ANOTHER STEP TOWARD AN OPTIMAL TWO-PARAMETER SOR METHOD

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Abstract. The SOR method is a well-known method obtained from a one-part splitting of the system matrix A, using one parameter ω for the diagonal. Using one parameter for the lower triangular matrix of A, M. Sisler introduced a new method. Later, he combined the standard SOR method and his method to get a two-parameter method.

Sisler proved that for cyclic and positive-definite matrices, if zero is an eigenvalue of the Jacobi iteration matrix, the two-parameter method is not superior to the SOR method.

In this paper we generalize Sisler's method and provide a range for the second parameter on which the two-parameter method is superior to the SOR method.

1. Introduction. We wish to find the solution vector x to the linear system $Ax = b$ where A is a sparse $n \times n$ matrix, and b is a given vector in the complex n-space. Usually A is not easy to invert. Let A_0 be an easy to invert part of A and write

$$A = A_0 - A_1 \tag{1.1.1}$$

or equivalently,

$$A = A_0(I - A_0^{-1}A_1) = A_0(I - B) \tag{1.1.2}$$

where $B = A_0^{-1}A_1$ is called the *iteration matrix.*

Display (1.1.1) is called an *additive splitting* which defines the sequence $\{x_k\}$ for an arbitrary vector x_0 via,

$$A_0 x_{k+1} - A_1 x_k = b \qquad k = 0, 1, 2, \ldots$$

or equivalently,

$$x_{k+1} = A_0^{-1}A_1 x_k + A_0^{-1}b \qquad k = 0, 1, 2, \ldots, \text{ and}$$

$$x_{k+1} = Bx_k + A_0^{-1}b \qquad k = 0, 1, 2, \ldots .$$

By (1.1.1) it is clear that if $\{x_k\}$ converges at all, it must converge to $x_{sol} = A^{-1}b$, the vector solution, where $Ax_{sol} = b$.

Display (1.1.2) shows that $\{x_k\}$ converges to $x_{sol} = A^{-1}b$ for each x_0 if and only if $\rho(B) < 1$ where $\rho(B)$ is the spectral radius of B [8].

We use (1.1.2) to measure the asymptotic convergence R_∞ of the sequence $\{x_k\}$, where R_∞ is defined by R_∞=-log $\rho(B)$, which carries information about how fast the sequence $\{x_k\}$ converges. In fact $\frac{1}{R_\infty}$ asymptotically represents the number of iterations that suffice to produce one additional decimal place of accuracy in the $x_k's$.

The above splitting is called *stationary* since there is no altering of the parameter from iteration to iteration, and it is a *one-part splitting* since each x_{k+1} depends on one previous vector x_k. The following well-known iteration methods are examples of one-part stationary splitting.

For the given matrix A, let $-L$, $-U$, and D denote the strictly lower triangular, strictly upper triangular, and diagonal part of A, respectively.

JACOBI Method. Choose $A_0 = D$ and $A_1 = L + U$. The Jacobi iteration matrix is represented by $B_J = A_0^{-1}A_1 = D^{-1}(L+U)$.

Successive Overrelaxation (SOR) Method. Choose $A_0 = \frac{1}{\omega}D - L$ and $A_1 = (\frac{1}{\omega} - 1)D + U$. The SOR iteration matrix is represented by

$$B_\omega = A_0^{-1}A_1 = (D - \omega L)^{-1}((1-\omega)D + \omega U). \tag{1.1.3}$$

The Successive Overrelaxation (SOR) method was developed independently in the 1950's by Frankel [2] and Young [12,13]. Since then there has been much interest in using more than one parameter for the SOR method to improve the convergence [3, 4, 5, 6, 7, 9]. In 1972 Sisler [9] used one more parameter, say α, for the lower triangular matrix L. Hence, $A_0 = D - \alpha L$ and $A_1 = (1-\alpha)L + U$.

A similar conclusion holds for the upper triangular matrix U. Later, Sisler combined his method and the SOR method [10] to obtain

$$A_0 = \frac{1}{\omega}D - \alpha L \text{ and } A_1 = \left(1 - \frac{1}{\omega}\right)D + (1-\alpha)L + U.$$

Then

$$B_{(\omega,\alpha)} = (D - \omega\alpha L)^{-1}((1-\omega)D + (1-\alpha)\omega L + \omega U). \tag{1.1.4}$$

He studied these methods for a cyclic matrix A and showed that if A is also positive-definite and $\mu = 0$ is an eigenvalue of the Jacobi iteration matrix B_J, then the two-parameter method (1.1.4) is not superior to the SOR method (1.1.3). His proof of Corollary 2.3 is rather complicated.

Later, Hadjidimos further developed Sisler's method and called the result the Accelerated Overrelaxation (AOR) method [3].

The modified Successive Overrelaxation (MSOR) method was first considered by Devogelaere [1]. Suppose the matrix A in the following form

$$A = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}$$

where D_1 and D_2 are square, non-singular matrices. Use ω for the "red" equations corresponding to D_1 and ω' for the "black" equations corresponding to D_2. The iteration matrix for the MSOR method is given by

$$B_{(\omega,\omega')} = A_0^{-1}A_1 = \begin{bmatrix} (1-\omega)I_1 & \omega F \\ \omega'(1-\omega)G & \omega\omega' GF + (1-\omega')I_2 \end{bmatrix}$$

where $F = -D_1^{-1}M$ and $G = -D_2^{-1}N$. Young [14] proved that if A is positive-definite then $\rho(B_{\omega_b}) < \overline{\rho}(B_{(\omega,\omega')})$, where ω_b is the optimal parameter for the SOR method and $\overline{\rho}(B_{(\omega,\omega')})$ is the virtual spectral radius of $B_{(\omega,\omega')}$. Young also showed that if A is positive-definite and $0 < \omega, \omega' \leq 1$ then the Gauss-Seidel (SOR with $\omega = 1$) iteration method converges faster than the MSOR method. In [5], we generalize Young's theorem for the case that the MSOR method converges faster than the Gauss-Seidel method and we combined the SOR and MSOR methods, which gives the following invertible part:

$$A = \begin{bmatrix} \frac{1}{\omega}D_1 & 0 \\ \alpha N & \frac{1}{\omega'}D_2 \end{bmatrix}.$$

In this paper we generalize Sisler's result in Theorem 2.2. Sisler's theorem with A positive-definite and $\mu = 0$ an eigenvalue of B_J [11] becomes Corollary 2.3 to our theorem. Furthermore, when the eigenvalues of the SOR method are restricted to a certain configuration in the complex plane, we introduce, in Theorem 2.4 and Theorem 2.5, a range value of α, for which the two-parameter method has faster convergence than the SOR method.

2. A Generalized Two-Parameter Method.

The matrix

$$B_{(\omega,\omega',\alpha)} = A_0^{-1}A_1 = \begin{bmatrix} (1-\omega)I_1 & \omega F \\ \omega'(1-\alpha\omega)G & \alpha\omega\omega' GF + (1-\omega')I_2 \end{bmatrix}$$

is the iteration matrix for the accelerated MSOR method [5].

It has been shown that λ, the eigenvalue of the accelerated MSOR iteration matrix, and μ, the eigenvalue of B_J, the Jacobi iteration matrix, are related by the following equation [5].

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = (\alpha\lambda + (1-\alpha))\omega\omega'\mu^2. \tag{2.1.5}$$

Theorem 2.1. Suppose that $A = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}$, where D_1 and D_2 are non-singular matrices. If ζ is an eigenvalue of the two-parameter iteration matrix $B_{(\frac{\delta}{\alpha},\frac{\delta}{\alpha},\alpha)}$ and λ is an eigenvalue of the SOR method, then

$$\zeta = \frac{1}{\alpha}\lambda + \left(1 - \frac{1}{\alpha}\right). \tag{2.1.6}$$

Proof. By (2.1.5), if ζ is an eigenvalue of $B_{(\frac{\delta}{\alpha},\frac{\delta}{\alpha},\alpha)}$ and μ is an eigenvalue of the Jacobi iterations matrix B_J, then

$$\left(\zeta + \frac{\delta}{\alpha} - 1\right)^2 = (\alpha\zeta + (1-\alpha))\frac{\delta^2}{\alpha^2}\mu^2. \tag{2.1.7}$$

If λ is an eigenvalue of the SOR method and μ is an eigenvalue of Jacobi iteration matrix B_J, then

$$(\lambda + \delta - 1)^2 = \lambda\delta^2\mu^2 \text{ [10]}. \tag{2.1.8}$$

By (2.1.7) and (2.1.8), we have

$$\frac{\alpha^2(\zeta + \frac{\delta}{\alpha} - 1)^2}{\alpha\zeta + (1-\alpha)} = \frac{(\lambda+\delta-1)^2}{\lambda}, \text{ and}$$

$$\lambda(\alpha(\zeta-1)+\delta)^2 = (\alpha(\zeta-1)+1)(\lambda+\delta-1)^2. \tag{2.1.9}$$

Let $\nu = \zeta - 1$ in (2.1.9). Then

$$(\lambda\alpha^2)\nu^2 + (2\alpha\delta\lambda - \alpha(\lambda+\delta-1)^2)\nu - (\lambda+\delta-1)^2 + \lambda\delta^2 = 0. \tag{2.1.10}$$

The discriminant, Δ, of (2.1.10) is

$$\begin{aligned}\Delta &= \alpha^2(\lambda+\delta-1)^4 - 4\alpha^2\lambda(\delta-1)(\lambda+\delta-1)^2 \\ &= \alpha^2(\lambda+\delta-1)^2(\lambda-\delta+1)^2\end{aligned}$$

and the solutions are

$$v_1 = \frac{\lambda-1}{\alpha} \text{ and } v_2 = \frac{(\delta-1)^2-\lambda}{\alpha\lambda}.$$

Since $v = \zeta - 1$,

$$\zeta = \frac{1}{\alpha}\lambda + \left(1 - \frac{1}{\alpha}\right) \text{ or} \tag{2.1.11}$$

$$\zeta = \frac{1}{\alpha}\frac{(\delta-1)^2}{\lambda} + \left(1 - \frac{1}{\alpha}\right). \tag{2.1.12}$$

We know that if $\alpha = 1$, then the two-parameter method becomes the standard SOR method. Therefore, (2.1.11) is the representation of the relationship between the eigenvalues of the SOR method and the two-parameter method.

Remark. By (2.1.11) an eigenvalue ζ of the two-parameter method is obtained by shifting λ, the eigenvalue of the SOR method, on the line that passes through the two points (1,0) and λ.

Theorem 2.2. Suppose A is consistently ordered and 2-cyclic with all the eigenvalues of B_J are real and $\rho(B_J) < 1$. If there is an eigenvalue of $\rho(B_{w_b})$ with real part less than $[\rho(B_J)]^2$, then the two-parameter method is not superior to the SOR method.

Proof. Let ζ be an eigenvalue of the two-parameter method, and let λ be an eigenvalue of the SOR method. By Theorem 2.1 we have

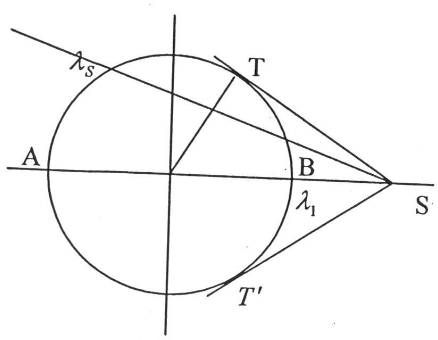
$$\zeta = \frac{1}{\alpha}\lambda + \left(1 - \frac{1}{\alpha}\right).$$

(1) If $\alpha < 1$, then an eigenvalue ζ of the two-parameter method can be obtained by shifting the eigenvalue of the SOR method to the left of λ on the line that passes through the two points S and λ where $S = (1, 0)$ and $\lambda = (\text{Re }\lambda, \text{Im }\lambda)$.

(2) If $\alpha > 1$, then an eigenvalue ζ of the two-parameter method is obtained by shifting λ the eigenvalue of the SOR method to the right of λ on the line that passes through the two points S and λ.

Since $\omega = \omega_b$, the optimum value of ω, all the eigenvalues of B_{ω_b} lie on the circle with radius $\omega_b - 1$ and center at origin [12].

We draw two tangent lines from the point S to this circle, and call the points of contact T and T' (Figure 1).



(Fig. 1)

If the location of the point T is determined by the complex number $x + yi$, then

$$[\rho(B_{\omega_b})]^2 + ((x-1)^2 + y^2) = 1$$

which leads to $x = [\rho(B_{\omega_b})]^2$. Thus, the real part of points T and T' is $[\rho(B_{\omega_b})]^2$. Therefore, if an eigenvalue of B_{ω_b}, say λ_s, has real part less than

$[\rho(B_{\omega_b}]^2$, then it must be located on the arc TAT' (Figure 1). We know that B_{ω_b} has an eigenvalue, say $\lambda_1 = \omega_b - 1$, at point B [14] (Figure 1).

The parameter $\alpha > 1$ slides λ_1 toward the point S, which causes a worse spectral radius for the two-parameter method. On the other hand, $\alpha < 1$ slides λ_s to the left, hence outside the circle, which causes a larger spectral radius for the two parameter method. Under the assumption of this theorem, $\alpha = 1$ (i.e., the SOR method) is optimal.

Corollary 2.3. (Sisler) Suppose A is consistently ordered and 2-cyclic, where all the eigenvalues of B_J are real and $\rho(B_J) < 1$. If $\mu = 0$ is an eigenvalue of the Jacobi iteration matrix B_J, then the two-parameter method is not superior to the SOR method.

Proof. Since $\mu = 0$ is an eigenvalue of B_J, $\lambda = 1 - \omega_b$ is an eigenvalue of the SOR iteration matrix B_{ω_b} [14]. Since $\lambda = 1 - \omega_b$ is located on the arc TAT' (Figure 1), by Theorem 2.2 the two-parameter method is not superior to the SOR method for $\omega = \omega_b$.

Theorem 2.4. Suppose A is consistently ordered and 2-cyclic, where all the eigenvalues of B_J are real and $\rho(B_J) < 1$. If all the eigenvalues of the SOR method have real parts greater than $[\rho(B_J)]^2$, then the two-parameter method is superior to the SOR method for

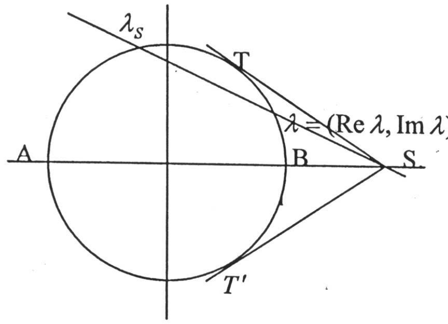
$$\frac{[\rho(B_{\omega_b})]^2 + 1 - 2\,\mathrm{Re}\,\lambda}{1 - [\rho(B_{\omega_b}]^2} < \alpha < 1,$$

where λ is the eigenvalue of B_{ω_b} with the smallest real part.

Proof. All the eigenvalues of the SOR iteration matrix including $\lambda_1 = \omega_b - 1$ are on the arc TBT' (Figure 2). The parameter $\alpha < 1$ shifts all the eigenvalues of the SOR method toward the inside of the circle. We will show that

$$\alpha = \frac{[\rho(B_{\omega_b})]^2 + 1 - 2\,\mathrm{Re}\,\lambda}{1 - [\rho(B_{\omega_b})]^2}$$

shifts the eigenvalue λ, which lies on TBT', to λ_s on the arc TAT' (Figure 2).



(Fig. 2)

The line $S\lambda$ that passes through the two points S and λ is given by

$$y = \frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda - 1}(x-1). \tag{2.4.13}$$

The circle with center at origin and radius $\rho(B_{\omega_b})$ is given by

$$x^2 + y^2 = [\rho(B_{\omega_b})]^2. \tag{2.4.14}$$

The intersection of the line and the circle is obtained by solving the following equation.

$$[(\mathrm{Im}\,\lambda)^2 + (\mathrm{Re}\,\lambda - 1)^2]x^2 - 2\,(\mathrm{Im}\,\lambda)^2 x + (\mathrm{Im}\,\lambda)^2 - (\mathrm{Re}\,\lambda - 1)^2[\rho(B_{\omega_b})]^2 = 0. \tag{2.4.15}$$

We get the two points

$$(\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda)$$

and

$$\left(\frac{2[\rho(B_{\omega_b})]^2 - \mathrm{Re}\,\lambda - \mathrm{Re}\,\lambda[\rho(B_{\omega_b})]^2}{[\rho(B_{\omega_b})]^2 - 2\,\mathrm{Re}\,\lambda + 1}, \frac{\mathrm{Im}\,\lambda(1 - [\rho(B_{\omega_b})]^2}{[\rho(B_{\omega_b})]^2 - 2\,\mathrm{Re}\,\lambda + 1}\right)$$

represented by λ and λ_s, respectively. Since

$$|\lambda_S|^2 = (\mathrm{Re}\,\lambda_S)^2 + (\mathrm{Im}\,\lambda_S)^2 = [\rho(B_{\omega_b})]^2,$$

λ_S lies on the circle.

Using the fact that $\lambda_s = \frac{1}{\alpha}\lambda + (1 - \frac{1}{\alpha})$, we can compute α by

$$\alpha = \frac{[\rho(B_{\omega_b})]^2 - 2\mathrm{Re}\,\lambda + 1}{1 - [\rho(B_{\omega_b})]^2}. \tag{2.4.16}$$

The parameter α (2.4.16) shifts λ, which lies on TBT', to λ_S on the arc TAT'. Therefore,

$$\frac{[\rho(B_{\omega_b})]^2 + 1 - 2\mathrm{Re}\,\lambda}{1 - [\rho(B_{\omega_b})]^2} < \alpha < 1$$

will shift all the eigenvalues of the SOR method toward the inside of the circle but never outside the circle. This implies that the spectral radius of the two-parameter method is smaller than $\rho(B_{\omega_b})$.

Example 2.4.1. The eigenvalues of the optimal SOR method, where $\omega_b = 1.50132$ are:

$$\begin{aligned} \lambda_1, \lambda_2 &= 0.4352 \pm 0.248842762i, \\ \lambda_3, \lambda_4 &= 0.4703 \pm 0.1736077544i, \\ \lambda_5, \lambda_6 &= 0.49 \pm 0.1055327258i, \text{ and} \\ \lambda_7 &= 0.50132. \end{aligned}$$

The spectral radius of the SOR method is $\rho(B_{\omega_b}) = 0.50132$.

Note that the two points T and T' (Figure 2) are located at $0.2513217427 \pm 0.433773125i$. All the eigenvalues of B_{ω_b} are located on the arc TBT'. Among the eigenvalues of B_{ω_b}, λ_1 and λ_2 have the smallest real part, $\mathrm{Re}\,\lambda = 0.4352$. Using (2.4.16) to calculate α, we get

$$\alpha = 0.508792313.$$

By Theorem 2.4, any α in the range of $0.508792313 < \alpha < 1$ provides a faster two-parameter method compared to the SOR method. We check the results for $\alpha = 0.65$ and $\alpha = 0.84$.

(1) Let $\alpha = 0.65$. The eigenvalues of the two-parameter method are

$$\begin{aligned} \zeta_1, \zeta_2 &= 0.1310769231 \pm 0.38288349634i, \\ \zeta_3, \zeta_4 &= 0.1850769231 \pm 0.2670888529i, \\ \zeta_5, \zeta_6 &= 0.2153846154 \pm 0.1629734243i, \text{ and} \\ \zeta_7 &= 0.2328. \end{aligned}$$

The spectral radius of the two-parameter method is $\rho(B_{(\omega_b,\alpha)}) = 0.40406526522$. Hence, $\rho(B_{(\omega_b,\alpha)}) < \rho(\omega_b)$.

(2) Let $\alpha = 0.84$. The eigenvalues of the two-parameter method are

$$\begin{aligned}\zeta_1, \zeta_2 &= 0.3276190476 \pm 0.2962413407i,\\ \zeta_3, \zeta_4 &= 0.3694047619 \pm 0.2066758981i,\\ \zeta_5, \zeta_6 &= 0.3928571429 \pm 0.1261103879i, \text{ and}\\ \zeta_7 &= 0.406333333.\end{aligned}$$

The spectral radius of the two-parameter method is $\rho(B_{(\omega_b,\alpha)}) = 0.4416935279$.

Hence, $\rho(B_{(\omega_b,\alpha)}) < \rho(\omega_b)$.

Theorem 2.8. Let $A = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}$, where D_1 and D_2 are non-singular matrices. Suppose the eigenvalues of the SOR method lie inside or on the circle with center at origin and radius $\rho(B_\omega)$. If all the eigenvalues that represent the spectral radius of the SOR method are located either on the arc TAT' or on the arc TBT' (Figure 3), then the two-parameter method has faster convergence than the SOR method for the following range of α:

Case 1. If all those eigenvalues of the SOR method that represent the spectral radius of B_ω are located on arc TAT' then the range for α is

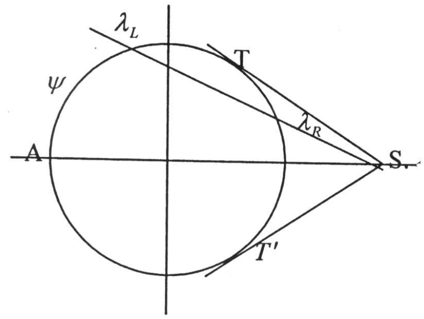
$$1 < \alpha < \frac{m}{1 - \operatorname{Re}\lambda - \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}},$$

where λ is the eigenvalue of the SOR method with the largest real part, and $m = 1 + |\lambda| - 2\operatorname{Re}\lambda$.

Case 2. If those eigenvalues of the SOR method which represent the spectral radius of B_ω are all located on arc TBT' then the range for α is

$$\frac{m}{1 - \operatorname{Re}\lambda + \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}} < \alpha < 1,$$

where λ is the eigenvalue of the SOR method with the smallest real part, and $m = 1 + |\lambda| - 2\operatorname{Re}\lambda$.



(Fig. 3)

Proof.

Case 1. Let the eigenvalue ψ of the SOR method which represents the spectral radius of B_ω lie on arc TAT', and let λ be the eigenvalue of the SOR method with the largest real part.

We will show that

$$\alpha = \frac{m}{1 - \operatorname{Re}\lambda \pm \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}}, \tag{2.5.17}$$

where $m = 1 + |\lambda| - 2\operatorname{Re}\lambda$ shifts the eigenvalue λ, which lies inside the circle, to λ_L on the arc to TAT' or to λ_R on the arc TBT' (Figure 3).

By (2.4.13) and (2.4.14) we have $[(\operatorname{Im}\lambda)^2 + (\operatorname{Re}\lambda - 1)^2]x^2 - 2(\operatorname{Im}\lambda)^2 x + (\operatorname{Im}\lambda)^2 - (\operatorname{Re}\lambda - 1)^2[\rho(B_{\omega_b})]^2 = 0$. Hence,

$$x = \frac{(\operatorname{Im}\lambda)^2 \pm (\operatorname{Re}\lambda - 1)\sqrt{[\rho(B_\omega)]^2((\operatorname{Im}\lambda)^2 + (1 - \operatorname{Re}\lambda)^2) - (\operatorname{Im}\ \lambda)^2}}{(\operatorname{Re}\lambda - 1)^2 + (\operatorname{Im}\lambda)^2}.$$

Let $m = (\operatorname{Re}\lambda - 1)^2 + (\operatorname{Im}\lambda)^2 = 1 + |\lambda| - 2\operatorname{Re}\lambda$. Hence, the real part of the new point ζ is

$$\operatorname{Re}\zeta = \frac{(\operatorname{Im}\lambda)^2 \pm (\operatorname{Re}\lambda - 1)\sqrt{[\rho(B_\omega)]^2 m - (\operatorname{Im}\lambda)^2}}{m}. \tag{2.5.18}$$

By (2.4.13),

$$\operatorname{Im}\zeta = \frac{(\operatorname{Im}\lambda)[(\operatorname{Im}\lambda)^2 \pm (\operatorname{Re}\lambda - 1)\sqrt{[\rho(B_\omega)]^2 m - (\operatorname{Im}\lambda)^2}}{m(\operatorname{Re}\lambda - 1)}. \tag{2.5.19}$$

Using (2.5.18), (2.5.19), and the fact that $\zeta = \frac{1}{\alpha}\lambda + (1 - \frac{1}{\alpha})$ (Theorem 2.1) to calculate α, we get

$$\alpha = \frac{m}{1 - \operatorname{Re}\lambda \pm \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}}, \text{ where } m = 1 + |\lambda| - 2\operatorname{Re}\lambda.$$

Let

$$\alpha_1 = \frac{m}{1 - \operatorname{Re}\lambda - \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}}. \tag{2.5.20}$$

Since $[\rho(B)]^2 > |\lambda|^2$,

$$[\rho(B)]^2 > 1 - 2(1 - \operatorname{Re}\lambda) + 1 - 2\operatorname{Re}\lambda + |\lambda| = 1 - 2(1 - \operatorname{Re}\lambda) + m.$$

Therefore,

$$[\rho(B)]^2 m > m - 2m(1 - \operatorname{Re}\lambda) + m^2 = 1 - 2\operatorname{Re}\lambda + |\lambda| - 2(1 - \operatorname{Re}\lambda)m + m^2, \text{ and}$$

$$[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2 > 1 - 2\operatorname{Re}\lambda + (\operatorname{Re}\lambda)^2 - 2(1 - \operatorname{Re}\lambda)m + m^2.$$

Hence,

$$\sqrt{[\rho(B)^2 m - (\operatorname{Im}\lambda)^2} > (1 - \operatorname{Re}\lambda) - m,$$

which implies

$$\alpha_1 = \frac{m}{1 - \operatorname{Re}\lambda - \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}} > 1.$$

Let

$$\alpha_2 = \frac{m}{1 - \operatorname{Re}\lambda + \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}}. \tag{2.5.21}$$

In an analogous way, we can show that $\alpha_2 < 1$.

Since in Case 1 the SOR method has at least one eigenvalue on the arc TAT', α has to be greater than 1 to ensure the shifting toward the inside of the circle. If we choose $\alpha = \alpha_1 > 1$ (2.5.20) then $\zeta = \lambda_R$ will be the result of the shifting. Hence, any α in the range

$$1 < \alpha < \frac{m}{1 - \operatorname{Re}\lambda - \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}}$$

keeps all the eigenvalues of the SOR method inside the circle $\rho(B_{(\omega,\alpha)}) < \rho(B_\omega)$.

Case 2. Since in this case the SOR method has at least one eigenvalue on the arc TBT', α has to be less than 1 to ensure the shifting toward the inside of the circle. If we choose $\alpha = \alpha_2 < 1$ (2.5.21), λ_L will be the result of the the shifting. Hence, any α in the range

$$\frac{m}{1 - \operatorname{Re}\lambda - \sqrt{[\rho(B)]^2 m - (\operatorname{Im}\lambda)^2}} < \alpha < 1$$

keeps all the eigenvalues of the the SOR method inside the circle and $\rho(B_{(\omega,\alpha)}) < \rho(B_\omega)$.

Example 2.5.1. The eigenvalues of the SOR iteration matrix B_ω are

$$\lambda_1, \lambda_2 = -0.65 \pm 0.466368952i,$$
$$\lambda_3, \lambda_4 = -0.3 \pm 0.5766282297i, \text{ and}$$
$$\lambda_5, \lambda_6 = 0.51 \pm 0.4794788838i.$$

The spectral radius of the SOR method is $\rho(B_\omega) = 0.8$.

Note that the two points T and T' are located at $0.64 \pm 0.48i$. The two eigenvalues λ_1 and λ_2 which represent the spectral radius of B_ω lie on the arc TAT' (Figure 3). Since λ_5 and λ_6 are the eigenvalues of B_ω with the largest real parts, using (2.5.20) to calculate α, we get

$$\alpha = \alpha_1 = 2.100751497.$$

By Theorem 2.5, any α in the range $1 < \alpha < 2.100751497$ provides a faster two-parameter method compared to the SOR method.

(1) Let $\alpha = 1.2$. The eigenvalues of the two-parameter method are

$$\zeta_1, \zeta_2 = -0.374999 \pm 0.3886407933i,$$
$$\zeta_3, \zeta_4 = -0.0833333 \pm 0.4805235247i, \text{ and}$$
$$\zeta_5, \zeta_6 = 0.591666 \pm 0.3995657365i.$$

The spectral radius of the two-parameter method is $\rho(B_{(\omega,\alpha)}) = 0.71394833$.
Therefore, $\rho(B_{(w,\alpha)}) < \rho(B_\omega)$.

(2) Let $\alpha = 1.92$. The eigenvalues of the two-parameter method are

$$\zeta_1, \xi_2 = 0.1406250001 \pm 0.2429004958i,$$
$$\zeta_3, \zeta_4 = 0.322916667 \pm 0.3003272029i, \text{ and}$$
$$\zeta_5, \zeta_6 = 0.74479166667 \pm 0.249728583i.$$

The spectral radius of the two-parameter method is $\rho(B_{(\omega,\alpha)}) = 0.7855437563$.
Therefore, $\rho(B_{(\omega,\alpha)}) < \rho(B_\omega)$.

Example 2.5.2. The eigenvalues of the SOR iteration matrix B_ω are

$$\lambda_1, \lambda_2 = 0.71 \pm 0.3686461718i,$$
$$\lambda_3, \lambda_4 = -0.3 \pm 0.5766282297i, \text{ and}$$
$$\lambda_5, \lambda_6 = 0.51 \pm 0.4794788838i.$$

The spectral radius of the SOR method is $\rho(B_\omega) = 0.8$.

Note that the two points T and T' are located at $0.64 \pm 0.48i$. The two eigenvalues λ_1 and λ_2 which represent the spectral radius of B_ω lie on the arc TBT' (Figure 3).

Since λ_3 and λ_4 are the eigenvalues of B_ω with the smallest real parts, using (2.5.21) to calculate α, we get

$$\alpha = \alpha_2 = 0.886739268.$$

By Theorem 2.8, any α in the range of $0.886739268 < \alpha < 1$ provides a faster two-parameter method compared to the SOR method.

Let $\alpha = 0.9$. The eigenvalues of the two-parameter method are

$$\begin{aligned}\zeta_1, \zeta_2 &= 0.6777778 \pm 0.4096068575i,\\ \zeta_3, \zeta_4 &= -0.444444 \pm 0.6406980329i, \text{ and}\\ \zeta_5, \zeta_6 &= 0.4555556 \pm 0.5327543153i.\end{aligned}$$

The spectral radius of the two-parameter method is $\rho(B_{(w,\alpha)}) = 0.791934$.

Therefore, $\rho(B_{(\omega,\alpha)}) < \rho(B_\omega)$.

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