and F as shown in Figure 1. It is well-known that the cevians AD, BE, and CF are concurrent at the Gergonne Point. In this note we want to consider the cevians through points D', E', and F' which are points of tangency of the incircle with lines parallel to the sides of $\triangle ABC$. These cevians are also concurrent.

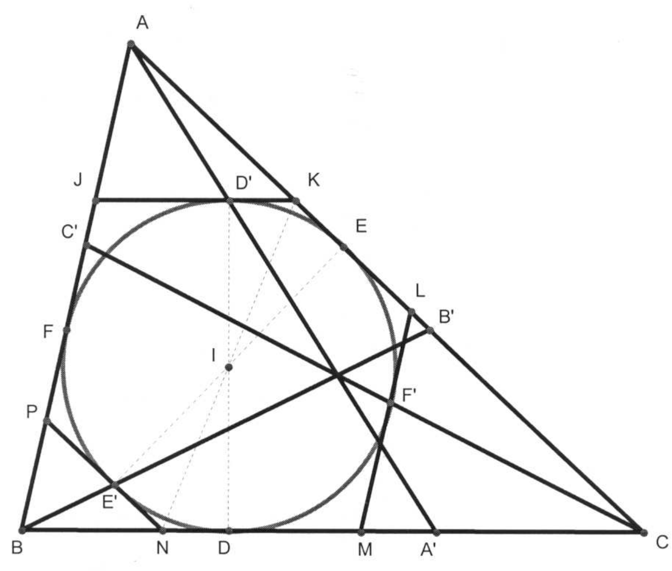

Figure 1

2. Proof of Concurrency. To prove that these cevians are indeed concurrent we name some additional points. Let A', B', and C' be the intersections of AD', BE', and CF' with sides BC, CA, and AB, respectively. Let J, K, L, M, N, and P be consecutive vertices of the hexagon circumscribing the incircle.

Since $\triangle ABA'$ is similar to $\triangle AJD'$ and $\triangle AA'C$ is similar to $\triangle AD'K$, then

$$\frac{BA'}{JD'} = \frac{AA'}{AD'} = \frac{A'C}{D'K}, \quad \text{or} \quad \frac{BA'}{A'C} = \frac{JD'}{D'K}.$$

In the same manner

$$\frac{CB'}{B'A} = \frac{NE'}{E'P} \quad \text{and} \quad \frac{AC'}{C'B} = \frac{LF'}{F'M}.$$

Let I be the incenter of the circle which is also the midpoint of diameters DD', EE', and FF'. Draw IK and IN. Then $\triangle KD'I \cong \triangle KEI$ and $\triangle NE'I \cong \triangle NDI$. Therefore, KN bisects $\angle D'IE$ and $\angle DIE'$ so that $\triangle KD'I \cong \triangle NDI$ and $\triangle KEI \cong \triangle NE'I$. Hence, $D'K = DN = E'N = EK$. In the same manner $D'J = DM = F'M = FJ$ and $E'P = EL = F'L = FP$. By substituting in the above equalities, we have

$$\begin{aligned} \frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} &= \frac{LF'}{F'M} \cdot \frac{JD'}{D'K} \cdot \frac{NE'}{E'P} \\ &= \frac{FP}{DM} \cdot \frac{DM}{EK} \cdot \frac{EK}{FP} \\ &= 1. \end{aligned}$$

Therefore, by Ceva's Theorem, AA', BB', and CC' are concurrent.

3. Checking Priority. A convenient way to judge whether a point of concurrency might be new is by checking *The Encyclopedia of Triangle Centers* [1]. On this website Kimberling has a list of over 2000 triangle centers. These are ranked by using trilinears (explained on the website) for the distance from the point of concurrency X_i to side 6 of a triangle having sides of length 6, 9, and 13 units. By using *Geometer's Sketchpad* [2] for the point above we get 4.50749. The closest number in Kimberling's list is

$$X_8 = 4.507489358552$$

which is the Nagel Point.

The Nagel Point is defined from "the lines joining the points of contact of an excircle with the sides of a triangle to the vertices opposite the respective sides, are concurrent" in [3]. Consider the following problem in [4]: "Let D, E, F be the points on the sides BC, CA, AB of triangle ABC such that D is half way around the perimeter from A, E half way around from B, and F half way around from C. Show that AD, BE, CF are concurrent." It is not clear that the point of concurrency above is really the Nagel Point.

To see that the point of concurrency, say X, is the Nagel Point, we construct the excircle with center I_a and let its point of tangency with BC be T.

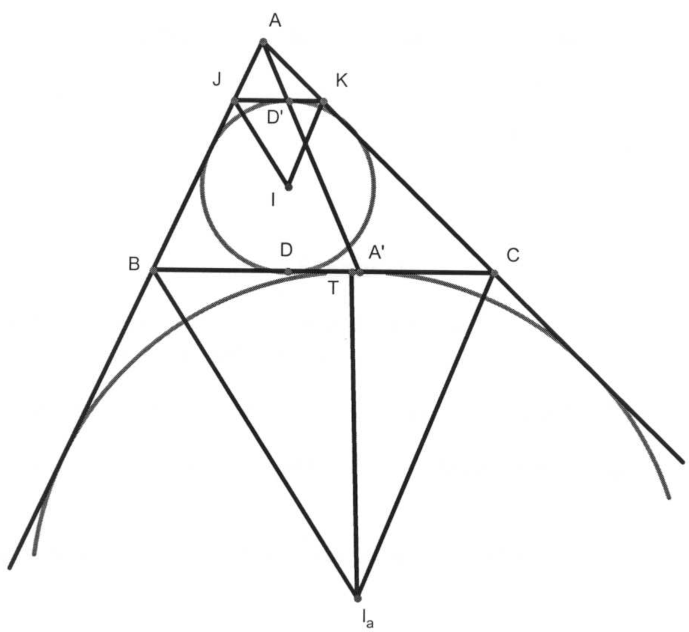

Figure 2

Since $\triangle BI_aC$ is similar to $\triangle JIK$ and since TI_a and $D'I$ are corresponding altitudes, then

$$\frac{BT}{TC} = \frac{JD'}{D'K}.$$

Since

$$\frac{BA'}{A'C} = \frac{JD'}{D'K}$$

from above, then

$$\frac{BT}{TC} = \frac{BA'}{A'C}.$$

Therefore, $T = A'$. In a similar manner the other two excircles are tangent to $\triangle ABC$ at B' and C'. Hence, X is the Nagel Point of $\triangle ABC$.

In addition to the two definitions above for the Nagel Point we now have if D', E', and F' are the points of tangency formed by the intersections of lines parallel to the sides of BC, CA, and AB, respectively with the incircle of $\triangle ABC$, then AD', BE', and CF' are concurrent.

This characterization considerably shortens the construction process for the Nagel Point. We need only construct the point of intersection of the cevians that pass through the opposite endpoints of diameters constructed at the points of tangency of the incircle with the triangle.

References

1. C. Kimberling, *Encyclopedia of Triangle Centers,* http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
2. N. Jackiw, Geometer's Sketchpad ® (version 4), Key Curriculum Press.
3. N. A. Court, *College Geometry* (2nd edition), Barnes & Noble, Inc., New York, NY, 1952.
4. H. Eves, *A Survey of Geometry* (revised edition), Allyn and Bacon, Inc., Boston, MA, 1972.

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