GENERALIZATION OF A GEOMETRIC INEQUALITY

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ABSTRACT. In this paper, using Bottema's inequality for two triangles and other results, the generalization of an inequality involving the medians and angle-bisectors of the triangle is proved. This settles affirmatively a problem posed by J-Liu.

1. Introduction and Main Result

In [1], the author posed 100 unsolved triangle inequality problems. Among his conjectures is an inequality for medians and angle-bisectors of a triangle and so-called Shc53:

$$(m_b+m_c)\sin\frac{A}{2}+(m_c+m_a)\sin\frac{B}{2}+(m_c+m_a)\sin\frac{C}{2}\geqslant w_a+w_b+w_c, \quad (1)$$

where m_a, m_b, m_c and w_a, w_b, w_c denote the medians and angle-bisector of $\triangle ABC$, A, B, C denote its angles.

Recently, we investigated inequality (1) again and found its generalization.

Theorem 1. *Let P be an arbitrary point in the plane of triangle ABC. Then*

$$(PB+PC)\sin\frac{A}{2}+(PC+PA)\sin\frac{B}{2}+(PA+PB)\sin\frac{C}{2}\geqslant\frac{2}{3}(w_a+w_b+w_c). \quad (2)$$

Equality holds if and only if the triangle ABC is equilateral and P is its center.

Obviously, if P is the centroid of $\triangle ABC$, then we easily obtain inequality (1) from (2).

2. Several Lemmas

In order to prove the theorem, we need some lemmas.

Besides the above notations, as usual, a, b, c denote the sides of triangle ABC; s, R, r, Δ denote its semi-perimeter, the radius of its circumcircle, the radius of its incircle, and its area, respectively. In addition, $\sum$ and

$\prod$ denote cyclic sum and product respectively (e.g., $\sum bc = bc + ca + ab$, $\prod(b+c) = (b+c)(c+a)(a+b)$).

Lemma 1. *For any $\triangle ABC$, the following inequality holds.*

$$\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \leqslant \frac{1}{2R} + \frac{3}{4r}. \tag{3}$$

Equality holds if and only if triangle ABC is equilateral.

Inequality (3) was proposed by the second author [2] of this paper and first proved by Jian-Ping Li [3]. It can also be derived expediently from a result of Xue-Zhi Yang [4]. Here, we give a convenient direct proof.

Proof. From the well known formula $w_a = \frac{2}{b+c}\sqrt{bcs(s-a)}$ and Heron's formula

$$\Delta = \sqrt{s(s-a(s-b)(s-c)}, \tag{4}$$

we have

$$\begin{aligned}\frac{1}{w_a} &= \frac{(b+c)\sqrt{bc(s-b)(s-c)}}{2bc\Delta}\\ &\leqslant \frac{b+c}{4bc\Delta}\left[\frac{abc}{b+c} + \frac{(b+c)(s-b)(s-c)}{a}\right]\\ &= \frac{a}{4\Delta} + \frac{1}{4abc\Delta}(s-b)(s-c)(b+c)^2.\end{aligned}$$

Hence,

$$\sum \frac{1}{w_a} \leqslant \frac{1}{4\Delta}\sum a + \frac{1}{4abc\Delta}\sum(s-b)(s-c)(b+c)^2. \tag{5}$$

Observe that

$$\begin{aligned}&\sum(s-b)(s-c)(b+c)^2\\ &= \frac{1}{4}\sum a^2(b+c)^2 - \frac{1}{4}\sum(b^2-c^2)^2\\ &= \frac{1}{2}\left[\sum b^2c^2 + abc\sum a - \left(\sum a^4 - \sum b^2c^2\right)\right]\\ &= \frac{1}{2}\left(abc\sum a + 2\sum b^2c^2 - \sum a^4\right)\\ &= 4(R+2r)rs^2.\end{aligned}$$

The last step was obtained using $\sum a = 2s, abc = 4Rrs$ and the equivalent form of Heron's formula:

$$16\Delta^2 = 2\sum b^2c^2 - \sum a^4.$$

Finally, we get

$$\sum \frac{1}{w_a} \leqslant \frac{1}{2r} + \frac{4(R+2r)rs^2}{4abc\Delta} = \frac{1}{2R} + \frac{3}{4r}.$$

Inequality (3) is proved and it is easy to show that equality occurs if and only if $a = b = c$. The proof of Lemma 1 is complete. □

Lemma 2. *For any triangle ABC, the following inequality holds.*

$$(w_a + w_b + w_c)^2 \leqslant \frac{9}{4}(s^2 + 9r^2). \tag{6}$$

Equality holds if and only if triangle ABC is equilateral.

Proof. From inequality (3) and the well-known identities

$$w_a w_b w_c = \frac{16Rr^2 s^2}{s^2 + 2Rr + r^2}, \tag{7}$$

and

$$\sum w_a^2 = \frac{s^6 + 3r^2 s^4 + (32R^2 + 40Rr + 3r^2)r^2 s^2 + r^4(4R+r)^2}{(s^2 + 2Rr + r^2)^2},$$

we have

$$\begin{aligned} \left(\sum w_a\right)^2 &= \sum w_a^2 + 2\sum w_b w_c = \sum w_a^2 + \frac{2}{w_a w_b w_c}\sum \frac{1}{w_a} \\ &\leqslant \frac{s^6 + 3r^2 s^4 + (32R^2 + 40Rr + 3r^2)r^2 s^2 + r^4(4R+r)^2}{(s^2 + 2Rr + r^2)^2} \\ &\quad + \frac{8r(3R+2r)s^2}{s^2 + 2Rr + r^2} \qquad (8) \\ &= \frac{s^6 + (24R + 19r)rs^4 + (80R^2 + 96Rr + 19r^2)r^2 s^2 + (4R+r)^2 r^4}{(s^2 + 2Rr + r^2)^2}. \end{aligned}$$

Now, we will prove that

$$\begin{aligned} &\frac{s^6 + (24R + 19r)rs^4 + (80R^2 + 96Rr + 19r^2)r^2 s^2 + (4R+r)^2 r^4}{(s^2 + 2Rr + r^2)^2} \\ &\leqslant \frac{9}{4}(s^2 + 9r^2). \end{aligned} \tag{9}$$

It is equivalent to

$$\begin{aligned} &5s^6 - (60R - 23r)rs^4 - (284R^2 + 24Rr - 95r^2)r^2 s^2 \\ &\quad + (260R^2 + 292Rr + 77r^2)r^4 \geqslant 0. \end{aligned} \tag{10}$$

This can be written as

$$\begin{aligned} &(s^2 - 16Rr + 5r^2)[5s^4 + (20Rr - 2r^2)s^2 + (12R + 39r)r^3] \\ &\quad + 4r^2(9s^2 + 17r^2)(R - 2r)^2 \geqslant 0. \end{aligned} \tag{11}$$

It follows from the well-known Gerretsen's inequality $s^2 \geqslant 16Rr - 5r^2$ (see [5] and also [6]) and Chapple-Euler's inequality $R \geqslant 2r$.

From (8) and (9), we obtain (6). Clearly, the equality in (6) occurs if and only if the triangle is equilateral. Lemma 2 is proved. □

Lemma 3. *The identity*

$$\sum a^2 \sin^2 \frac{A}{2} = \frac{(2R-3r)s^2 + (4R+r)r^2}{2R} \tag{12}$$

holds for all triangles ABC.

Proof. This identity follows from

$$\begin{aligned} &\sum a^2 \sin^2 \frac{A}{2} \\ &= \frac{1}{2}\left[\sum a^2 - 4R^2 \sum (1-\cos^2 A)\cos A\right] \\ &= \frac{1}{2}\sum a^2 - 2R^2\left(\sum \cos A - \sum \cos^3 A\right), \end{aligned}$$

and the following identities [6]:

$$\sum a^2 = 2(s^2 - 4Rr - r^2), \tag{13}$$

$$\sum \cos A = 1 + \frac{r}{R}, \tag{14}$$

$$\sum \cos^3 A = \frac{(2R+r)^3 - 3rs^2}{4R^3} - 1. \tag{15}$$

□

Lemma 4. *For any triangle ABC, we have*

$$\sqrt{\prod \sin \frac{A}{2} \sum \sin \frac{A}{2}} \geqslant \frac{r(4R+r)}{2sR}. \tag{16}$$

Equality holds if and only if triangle ABC is equilateral.

Proof. By the simple inequality $\cos B + \cos C \leqslant 2\sin \frac{A}{2}$, etc. It is deduced $\sum \sin \frac{A}{2} \geqslant \sum \cos A$. Hence, using identity (14), we have

$$\sum \sin \frac{A}{2} \geqslant 1 + \frac{r}{R}. \tag{17}$$

According to the above inequality and the known relation

$$\prod \sin \frac{A}{2} = \frac{r}{4R}, \tag{18}$$

to prove (16) we need to show that

$$\sqrt{\frac{r}{4R}\left(1 + \frac{r}{R}\right)} \geqslant \frac{r(4R+r)}{2sR}.$$

After squaring both of sides and simplifying, it becomes

$$(R+r)s^2 - r(4R+r)^2 \geqslant 0,$$

i.e.,

$$(R+r)(s^2 - 16Rr + 5r^2) + 3(R-2r)r^2 \geqslant 0.$$

This follows from $s^2 \geqslant 16Rr - 5r^2$ and $R \geqslant 2r$. Thus, inequality (16) is true. $\square$

Lemma 5. *For any triangle ABC, the following inequality holds.*

$$\sum (b^2 + c^2 - a^2) \sin \frac{B}{2} \sin \frac{C}{2} \geqslant \frac{s^4 - 10Rrs^2 - (8R^2 + 6Rr + r^2)r^2}{4R^2}. \quad (19)$$

Equality holds if and only if triangle ABC is equilateral.

Proof. If $\triangle ABC$ is a non-obtuse triangle, using the simple well-known inequality $\sin \frac{A}{2} \leqslant \frac{a}{b+c}$, etc. we have

$$\sum \frac{b^2 + c^2 - a^2}{\sin \frac{A}{2}} \geqslant \sum \frac{b+c}{a}(b^2 + c^2 - a^2). \quad (20)$$

Indeed, the above inequality holds for all triangles. Next, we shall prove our result.

Since $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, inequality (20) is also

$$\sum (b^2 + c^2 - a^2) \left[\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}} - \frac{b+c}{a} \right] \geqslant 0,$$

or equivalently

$$\sum \frac{(s-a)(b^2 + c^2 - a^2)(b-c)^2}{a\left[a\sqrt{bc(s-b)(s-c)} + (b+c)(s-b)(s-c)\right]} \geqslant 0. \quad (21)$$

Without loss of generality, we may assume that A is an obtuse angle and $a > b \geqslant c$, then we easily know that

$$a\sqrt{bc(s-b)(s-c)} > b\sqrt{ca(s-c)(s-a)},$$
$$(b+c)(s-b)(s-c) > (c+a)(s-c)(s-a).$$

Putting

$$X = a\sqrt{bc(s-b)(s-c)} + (b+c)(s-b)(s-c),$$
$$Y = b\sqrt{ca(s-c)(s-a)} + (c+a)(s-c)(s-a),$$

then $X > Y$. In addition, from

$$\begin{aligned}\frac{s-b}{bY} - \frac{s-a}{aX} &= \frac{(aX-bY)s - ab(X-Y)}{abXY} \\ &> \frac{(bX-bY)s - ab(X-Y)}{abXY} = \frac{(s-a)(X-Y)}{aXY} \geqslant 0,\end{aligned}$$

we find

$$\frac{s-b}{bY} > \frac{s-a}{aX}.$$

According to this and $a^2+b^2-c^2>0$, $c^2+a^2-b^2>0$, $s-b>s-a$, $(a-c)^2>(b-c)^2$, we have that

$$\begin{aligned}&\sum \frac{(s-a)(b^2+c^2-a^2)(b-c)^2}{a\left[a\sqrt{bc(s-b)(s-c)}+(b+c)(s-b)(s-c)\right]} \\ &\geqslant \frac{s-a}{aX}(b^2+c^2-a^2)(b-c)^2 + \frac{s-b}{bY}(c^2+a^2-b^2)(a-c)^2 \\ &\geqslant \frac{s-a}{aX}(b^2+c^2-a^2)(b-c)^2 + \frac{s-a}{aX}(c^2+a^2-b^2)(b-c)^2 \\ &= \frac{2(s-a)}{aX}(b-c)^2c^2 \geqslant 0.\end{aligned}$$

Therefore, the inequality (20) holds for obtuse triangles. Furthermore, we know that (20) is valid for all triangles.

Now, by (20) and (18), we obtain

$$\begin{aligned}&\sum (b^2+c^2-a^2)\sin\frac{B}{2}\sin\frac{C}{2} \\ &\geqslant \frac{r}{4R}\sum\frac{b+c}{a}(b^2+c^2-a^2). \\ &= \frac{r}{4abcR}\left[\sum bc(b+c)\sum a^2 - 2abc\sum a(b+c)\right] \\ &= \frac{r}{4abcR}\left[\left(\sum a\sum bc - 3abc\right)\sum a^2 - 4abc\sum bc\right] \\ &= \frac{s^4 - 10Rrs^2 - (8R^2+6Rr+r^2)r^2}{4R^2}.\end{aligned}$$

Lemma 5 is proved. □

Lemma 6. *Let P is an arbitrary point in the plane of triangle ABC, a', b', c' denote the sides of $\triangle A'B'C'$ and Δ' denote its area. Then*

$$\begin{aligned}&(a'PA+b'PB+c'PC)^2 \geqslant \qquad (22)\\ &\frac{1}{2}\left[a^2(b'^2+c'^2-a'^2)+b^2(c'^2+a'^2-b'^2)+c^2(a'^2+b'^2-c'^2)\right]+8\triangle\triangle'.\end{aligned}$$

Equality holds in one of the following cases: (i) $\triangle ABC \sim \triangle A'B'C'$, P lies inside of $\triangle ABC$, and $A' + \angle BPC = B' + \angle CPA = C' + \angle APB = \pi$; (ii)

P coincides with one of the vertices of $\triangle ABC$, the sum of the angle where lies this vertices of triangle ABC and the relevant angle of triangle $A'B'C'$ is π.

Inequality (25) is Bottema's inequality for two triangles [6, 7].

3. Proof of Theorem

Proof. Inequality (2) is also

$$\sum\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)PA \geqslant \frac{2}{3}\sum w_a. \tag{23}$$

By Heron's formula (4), it is easily known that $\sin\frac{B}{2}+\sin\frac{C}{2}, \sin\frac{C}{2}+\sin\frac{A}{2}, \sin\frac{A}{2}+\sin\frac{B}{2}$ form a triangle with area $\sqrt{\prod\sin\frac{A}{2}\sum\sin\frac{A}{2}}$. Hence, by using Lemma 6, we get

$$\begin{aligned}
&\left[\sum\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)PA\right]^2\\
&\geqslant \frac{1}{2}\sum(b^2+c^2-a^2)\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2+8\Delta\sqrt{\prod\sin\frac{A}{2}\sum\sin\frac{A}{2}}\\
&= \frac{1}{2}\sum(b^2+c^2-a^2)\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)\\
&\quad +\sum(b^2+c^2-a^2)\sin\frac{B}{2}\sin\frac{C}{2}+8\Delta\sqrt{\prod\sin\frac{A}{2}\sum\sin\frac{A}{2}}\\
&= \sum a^2\sin^2\frac{A}{2}+\sum(b^2+c^2-a^2)\sin\frac{B}{2}\sin\frac{C}{2}\\
&\quad +8\Delta\sqrt{\prod\sin\frac{A}{2}\sum\sin\frac{A}{2}}.
\end{aligned}$$

In order to prove (23), we need to show that

$$\begin{aligned}
&\sum a^2\sin^2\frac{A}{2}+\sum(b^2+c^2-a^2)\sin\frac{B}{2}\sin\frac{C}{2}\\
&\quad +8\Delta\sqrt{\prod\sin\frac{A}{2}\sum\sin\frac{A}{2}} \geqslant \frac{4}{9}\left(\sum w_a\right)^2.
\end{aligned} \tag{24}$$

According to Lemma 5, it suffices to prove that

$$\begin{aligned}
&\frac{(2R-3r)s^2+(4R+r)r^2}{2R}+\frac{s^4-10Rrs^2-(8R^2+6Rr+r^2)r^2}{4R^2}\\
&+\frac{4(4R+r)r^2}{R} \geqslant s^2+9r^2.
\end{aligned}$$

One may simplify this to

$$s^4-16Rrs^2+(28R^2+12Rr-r^2)r^2 \geqslant 0, \tag{25}$$

which is equivalent to

$$(s^2 - 5r^2)(s^2 - 16Rr + 5r^2) + 4(R - 2r)(7R - 3r)r^2 \geqslant 0.$$

This follows from Gerretsen's inequality $s^2 \geqslant 16Rr - 5r^2$ and Chapple-Euler's inequality $R \geqslant 2r$. Hence, inequality (23), i.e., (2) is proved. It is easy to obtain the condition when equality occurs in (2). This completes the proof of Lemma 6. □

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References

[1] J.-Liu, *100 Problems to be Solved About Triangular Inequality*, Geometric Inequality in China (chief editor-Zun Shan), Jiangsu Educational Press, NanJing, China, 1996, 137–161 (Chinese).

[2] J.-Liu, *Some New Inequalities for the Triangle*, Zhongxue Shuxue, **5** (1994), 9–12 (Chinese).

[3] J.-P. Li, *A Proof of a Conjecture*, Hunan Mathematical Communication, **2** (1995), 39–40 (Chinese).

[4] X.-Zh. Yang, *A Inequality of the Bisector*, Shuxue Tongxun, **8** (1995), 17 (Chinese).

[5] O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

[6] D. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1989.

[7] O. Bottema, *Hoofdstukken uit de Elementaire Meetkunde*, Den Haag, (1944), 97–99.

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