VOLUME OF A TETRAHEDRON IN THE TAXICAB SPACE

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Abstract. In this paper, we give the taxicab version of the Heron-Tartaglia formula to calculate the volume of a tetrahedron using the taxicab distance.

1. Introduction. The three dimensional taxicab space is constructed by using the taxicab metric

$$d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$$

in three dimensional Cartesian space instead of the well-known Euclidean metric

$$d_E(P,Q) = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right]^{1/2},$$

where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. In the taxicab space, the taxicab distance d_T between points P and Q is the length of one of the shortest paths from P to Q composed of line segments, each parallel to a coordinate axis. Since taxicab geometry has a distance function different from that in Euclidean geometry, it is interesting to study the taxicab analogues of topics that include the distance concept in Euclidean geometry [1, 2, 3, 4, 5, 7]. Here, we study the following problem: How can one compute the volume of a tetrahedron using the taxicab distance?

2. Preliminaries. In Euclidean geometry the well-known Heron's Formula, which expresses the area A of a triangle in terms of the lengths of its edges a, b, c , can be given by

$$A^2 = \frac{1}{16}\left(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\right)$$

or

$$A^2 = p(p-a)(p-b)(p-c),$$

where $p = (a + b + c)/2$ which is the semiperimeter of the triangle. The taxicab version of Heron's Formula was given in [4] by Ozcan and Kaya.

The generalization of Heron's Formula to the volume of a tetrahedron is also well-known in Euclidean geometry: Let l_i, $1 \leq i \leq 6$, denote the the Euclidean lengths of the edges of tetrahedron $ABCD$ such that $l_1 = d_E(B,C)$, $l_2 = d_E(A,C)$, $l_3 = d_E(A,B)$, $l_4 = d_E(D,C)$, $l_5 = d_E(D,A)$, and $l_6 = d_E(D,B)$ (see Figure 1). Then the volume V of tetrahedron $ABCD$ can be calculated by the formula

$$V^2 = \frac{1}{144}$$
$$(l_1^2 l_5^2 (l_2^2 + l_3^2 + l_4^2 + l_6^2 - l_1^2 - l_5^2) + l_2^2 l_6^2 (l_1^2 + l_3^2 + l_4^2 + l_5^2 - l_2^2 - l_6^2)$$
$$+ l_3^2 l_4^2 (l_1^2 + l_2^2 + l_5^2 + l_6^2 - l_3^2 - l_4^2) - l_1^2 l_2^2 l_3^2 - l_1^2 l_4^2 l_6^2 - l_2^2 l_4^2 l_5^2 - l_3^2 l_5^2 l_6^2)$$

which is known as Heron-Tartaglia Formula [6]. The aim of this work is to give a taxicab version of this formula.

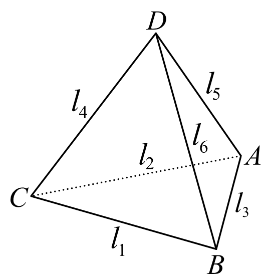

Figure 1

3. Taxicab Version of Heron-Tartaglia Formula. In the taxicab space, there are cases in which it is possible to construct infinitely many tetrahedra having given taxicab lengths of its edges t_i , $1 \leq i \leq 6$, such that no two have the same volume (see Example). Therefore, knowing only the taxicab lengths of edges of a tetrahedron is not sufficient to calculate its volume generally. Moreover, taxicab lengths of these edges are not invariant, in general, under rotations and reflections. Thus, we use the coordinates of vertices of the tetrahedron additionally to give a general volume formula for the tetrahedron in taxicab geometry.

The following example refers to Figure 2, in which two different taxicab circles are shown. Recall that a taxicab circle with center C and radius r is the set of all points whose taxicab distance to C is r. This locus of points is a square with center C, each side having slope ± 1, and each diagonal having length $2r$. Just as for a Euclidean circle, the center C and one point at a taxicab distance r from C completely determine the taxicab circle.

Example. Let a, b, and c be positive real numbers, and let $B = (0,-a,0)$, $C = (0,0,0)$, $D = (0,0,c)$, $P = (0,b,0)$ and $P' = (-b,0,0)$ be points in the taxicab space. Choose a point A in the line segment PP', such that $ABCD$ is a tetrahedron. Let C_1 and C_2 be taxicab circles in the plane $z = 0$, with centers C and B, and with radius b and $(a+b)$, respectively (see Figure 2). Since A is on both C_1 and C_2, it is obvious that $d_T(C,A) = b$ and $d_T(B,A) = a + b$. Now, it is easy to see that if the point A moves on the line segment PP', the taxicab lengths of edges of tetrahedron $ABCD$ do not change, but the volume of tetrahedron $ABCD$ changes since the area of triangle ABC changes. Thus, there are infinitely many tetrahedra $ABCD$ having taxicab lengths of its edges $d_T(B,C) = a$, $d_T(A,C) = b$, $d_T(A,B) = a+b$, $d_T(D,C) = c$, $d_T(D,A) = c+b$, and $d_T(D,B) = c+a$, such that no two have the same volume.

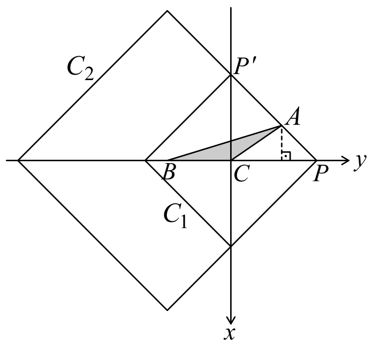

Figure 2

The following theorem gives a general volume formula for the tetrahedron in the taxicab space.

Theorem. The volume V of tetrahedron $ABCD$ with the vertices $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$, and $D = (d_1, d_2, d_3)$ in the taxicab space, can be calculated by the formula

$$
\begin{aligned}
V^2 = {} & \frac{1}{144} \\
& ((\alpha_1 t_1)^2(\alpha_5 t_5)^2((\alpha_2 t_2)^2 + (\alpha_3 t_3)^2 + (\alpha_4 t_4)^2 + (\alpha_6 t_6)^2 - (\alpha_1 t_1)^2 - (\alpha_5 t_5)^2) \\
& + (\alpha_2 t_2)^2(\alpha_6 t_6)^2((\alpha_1 t_1)^2 + (\alpha_3 t_3)^2 + (\alpha_4 t_4)^2 + (\alpha_5 t_5)^2 - (\alpha_2 t_2)^2 - (\alpha_6 t_6)^2) \\
& + (\alpha_3 t_3)^2(\alpha_4 t_4)^2((\alpha_1 t_1)^2 + (\alpha_2 t_2)^2 + (\alpha_5 t_5)^2 + (\alpha_6 t_6)^2 - (\alpha_3 t_3)^2 - (\alpha_4 t_4)^2) \\
& - (\alpha_1 t_1)^2(\alpha_2 t_2)^2(\alpha_3 t_3)^2 - (\alpha_1 t_1)^2(\alpha_4 t_4)^2(\alpha_6 t_6)^2 \\
& - (\alpha_2 t_2)^2(\alpha_4 t_4)^2(\alpha_6 t_6)^2 - (\alpha_3 t_3)^2(\alpha_5 t_5)^2(\alpha_6 t_6)^2),
\end{aligned}
$$

where

$$\alpha_1 = \left(\sum_{j=1}^{3}(b_j - c_j)^2\right)^{1/2} \Big/ \sum_{j=1}^{3}|b_j - c_j|\,,$$

$$\alpha_2 = \left(\sum_{j=1}^{3}(a_j - c_j)^2\right)^{1/2} \Big/ \sum_{j=1}^{3}|a_j - c_j|\,,$$

$$\alpha_3 = \left(\sum_{j=1}^{3}(a_j - b_j)^2\right)^{1/2} \Big/ \sum_{j=1}^{3}|a_j - b_j|\,,$$

$$\alpha_4 = \left(\sum_{j=1}^{3}(c_j - d_j)^2\right)^{1/2} \Big/ \sum_{j=1}^{3}|c_j - d_j|\,,$$

$$\alpha_5 = \left(\sum_{j=1}^{3}(a_j - d_j)^2\right)^{1/2} \Big/ \sum_{j=1}^{3}|a_j - d_j|\,,$$

$$\alpha_6 = \left(\sum_{j=1}^{3}(b_j - d_j)^2\right)^{1/2} \Big/ \sum_{j=1}^{3}|b_j - d_j|\,,$$

and $t_1 = d_T(B,C)$, $t_2 = d_T(A,C)$, $t_3 = d_T(A,B)$, $t_4 = d_T(D,C)$, $t_5 = d_T(D,A)$, $t_6 = d_T(D,B)$.

Proof. Let $ABCD$ be a tetrahedron in the taxicab space with vertices $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$, $D = (d_1, d_2, d_3)$ such that the taxicab lengths of its edges $t_1 = d_T(B,C)$, $t_2 = d_T(A,C)$, $t_3 = d_T(A,B)$, $t_4 = d_T(D,C)$, $t_5 = d_T(D,A)$, $t_6 = d_T(D,B)$ (see Figure 3). Let l_i be Euclidean lengths of the edges with taxicab lengths t_i, $1 \le i \le 6$, respectively.

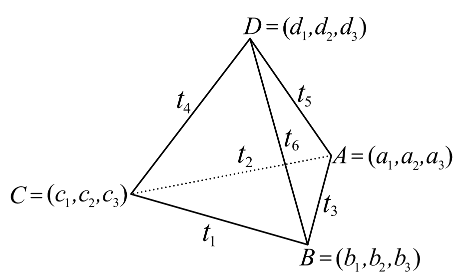

Figure 3

The following equation, which relates the Euclidean distance to the taxicab distance between two points in the Cartesian space, can be easily derived by a straightforward calculation with the coordinate definitions of $d_E(P,Q)$ and $d_T(P,Q)$ given in Section 1. If the line through P and Q has direction vector $\langle p,q,r\rangle$, then

$$d_E(P,Q) = \left[\left(p^2+q^2+r^2\right)^{1/2}/(|p|+|q|+|r|)\right] d_T(P,Q).$$

Since $\langle b_1-c_1, b_2-c_2, b_3-c_3\rangle$ is a direction vector of the line through $B=(b_1,b_2,b_3)$ and $C=(c_1,c_2,c_3)$, one can obtain using the last equation that

$$d_E(B,C) = \left[\left(\sum_{j=1}^{3}(b_j-c_j)^2\right)^{1/2} / \sum_{j=1}^{3}|b_j-c_j|\right] d_T(B,C).$$

Thus, $l_1 = \alpha_1 t_1$, where

$$\alpha_1 = \left(\sum_{j=1}^{3}(b_j-c_j)^2\right)^{1/2} / \sum_{j=1}^{3}|b_j-c_j|\,.$$

Similarly, one obtains that

$$l_i = \alpha_i t_i,\ 1 \le i \le 6,$$

where each α_i is as in the statement of the theorem. Replacing l_i with $\alpha_i t_i$, $1 \le i \le 6$, in the Heron-Tartaglia Formula, one gets the equation in the statement of the theorem, which is the taxicab version of the Heron-Tartaglia Formula.

Remark. Every convex polyhedron can be thought of as the union of a finite number of tetrahedra with a common vertex and with bases in the faces of the polyhedron. Therefore, the volume of any convex polyhedron given by its vertices in the taxicab space can be calculated using the taxicab version of the Heron-Tartaglia Formula.

Considering the Remark, one can easily obtain the volume formula for the taxicab sphere in terms of the length of its edge or its radius, using the taxicab version of the Heron-Tartaglia Formula, as follows.

The volume V of the taxicab sphere with radius r (see Figure 4) can be calculated by the formula

$$V = \frac{4}{3}r^3 \quad \text{or} \quad V = \frac{1}{6}l^3,$$

where l is the taxicab length of an edge of the taxicab sphere.

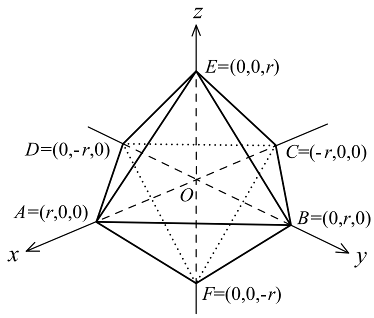

Figure 4

References

1. Z. Akca, R. Kaya, "On the Distance Formulae in Three Dimensional Taxicab Space," *Hadronic Journal*, 27 (2004), 521–532.
2. Y. P. Ho and Y. Liu, "Parabolas in Taxicab Geometry," *Missouri Journal of Mathematical Sciences*, 10 (1997), 151–159.
3. E. F. Krause, *Taxicab Geometry*, Addision-Wesley, Menlo Park, California, 1975.
4. M. Özcan, R. Kaya, "Area of a Triangle in Terms of the Taxicab Distance," *Missouri Journal of Mathematical Sciences*, 15 (2003), 178–185.
5. M. Özcan, R. Kaya, "On the Ratio of Directed Lengths in the Taxicab Plane and Related Properties," *Missouri Journal of Mathematical Sciences*, 14 (2002), 107–117.
6. I. K. Sabitov, "The Generalized Heron-Tartaglia Formula and Some of its Consequences," *Sb. Math.*, 189 (1998), 1533–1561.
7. D. J. Schattschneider, "The Taxicab Group," *American Mathematical Monthly*, 91 (1984), 423–428.

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