ONE STEP CLOSER TO AN OPTIMAL TWO-PARAMETER SOR METHOD: A GEOMETRIC APPROACH

Saadat Moussavi

Abstract. The well-known SOR method is obtained from a one-part splitting of the system matrix A, using one parameter ω for the diagonal.

A strong interest in using more than one parameter for the SOR method to improve the convergence has been developed. Sisler, Niethammer, and Hadiidimos worked on the two-parameter method in the seventies. This author has generalized Sisler's method and introduced a range for the second parameter, providing a faster two-parameter method compared to the SOR method.

In this paper, we go one step further by removing the hypothesis that requires the eigenvalues of the Jacobi iteration matrix to be real. The result is an optimal value for the second parameter when the eigenvalues of the SOR method are in a certain well-defined region.

1. Introduction. We wish to find the solution vector x to the linear system $Ax = b$, where A is a sparse $n \times n$ matrix and b is a given n-vector of complex n-space. Usually A is not easy to invert. Therefore, we seek an easy way to invert part of A, say A_0, and we write

$$A = A_0 - A_1 \tag{1.1.1}$$

or equivalently,

$$A = A_0(I - A_0^{-1}A_1) = A_0(I - B), \tag{1.1.2}$$

where $B = A_0^{-1}A_1$ is called the *iteration matrix.*

Display (1.1.1) defines the sequence $\{x_k\}$ for an arbitrary vector x_0 via

$$A_0 x_{k+1} - A_1 x_k = b \qquad k = 0, 1, 2, \ldots$$

or equivalently,

$$x_{k+1} = A_0^{-1}A_1 x_k + A_0^{-1}b \qquad k = 0, 1, 2, \ldots\,, \text{ and}$$
$$x_{k+1} = Bx_k + A_0^{-1}b \qquad k = 0, 1, 2, \ldots\,.$$

By (1.1.1) it is clear that if $\{x_k\}$ converges at all, it must converge to $x_{sol} = A^{-1}b$. Display (1.1.2) shows that $\{x_k\}$ converges to $x_{sol} = A^{-1}b$ for each x_0 if and only if $\rho(B) < 1$, where $\rho(B)$ is the spectral radius of B [9].

We use (1.1.2) to measure the asymptotic convergence R_∞ of the sequence $\{x_k\}$, where R_∞ is defined by $R_\infty = -\log \rho(B)$, which carries information about how fast the sequence $\{x_k\}$ converges. In fact, $\frac{1}{R_\infty}$ asymptotically represents the number of iterations that suffice to produce one additional decimal place of accuracy in the x_k's.

The following well-known iteration methods are two examples of such a splitting. For the given matrix A, let $-L$, $-U$, and D denote the strictly lower triangular, upper triangular, and diagonal part of A, respectively.

JACOBI Method. Choose $A_0 = D$ and $A_1 = L + U$, where D is the diagonal part of A and $-L$, $-U$ are the strictly lower and upper triangular parts of A, respectively.

Successive Overrelaxation (SOR) Method. Choose $A_0 = \frac{1}{\omega}D - L$ and $A_1 = (\frac{1}{\omega} - 1)D + U$.

The Successive Overrelaxation (SOR) method was developed independently in the fifties by Frankel [2] and Young [13, 14]. Since then there has been strong interest in using more than one parameter for the SOR method to improve the convergence [3, 4, 6, 7, 8, 10, 11, 12].

The modified Successive Overrelaxation (MSOR) method was first considered by Devogelaere [1]. Consider the matrix A in the following form

$$A = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix},$$

where D_1 and D_2 are square, non-singular matrices. We use ω and ω' to create the easy to invert part of A given by

$$A_0 = \begin{bmatrix} \frac{1}{\omega}D_1 & 0 \\ N & \frac{1}{\omega'}D_2 \end{bmatrix}.$$

Young [15] has shown that if A is positive-definite, $0 < \omega \leq 1$, and $0 < \omega' \leq 1$, then the Gauss-Seidel iteration method converges faster than the MSOR method. In [5], Young's Theorem has been generalized for the case where the MSOR method converges faster than the Gauss-Seidel method.

In the case where the eigenvalues of the SOR method are restricted to a certain configuration in the complex plane, we attempt in Theorem 2.8 to find the optimum value for α, the second parameter. Moreover, the result will be a generalization of the dePillis result given in Corollary 2.9.

2. A Geometric Approach. In [6], it has been shown that λ, the eigenvalue of the SOR iteration matrix, and ζ, the eigenvalue of $B_{(\frac{\delta}{\alpha},\frac{\delta}{\alpha},\alpha)}$, the two-parameter iteration matrix, are related by

$$\zeta = \frac{1}{\alpha}\lambda + \left(1 - \frac{1}{\alpha}\right) \cdot 1.$$

Remark 2.1.1. If λ is a point in the complex plane and $\zeta = \frac{1}{\alpha}\lambda + (1 - \frac{1}{\alpha})$, then

$$\alpha = \frac{(\operatorname{Im}\lambda)^2 + (1 - \operatorname{Re}\lambda)^2}{1 - \operatorname{Re}\lambda}, \tag{2.1.3}$$

where $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ represent the real and imaginary parts of λ, respectively, and it produces ζ with the smallest magnitude.

Remark 2.1.2. If λ_1 and λ_2 are two points in the complex plane and $\zeta_k = \frac{1}{\alpha}\lambda_k + (1 - \frac{1}{\alpha}) \cdot 1$ for $k = 1, 2$, then

$$\alpha = 1 + \frac{|\lambda_1|^2 - |\lambda_2|^2}{2\,(\operatorname{Re}\lambda_1 - \operatorname{Re}\lambda_2)}, \tag{2.1.4}$$

where $\operatorname{Re}\lambda_1$ and $\operatorname{Re}\lambda_2$ represent the real parts of $\lambda_1$and λ_2, respectively, and it produces ζ_1 and ζ_2 such that $|\zeta_1| = |\zeta_2|$.

Theorem 2.2. Suppose that $A_0 = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}$, where D_1 and D_2 are non-singular matrices. If all the eigenvalues of the SOR method lie in the shaded area in Figure 1, where λ and ρ belong to $\sigma(B_\omega)$, the set of eigenvalues of the SOR method, and

$$\alpha_1 = 1 + \frac{|\rho|^2 - |\lambda|^2}{2\,(\operatorname{Re}\rho - \operatorname{Re}\lambda)} \text{ and } \alpha_2 = \frac{(\operatorname{Im}\lambda)^2 + (1 - \operatorname{Re}\lambda)^2}{1 - \operatorname{Re}\lambda},$$

then $\alpha = \max\{\alpha_1, \alpha_2\}$ is the optimal parameter for the two-parameter method $B_{(\frac{\delta}{\alpha}, \frac{\delta}{\alpha}, \alpha)}$.

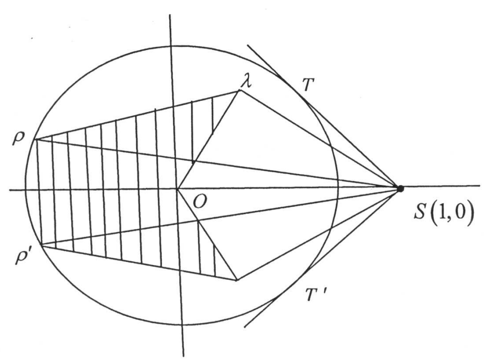

Figure 1

Proof. By Remarks 2.1.1 and 2.1.2 we know that

(1) the parameter α_2 shifts λ to the point H on the line $S\lambda$ that passes through the two points S and λ, where $S = (1, 0)$, $\lambda = (\operatorname{Re}\lambda, \operatorname{Im}\lambda)$, and OH is perpendicular to the line $S\lambda$ (Figure 2), and

(2) the parameter α_1 shifts λ to the point B on the line $S\lambda$ and moves ρ to the point A on the line $S\rho$ that passes through the two points S and ρ, where $S = (1, 0)$, $\rho = (\text{Re}\rho, \text{Im}\rho)$, and $OA = OB$ (Figure 2).

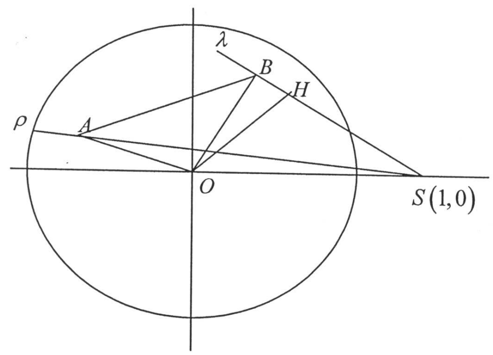

Figure 2

Case 1. Suppose $\alpha_1 = \max\{\alpha_1, \alpha_2\}$ or $\alpha_1 > \alpha_2$. Then point B must lie to the right of point H on the line $S\lambda$, the line that passes through the two points S:(1,0) and λ.

(i) Let α_3 be any parameter that shifts λ to the point B' which lies to the right of B on the line $S\lambda$. The parameter α_3 shifts ρ to the point A' to the right of A, on the line $S\rho$, the line that passes through the two points S and ρ. This occurs because AB and $A'B'$ are parallel (Figure 3).

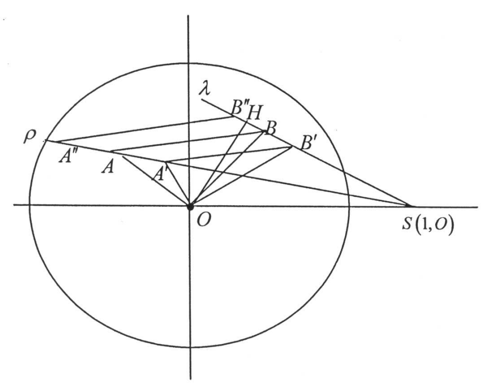

Figure 3

Since $OA' < OB'$ and $OB' > OB = OA$, OB' represents the spectral radius of the two-parameter method using α_3. In this case, $\rho(B_{(\omega,\alpha_3)}) > \rho(B_{(\omega,\alpha_1)})$.

(ii) Let α_3 be a parameter that shifts λ to the point B'', lying to the left of H on the line $S\lambda$ (Figure 3). This parameter, α_3, slides ρ to the point A'' on the line $S\rho$. This shift occurs because AB and $A''B''$ are parallel. Since $OA'' > OA = OB$, OA'' represents the spectral radius of the two-parameter method using α_3. Again we conclude that $\rho(B_{(\omega,\alpha_3)}) > \rho(B_{(\omega,\alpha_1)})$.

By (i) and (ii), we can conclude that α_1 is optimal under the conditions of Case 1 wherein $\alpha_1 = \max\{\alpha_1, \alpha_2\}$.

Case 2. Suppose $\alpha_2 = \max\{\alpha_1, \alpha_2\}$ or $\alpha_2 > \alpha_1$. Then the point B must lie to the left of the point H on the line $S\lambda$, the line that passes through the two points S and λ. The parameter α_2 also moves ρ to the point A on the line $S\rho$ such that $OA = OB$ (Figure 4).

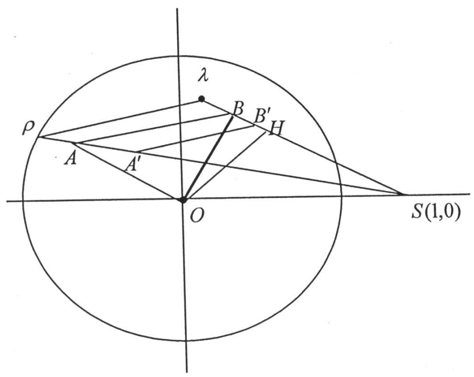

Figure 4

For any α, say α_3, preceding in the same manner as in parts (i) and (ii), one can show that OH is the smallest spectral radius in Case 2, that is

$$\rho(B_{(\omega,\alpha)}) > \rho(B_{(\omega,\alpha_2)}) \text{ for any } \alpha.$$

Therefore, α_2 is optimal under the conditions of Case 2 wherein $\alpha_2 = \max\{\alpha_1, \alpha_2\}$.

Case 3. If the point H lies to the left of λ on the line $S\lambda$ that passes through the two points S and λ, then α_1 is the optimal parameter. This is true because if α_2 shifts λ to the left, it will also shift ρ to the left along the line $S\rho$, hence outside the circle. In this case, $\alpha_1 = \max\{\alpha_1, \alpha_2\}$ since $\alpha_1 < 1$, but α_1 is always greater than 1 in the given shaded region.

Cases 1, 2, and 3 show that $\alpha = \max\{\alpha_1, \alpha_2\}$ is the optimal parameter for the two-parameter method.

Remark 2.2.1. Suppose $A_0 = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}$, where D_1 and D_2 are non-singular matrices. If all the eigenvalues of the SOR method lie in the shaded area of Figure 1, where λ and ρ belong to $\sigma(B_\omega)$, the set of eigenvalues of the SOR method, and

$$\alpha_1 = 1 + \frac{|\rho|^2 - |\lambda|^2}{2\,(\operatorname{Re}\rho - \operatorname{Re}\lambda)} \quad \text{and} \quad \alpha_2 = \frac{(\operatorname{Im}\lambda)^2 + (1 - \operatorname{Re}\lambda)^2}{1 - \operatorname{Re}\lambda},$$

then $\alpha = \max\{\alpha_1, \alpha_2\}$ shifts the given shaded region bounded by $\lambda O \lambda' \rho' \rho$ to the shaded area bounded by $BOB'A'A$ (Figure 5).

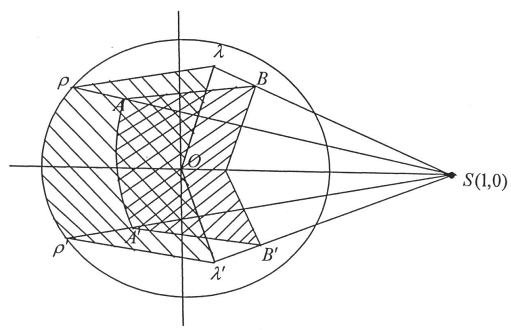

Figure 5

Corollary 2.3 (dePillis). If the eigenvalues of the SOR method are inside the shaded area TKT' in Figure 6, and ρ, an eigenvalue of the SOR method, is on the arc TKT', where T and T' are the intersecton points of the tangent lines to the circle from point S, then the parameter that shifts ρ to point H is optimal, where OH is perpendicular to $S\rho$.

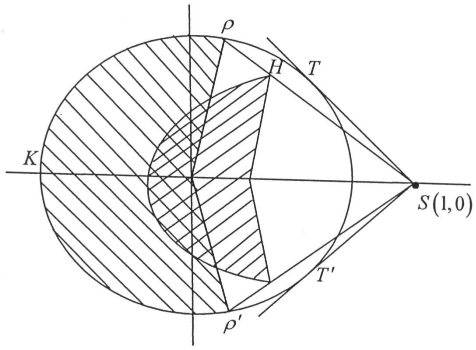

Figure 6

Proof. In this case since λ and ρ coincide, by (2.7.16), $\alpha_1 = 1$ and $\alpha_2 > 1$. Hence, $\alpha_2 = \max\{\alpha_1, \alpha_2\}$.

Examples.

(1) The eigenvalues of the SOR method are $\lambda_1, \lambda_2 = 0.4 \pm 0.4i$ and $\rho, \rho' = -0.7 \pm 0.3873i$. By Remarks 2.1.1 and 2.1.2

$$\alpha_1 = 1.1454513 \text{ and } \alpha_2 = 0.8666666.$$

Thus, by Theorem 2.2, α_1 is optimal. The spectral radii of the SOR method and the two-paramater methods are

$$\rho(B_\omega) = 0.8$$
$$\rho(B_{(\omega,\alpha_1)}) = 0.5905$$
$$\rho(B_{(\omega,\alpha_2)}) = 1.0603.$$

Thus, $\rho(B_{(\omega,\alpha_1)}) < \rho(B_\omega)$.

(2) The eigenvalues of the SOR method are $\lambda_1, \lambda_2 = 0.4 \pm 0.68i$ and $\rho, \rho' = -0.7 \pm 0.3873i$. By Remarks 2.1.1 and 2.1.2,

$$\alpha_1 = 1.02933548 \text{ and } \alpha_2 = 1.7302857.$$

Thus, by Theorem 2.2, α_2 is optimal. The spectral radii of the SOR method and the two-paramater methods are

$$\rho(B_\omega) = 0.8$$
$$\rho(B_{(\omega,\alpha_1)}) = 0.7524$$
$$\rho(B_{(\omega,\alpha_2)}) = 0.4369.$$

Thus, $\rho(B_{(\omega,\alpha_2)}) < \rho(B_{(\omega,\alpha_1)}) < \rho(B_\omega)$.

References

1. R. Devogelaere, "Overrelaxation, An Abstract," *Amer. Math. Soc. Notices,* (1950), 147–273.
2. S. P. Frankel, "Convergence Rate of Iterative Treatments of Partial Differential Equations," *Math. Tables Aids Comp.,* 4 (1950), 65–75.
3. A. Hadjidimos, "Accelerated Overrelaxation Method," *Mathematics of Computation,* 32 (1978), 149–157.
4. S. Moussavi, "More About the Two Parametric SOR Method," *Rendiconti dell' Instituto di Mathematics dell' Univ. di Trieste,* 22 (1991), 7–27.
5. S. Moussavi, "A Generalization of Young's Theorem," *MJMS*, 4 (1992), 76–87.
6. S. Moussavi, "Another Step Toward an Optimal Two-Parameter SOR Method," *MJMS*, 21 (2009), 42–55.
7. W. Niethammer and J. Schade, "On a Relaxed SOR Method Applied to Non-Symmetric Linear Systems," *J. Comp. Appl. Math.,* 1 (1975) 133–136.
8. W. Niethammer, "On Different Splittings and the Associated Iterations Methods," *SIAM J. Numer. Anal.,* 16 (1979), 186–200.
9. R. Plemmons and A. Berman, *Nonnegative Matrices in the Mathematical Sciences,* Academic Press, New York, London, 1979.
10. M. Sisler, "Uberein Iterationsverfahren fur Zyklische Matrizen," *Apl. Mat.*, 17 (1972), 225–233.
11. M. Sisler, "Uber die Knvergenez Eines Gewissen Iterationsverfasrens fur Zyklisch Matrizen," *Apl. Mat.*, 18 (1973), 89–98.
12. M. Sisler, "Uber die Optimierung Eines Zweiparametrigen Iterationsverfasrens," *Apl. Math.*, 20 (1975), 126–142.
13. D. Young, *Iterative Methods for Solving Partial Difference Equations of Elliptic Type,* Doctoral Thesis, Harvard Univ., Cambridge, MA, 1950.
14. D. Young, "Iterative Methods for Solving Partial Differential Equations of Elliptic Type," *Trans. Amer. Math. Soc.,* 76 (1954), 92–111.
15. D. Young, *Iterative Solutions of Large Linear Systems of Equations,* Academic Press, New York, London, 1971.

Mathematics Subject Classification (2000): 65F10

Saadat Moussavi
Department of Mathematics
University of Wisconsin-Oshkosh
Oshkosh, WI 54901
email: moussavi@uwosh.edu