TAYLOR SERIES ARE LIMITS OF LEGENDRE EXPANSIONS

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Abstract. Next to a power series, the classical Legendre series offers the simplest method of representing a function using polynomial expansion means. In 1862, Neumann established results for complex Legendre expansions that are analogous to Taylor's Theorem and the Cauchy-Hadamard Formula for power series, the primary difference being that results are stated in terms of ellipses, as opposed to discs, of convergence. After a simple change of variable, the foci of these ellipses may vary, each leading to a modified Legendre expansion of the original function. Our main result is that as the foci of these ellipses tend to one another, the limit of the corresponding Legendre expansions is the Taylor series representation.

1. Introduction. In [1], Askey and Haimo pose the following question:

> "What are some basic similarities between power series and Fourier series beyond the fact that they are both infinite series?"

A slight generalization of this question might consider comparing power series and other types of series expansions involving orthogonal functions. Perhaps the simplest type of such a series expansion is a Legendre series. Such series are constructed using the classical Legendre polynomials:

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
&\vdots \\
P_n(x) &= \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x).
\end{aligned}
$$

Each Legendre polynomial P_n is a solution of the second order Sturm-Liouville differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+n(n+1)y=0.$$

Taken together, the family of functions

$$\left\{\sqrt{\frac{2n+1}{2}}P_n\right\}_{n=0}^{n=\infty}$$

forms a complete orthonormal basis for the Hilbert space $L^2([-1,1])$.
A classical result first published in 1862 by K. Neumann [2], but also cited as a special case of Theorem 9.1.1 in [5], states that for a function f analytic in an open disk of radius $R>1$, the Legendre series of f,

$$\sum_{n=0}^{n=\infty}\left(\frac{2n+1}{2}\int_{t=-1}^{t=1}f(t)P_n(t)\,dt\right)P_n(z),$$

converges to $f(z)$ at all z interior to the largest ellipse having foci at $(\pm 1,0)$ in which f is analytic. In addition, if ρ denotes the semiaxis of this ellipse, then

$$\frac{1}{\rho}=\overline{\lim}_{n\to\infty}\left|\frac{2n+1}{2}\int_{t=-1}^{t=1}f(t)P_n(t)\,dt\right|^{\frac{1}{n}}.$$

This formula is analogous to the Cauchy-Hadamard Formula for the radius of convergence of a power series. By considering the smallest possible ellipse above, which is inscribed in the circle $|z|=R$, one obtains the inequality

$$\overline{\lim}_{n\to\infty}\left|\frac{2n+1}{2}\int_{t=-1}^{t=1}f(t)P_n(t)\,dt\right|^{\frac{1}{n}}\leq\frac{1}{R+\sqrt{R^2-1}}.$$

If instead, one only assumes that f is analytic in a disk of radius $R>0$ about the origin, then for h sufficiently small and positive, $\frac{R}{h}>1$, and f possesses a Legendre series relative to the interval $[-h,h]$ given by

$$\sum_{n=0}^{n=\infty}\left(\frac{2n+1}{2h}\int_{t=-h}^{t=h}f(t)P_n(t/h)\,dt\right)P_n(z/h)$$

[4]. A simple change of variable, together with Neumann's result above, implies that this series converges to $f(z)$ for all z interior to an ellipse centered at the origin and having foci at $(\pm h, 0)$. Moreover,

$$\overline{\lim}_{n\to\infty} \left| \frac{2n+1}{2h} \int_{t=-h}^{t=h} f(t) P_n(t/h)\, dt \right|^{\frac{1}{n}} \le \frac{1}{\frac{R}{h} + \sqrt{\left(\frac{R}{h}\right)^2 - 1}}. \tag{1}$$

2. A Limit Theorem. In one sense then, a fundamental difference exists between the Taylor and Legendre series expansion, with one being valid in a disk centered at the origin and the other in an ellipse having foci at $(\pm h, 0)$. However, as h approaches zero, the foci of this ellipse move toward the origin. Given this fact, one might wonder whether the corresponding limiting value of the Legendre series is simply the Taylor series. The following theorem establishes that this is precisely the case. Surprisingly, a proof of this somewhat intuitively obvious result, to the best of the author's knowledge, is not present in the literature.

Theorem 1. For f analytic in the open disk of radius $R > 0$ at the origin and for z interior to this disk,

$$\lim_{h\to 0^+} \sum_{n=0}^{n=\infty} \left(\frac{2n+1}{2h} \int_{t=-h}^{t=h} f(t) P_n(t/h)\, dt \right) P_n(z/h) = \sum_{n=0}^{n=\infty} \frac{f^{(n)}(0)}{n!} z^n. \tag{2}$$

In other words, the limiting value of the Legendre series expansion at z is merely the Taylor series expansion.

Proof. Define the function

$$g(h) = \frac{|z| + h}{R + \sqrt{R^2 - h^2}},$$

which is continuous on $[0, R]$ and satisfies $g(0) = \frac{|z|}{2R} < \frac{1}{2}$. By continuity, there exists $0 < h_0 < R$ so that for all $0 < h \le h_0$, $g(h) \le \frac{r}{2}$ for some fixed r in $[0, 1)$.

The proof will proceed in two primary steps. In the first step, we will assume $h < h_0$ and establish that

$$\overline{\lim}_{n\to\infty} \left| \left(\frac{2n+1}{2h} \int_{t=-h}^{t=h} f(t) P_n(t/h)\, dt \right) P_n(z/h) \right|^{\frac{1}{n}} \le r < 1.$$

This inequality will justify passing the limit as $h \to 0^+$ inside the series above.
The key ingredient in this process will be a result of Mauro Picone [3] concerning the maximum modulus of P_n at a complex input w. If ϵ denotes the eccentricity of the ellipse having center at the origin, foci at $(\pm 1, 0)$, and major axis of length $2|w|$, then

$$|P_n(w)| \le \frac{(2n-1)!!}{n!}\left(\frac{1}{\epsilon}+1\right)^n.$$

Here the double factorial $(2n-1)!!$ denotes $(2n-1)(2n-3)\cdots 3\cdot 1$. Applying Picone's result and simplifying the resulting eccentricity to $|z|/h$, we arrive at

$$|P_n(z/h)| \le \frac{(2n-1)!!}{n!}\left(\frac{|z|}{h}+1\right)^n.$$

This inequality, combined with (1), yields

$$\begin{aligned}
&\overline{\lim}_{n\to\infty}\left|\left(\frac{2n+1}{2h}\int_{t=-h}^{t=h} f(t)P_n(t/h)\,dt\right)P_n(z/h)\right|^{\frac{1}{n}}\\
&\le \overline{\lim}_{n\to\infty}\left(\frac{1}{\frac{R}{h}+\sqrt{\left(\frac{R}{h}\right)^2-1}}\right)\cdot\left(\frac{(2n-1)!!}{n!}\right)^{\frac{1}{n}}\left(\frac{|z|}{h}+1\right)\\
&= \overline{\lim}_{n\to\infty}\left(\frac{(2n-1)!!}{n!}\right)^{\frac{1}{n}}\cdot\frac{|z|+h}{R+\sqrt{R^2-h^2}}\\
&= \overline{\lim}_{n\to\infty}\left(\frac{(2n-1)!!}{n!}\right)^{\frac{1}{n}}\cdot g(h)\\
&\le \overline{\lim}_{n\to\infty}\left(\frac{(2n-1)!!}{n!}\right)^{\frac{1}{n}}\cdot\frac{r}{2}. \qquad (3)
\end{aligned}$$

Rewriting

$$\frac{(2n-1)!!}{n!} = \frac{(2n-1)!}{2^{n-1}(n-1)!n!}$$

and using Stirling's approximation, $n! \sim n^n e^{-n}\sqrt{2\pi n}$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\left(\frac{(2n-1)!!}{n!}\right)^{\frac{1}{n}} = 2,$$

so that (3) reduces to r, thereby completing the first step of the proof.
Now consider the limit

$$\lim_{h\to 0}\left(\frac{2n+1}{2h}\int_{t=-h}^{t=h} f(t)P_n(t/h)\,dt\right) P_n(z/h),$$

which may be rewritten as

$$\lim_{h\to 0}\left(\frac{2n+1}{2}\int_{t=-1}^{t=1} f(th)P_n(t)\,dt\right) P_n(z/h).$$

For each n, an application of Taylor's Theorem yields

$$f(th) = f(0)+f'(0)th+\frac{f''(0)}{2}(th)^2+\ldots+\frac{f^{(n)}(0)}{n!}(th)^n+\frac{f^{(n+1)}(c_n)}{(n+1)!}(th)^{n+1}$$

for some c_n between 0 and t.
For each $0 \le m \le n-1$, t^m is orthogonal to P_n. Also,

$$\int_{t=-1}^{t=1} t^n P_n(t)\,dt = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

[6]. Thus,

$$\frac{2n+1}{2}\int_{t=-1}^{t=1} f(th)P_n(t)\,dt = h^n \cdot \frac{2^n(n!)^2}{(2n)!}\cdot\frac{f^{(n)}(0)}{n!} + O(h^{n+1}). \qquad (4)$$

On the other hand,

$$P_n(z/h) = \frac{1}{h^n}\left(\frac{(2n)!}{2^n(n!)^2}\cdot z^n + O(h)\right). \qquad (5)$$

Combining (4) and (5), we arrive at

$$\lim_{h \to 0^+} \left(\frac{2n+1}{2h} \int_{t=-h}^{t=h} f(t) P_n(t/h)\, dt \right) P_n(z/h) = \frac{f^{(n)}(0)}{n!} z^n,$$

which completes the proof of the theorem.

Acknowledgement. The author wishes to express his gratitude to Paul Nevai for his suggestion that simplified the proof.

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Mathematics Subject Classification (2000): 42C10, 30B10

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