RECOGNIZABILITY OF THE SIMPLE $\mathbf{K_n}$-GROUPS $(\mathbf{n = 3, 4})$

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Abstract. Let G be a finite group. Based on the prime graph of G, the order of G can be divided into a product of co-prime positive integers. These integers are called order components of G and the set of order components is denoted by $OC(G)$. Some non-abelian simple groups are known to be uniquely determined by their order components. In this paper we discuss the recognizability of simple K_n-groups $(n = 3, 4)$ by their order components.

1. Introduction. We denote by $\pi(m)$ the set of all prime divisors of $m \in \mathbb{N}$ and if G is a finite group, then $\pi(G)$ is defined to be $\pi(|G|)$. We say that G is a *K_n-group* when $|\pi(G)| = n$.

We construct the prime graph of G as follows. *The prime graph* (Gruenberg-Kegel graph) $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. Now $|G|$ can be expressed as a product of co-prime positive integers m_i, $i = 1, 2, \ldots, t(G)$, where $\pi(m_i) = \pi_i$. These integers are called *the order components* of G. The set of order components of G will be denoted by $OC(G)$. Also, if $2 \mid |G|$, then $2 \mid m_1$ and we call $m_2, \ldots, m_{t(G)}$ *the odd order components* of G. The order components of non-abelian simple groups can be obtained by using the tables in [5, 7, 9]. It is obvious that $OC(G)$ is a finite subset of $\mathbb{N}$ consisting of co-prime integers. But not every subset of co-prime elements of $\mathbb{N}$ can be the order components of a finite group. That is, for a given subset A of $\mathbb{N}$, groups G such that $OC(G) = A$ do not necessarily exist. We denote by $k(A)$ the number of isomorphism classes of finite groups G satisfying $OC(G) = A$. Hence, if G is a finite group, then $k(OC(G)) \geq 1$. Using this function, we introduce the following definition.

Definition 1.1. A group G is called an *r-recognizable group* if $k(OC(G)) = r$. Usually a 1-recognizable group is called a *characterizable group*.

If p is a prime number, then $k(OC(\mathbb{Z}_p)) = 1$. Similarly, $k(OC(\mathbb{Z}_4)) = 2$.

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Therefore, we divide all groups into the following classes: characterizable groups and r-recognizable groups, where $r > 1$.

In this paper, we give some new classes of finite simple groups which are characterizable by their order components. In particular, we discuss the simple K_n-groups for $n = 3, 4$.

2. Preliminary Results. For finite characterizable groups, Chen, the first author and their collaborators, and some other authors obtained many results, which can be summarized as follows.

Theorem 2.1. (See the references of [6]) The following groups are characterizable:

(1) Alternating groups A_n, where $n = p$, $p+1$, or $p+2$, and p is an odd prime number,
(2) Symmetric groups S_n, where $n = p$ or $p+1$, and p is an odd prime number,
(3) All almost sporadic simple groups except $Aut(McL)$ and $Aut(J_2)$,
(4) Simple groups of Lie type: $E_6(q)$, ${}^2E_6(q)$, $E_8(q)$, $F_4(q)$, ${}^2D_n(q)$, $PSL(p,q)$, $PSU(p,q)$, $G_2(q)$ where $q \equiv 0 \pmod 3$, Suzuki-Ree groups, ${}^3D_4(q)$ where $q > 2$.

Lemma 2.1. ([6]) Let G be a finite group and let M be a finite group with $t(M) = 2$ satisfying $OC(G) = OC(M)$. Let $OC(M) = \{m_1, m_2\}$. Then one of the following holds:

(a) G is a Frobenius or 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group.

Moreover, $OC(K/H) = \{m'_1, m'_2, \ldots, m'_s, m_2\}$, where $m'_1 m'_2 \ldots m'_s | m_1$. Also, $G/K \leq Out(K/H)$.

Lemma 2.2. ([3]) Let G be a finite group with $t(G) \geq 2$ and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some order components of G but not π_i-numbers, then $m_1 m_2 \cdots m_r$ is a divisor of $|N| - 1$.

Theorem 2.2. [1] If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, and $U_4(2)$.

Theorem 2.3. [1] If G is a simple K_4-group, then G is isomorphic to one of the following groups: A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_3(4)$, $L_3(5)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $O_5(4)$, $O_5(5)$, $O_5(7)$, $O_5(9)$, $O_7(2)$, $O_8^+(2)$, $G_2(3)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, $Sz(8)$, $Sz(32)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, and $L_2(q)$, where q is a prime power satisfying

$$q(q^2-1) = \gcd(2, q-1) 2^{\alpha_1} 3^{\alpha_2} p^{\alpha_3} r^{\alpha_4},$$

where $\alpha_i \in \mathbb{N}$ $(1 \leq i \leq 4)$ and p, r are distinct primes.

3. Recognizability of K_n-groups.

Theorem 3.1. Let M be a simple K_3-group. Then M is characterizable by its order components.

Proof. By using Theorems 2.1 and 2.2, it follows that if $M \neq U_4(2)$, then M is characterizable by its order components. Now let $M = U_4(2)$ and $OC(G) = OC(M)$. Then $OC(G) = \{2^6 \times 3^4, 5\}$ and hence, G has a non-connected prime graph. Therefore, we can use Lemma 2.1.

If G is a Frobenius group, then $|G|$ is even and by using Lemma 2.3 in [6], we have $OC(G) = \{|H|, |K|\}$, where H and K are the Frobenius complement and the Frobenius kernel of G, respectively. Since $|H| < |K|$, we conclude that $|H| = 5$ and $|K| = 2^6 \times 3^4$. By using the properties of Frobenius groups we have $|H| \mid (|K| - 1)$, which is a contradiction. Therefore, G is not a Frobenius group.

If G is a 2-Frobenius group, then there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Therefore, by using Lemma 2.4 in [6], we have $|K/H| = 5$ and $|G/K| \mid 4$ with $|G/K| \neq 1$. If $|G/K| = 4$, then since G/H is a Frobenius group with kernel K/H, it follows that $5 \mid (4-1)$, which is a contradiction. If $|G/K| = 2$, then $|H| = 2^5 \times 3^4$ and $|K| = 2^5 \times 3^4 \times 5$. Since K is a Frobenius group with kernel H, it follows that $5 \mid (2592 - 1)$, which is impossible. Therefore, G is not a 2-Frobenius group. Hence, by using Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group and K/H is a non-abelian simple group. Also, 5 will be an odd order component of K/H. Moreover, K/H will be a simple K_3-group since G is a simple K_3-group. Obviously, $\pi(G) = \{2, 3, 5\}$ and hence, $\pi(K/H) = \{2, 3, 5\}$. Therefore, K/H is A_5, A_6, or $U_4(2)$. If $K/H = A_5$, then $|G/K| \cdot |H| = 2^4 \times 3^3$. Also, $|OUT(K/H)| = 2$ and hence, $|G/K| \mid 2$, by Lemma 2.1, and so $3^3 \mid |H|$. Let P be a 3-Sylow subgroup of H. Since H is nilpotent and $H \triangleleft G$, we conclude that $P \triangleleft G$. Hence, $5 \mid (|P| - 1)$, by Lemma 2.2, which is a contradiction.

If $K/H = A_6$, then $|G/K| \cdot |H| = 2^3 \times 3^2$. Similar to the last case, it follows that $|G/K|$ divides 2 and if P is a 3-Sylow subgroup of H, then $P \triangleleft G$ and $5 \mid (3^2 - 1)$, which is a contradiction. Therefore, $K/H = U_4(2)$ and hence, $K = G$ and $H = 1$. Hence, $U_4(2)$ is uniquely determined by its order components.

Theorem 3.2. Let M be a simple K_4-group. Then M is characterizable by its order components, except when $M = A_{10}$.

Proof. If M is a simple K_4-group, then M is one of the finite groups listed in Theorem 2.3. Also, by using Theorems 2.1 and 2.3, we conclude that the theorem is proved for the following groups:
A_7, A_8, A_9, M_{11}, M_{12}, J_2, $L_2(q)$, $L_3(4)$, $L_3(8)$, $L_3(5)$, $L_3(17)$, $C_2(7)$, $C_2(9)$, $G_2(3)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_5(2)$, $Sz(8)$, $Sz(32)$. Also, since A_{10} has a connected prime graph, it follows that $OC(A_{10}) = OC(\mathbb{Z}_{1814400}) = \{1814400\}$.

Therefore, A_{10} is not characterizable by its order component. Now we prove the theorem for the rest of them. Since the technique is similar for some of these groups, we only state a few of them.

Case I. $M = L_4(3)$

Let G be a finite group such that $OC(G) = OC(M)$. We claim that G is not a Frobenius group nor a 2-Frobenius group.

If G is a Frobenius group, then by using [6], it follows that $|H| = 13$ and $|K| = 2^7 \times 3^6 \times 5$, where H and K are the Frobenius complement and the Frobenius kernel of G, respectively. But $|H| \nmid (|K| - 1)$, which is a contradiction. Therefore, G is not a Frobenius group.

If G is a 2-Frobenius group, then there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups. Therefore, $|K/H| = 13$ and $|G/K| \mid 12$ with $|G/K| \neq 1$. Let P be a 5-Sylow subgroup of H. Then $P \trianglelefteq G$ and hence, $13 \mid (5-1)$, which is a contradiction. Therefore, by Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group such that 13 is an odd order component of K/H. Also, $|\pi(K/H)| \leq 4$ and hence, K/H is a simple K_n-group, where $n = 3$ or 4. Moreover, $\pi(K/H) \subseteq \{2,\ 3,\ 5,\ 13\}$. Therefore, K/H can be equal to $L_4(3)$, $C_2(5)$, ${}^2F_4(2)'$, $U_3(4)$, $L_2(25)$, or $L_3(3)$. Now we consider these cases separately.

If $K/H = L_3(3)$, then $|G/K| \cdot |H| = 2^3 \times 3^3 \times 5$ and $|G/K| \mid 2$. Let P be a 5-Sylow subgroup of H. Then $P \triangleleft G$ and hence, $13 \mid (5-1)$, which is a contradiction.

If $K/H = C_2(5)$, ${}^2F_4(2)'$, $U_3(4)$ or $L_2(25)$, then $5^2 \mid |K/H|$ but $5^2 \nmid |G|$, which is a contradiction. Therefore, $K/H = L_4(3)$ and hence, $K = G = L_4(3)$ and $H = 1$. Hence, $L_4(3)$ is characterizable by its order components.

Case II. $M = C_2(4)$

If $OC(G) = OC(M)$, then G is not a Frobenius group, since $17 \nmid (2^8 \times 3^2 \times 5^2 - 1)$. Also, by using the 5-Sylow subgroup of H, it follows that G is not a 2-Frobenius group. Therefore, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a non-abelian simple group and 17 is an odd order component of K/H, $|\pi(K/H)| \leq 4$, and $\pi(K/H) \subseteq \{2,\ 3,\ 5,\ 17\}$. Therefore, K/H is $L_2(17)$, $L_2(16)$, or $C_2(4)$.

If $K/H = L_2(17)$, then $|G/K| \cdot |H| = 2^4 \times 5^2$ and $|G/K| \mid 2$. Let P be a 5-Sylow subgroup of H. Then $17 \nmid (5^2 - 1)$, which is a contradiction.

If $K/H = L_2(16)$, then $|G/K| \cdot |H| = 2^4 \times 3 \times 5$ and $|G/K| \mid 4$. Let P be a 3-Sylow subgroup of H. Then $17 \nmid (3-1)$, which is a contradiction.

Hence, $K/H = C_2(4)$ and hence, $K = G = C_2(4)$ and $H = 1$. Hence, $C_2(4)$ is characterizable by its order components.

Since the proofs of the other cases are similar, we omit the details.

Order components have an important role in research in group theory. For example, by using the order components of finite groups, we can prove the validity of Thompson's conjecture and Shi-Bi's conjecture [8] for some finite groups.

Corollary 3.1. (Thompson's conjecture) Let G be a finite group with $Z(G) = 1$ and let M be a simple K_n-group, where $n = 3, 4$. Let $N(G) = \{k \mid G$ has a conjugacy class of size $k\}$. If $N(G) = N(M)$, then $G \cong M$.

Corollary 3.2. (Shi-Bi's conjecture) Let G be a group and let M be a simple K_n-group, where $n = 3, 4$. Then $G \cong M$ if and only if

(i) $|G| = |M|$ and
(ii) $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of orders of elements in G.

Proof. The assumptions of these two corollaries imply that $OC(G) = OC(M)$, by Lemmas 2.6 and 2.7 in [6]. Thus, the corollaries follow by Theorems 3.1 and 3.2.

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