SEPARATION AXIOMS AND LATTICE EQUIVALENCE

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ABSTRACT. This paper deals with the relation between lattice-equivalence and some separation axioms.

We are concerned with two questions:

The first one is to characterize topological spaces X such that X and $\mathbf{F}(X)$ are lattice equivalent for some covariant functors $\mathbf{F}$ from **TOP** to itself.

In the second question, it is proved that $T_{(0,2)}, T_{(S,D)}, T_{(S,1)}$ and $T_{(0,3\frac{1}{2})}$ are lattice-invariant properties but $S, T_{(0,1)}, T_{(0,S)}, T_{(1,2)}, T_{(1,S)}$, $T_{(1,3\frac{1}{2})}$ and $T_{(0,D)}$ are not.

1. Introduction

Among the oldest separation axioms in topology there are some famous ones, T_0, T_1, T_2, T_D, S and $\rho = T_{3\frac{1}{2}}$ (where S designates sober and ρ Tychonoff) and we have the following implications.

- $T_{3\frac{1}{2}} \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_D \Longrightarrow T_0$.
- $T_{3\frac{1}{2}} \Longrightarrow T_2 \Longrightarrow S \Longrightarrow T_0$.

In [1] and [4] the authors have introduced some new separation axioms namely $T_{(0,1)}$, $T_{(0,2)}$, $T_{(0,S)}$, $T_{(0,3\frac{1}{2})}$, $T_{(0,D)}$, $T_{(S,1)}$, $T_{(S,2)}$, $T_{(S,D)}$, $T_{(S,3\frac{1}{2})}$, $T_{(1,2)}$, $T_{(1,3\frac{1}{2})}$, $T_{(1,S)}$ and $T_{(2,3\frac{1}{2})}$.

Recall that, for $i \in \{1, 3\frac{1}{2}, 2, D\}$, a topological space X is said to be a $T_{(0,i)}$-space (resp., $T_{(S,i)}$-space) if its T_0-reflection (resp., Sober-reflection) is a T_i-space.

Some of the "new" separation axioms which we have introduced here are well-known. For example, what we have called $T_{(0,1)}$-spaces are nothing but R_0-spaces, sometimes also called symmetric spaces. They were introduced by N. A. Shanin in [9] and rediscovered by A. S. Davis in [2]. Since the underlying topology of nearness spaces is always R_0, they are widely used. Similarly, what we have called $T_{(0,2)}$-spaces are nothing but R_1-spaces in

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the sense of Davis. The properties used to define $T_{(0,1)}$-spaces (resp., $T_{(0,2)}$-spaces) are well-known characterizations of R_0-spaces (resp., R_1-spaces) which can be found in [3].

The present paper is devoted to shedding some light on separation axioms and gives other characterizations using the notion of lattice equivalent topological spaces.

Let us first recall some notions which were introduced by the Grothendieck school [6, 7], such as quasihomeomorphisms and sober spaces.

Recall that a continuous map $q : X \longrightarrow Y$ is said to be a *quasihomeomorphism* if $U \longmapsto q^{-1}(U)$ (resp., $C \longmapsto q^{-1}(C)$) defines a bijection $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ (resp., $\mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$), where $\mathcal{O}(X)$ (resp., $\mathcal{F}(X)$) is the collection of all open sets (resp., closed sets) of X [7].

A set F of a topological space X is said to be *irreducible* if for each open sets U and V of X such that $F \cap U \neq \emptyset$ and $F \cap V \neq \emptyset$, we have $F \cap U \cap V \neq \emptyset$ (equivalently, if C_1 and C_2 are two closed sets of X such that $F \subseteq C_1 \cup C_2$, then $F \subseteq C_1$ or $F \subseteq C_2$).

A topological space X is called *sober* if each nonempty irreducible closed set F of X has a unique generic point (i.e., there exists a unique $x \in X$ such that $F = \overline{\{x\}}$).

A topological space X is called a T_D-space if each point $\{x\}$ in X is locally closed (i.e., there exists an open set U of X such that $\{x\} = \overline{\{x\}} \cap U$).

A space X is said to be *completely regular* if every closed set F of X is completely separated from any point x not in F (i.e, there exists a continuous map $f : X \longrightarrow \mathbb{R}$ such that $f(x) = 0$ and $f(F) = \{1\}$). Recall that a topological space X is called a T_1-*space* if each singleton of X is closed. A completely regular T_1-space is called a *Tychonoff space* [11]. We remark here that a Tychonoff space is a Hausdorff space (T_2-space).

This paper is composed of an introduction and three sections. The first one deals with some remarks about separation axioms. The main result of the second section is the characterizations of $T_{(0,1)}$, $T_{(0,2)}$ and $T_{(0,3\frac{1}{2})}$ by the notion of lattice equivalence (see Theorem 3.5). In Section three it is proved that $T_{(0,2)}$, $T_{(S,D)}$, $T_{(S,1)}$ and $T_{(0,3\frac{1}{2})}$ are lattice invariant properties but S, $T_{(0,1)}$, $T_{(0,S)}$, $T_{(1,2)}$, $T_{(1,S)}$, $T_{(1,3\frac{1}{2})}$ and $T_{(0,D)}$ are not.

2. $T_{(0,1)}$, $T_{(0,2)}$, AND $T_{(0,3\frac{1}{2})}$-SPACES

We denote by Top the category of topological spaces with continuous maps as morphisms and by Top_i, for $i = 0, 1, 2, 3\frac{1}{2}$, the full subcategory of Top whose objects are T_i-spaces. Recall that Top_i is a reflective subcategory of Top (see for example [8] and [12]). In other words, there exists a universal T_i-space for every topological space X; we denote it by $T_i(X)$. On the other side, the T_D property is not reflective in Top (for more details see [1, p. 3718]).

First, let us give some straightforward remarks about quasihomeomorphisms.

Remark 2.1. Let $q : X \longrightarrow Y$ be a quasihomeomorphism. Then, according to [1, Lemma 3.7], the following properties hold.

(a) If X is a T_0-space, then q is one-one.
(b) If Y is a T_D-space, then q is onto.
(c) If Y is a T_D-space and X is a T_0-space, then q is a homeomorphism.
(d) If X is sober and Y is a T_0-space, then q is a homeomorphism.

Example 2.2. Let X be a topological space.

(1) The canonical surjection $\mu_0 : X \longrightarrow T_0(X)$ (resp., $\theta_X : X \longrightarrow S(X)$) is a quasihomeomorphism.
(2) For $i \in \{1, 3\frac{1}{2}, 2\}$, the canonical surjection $\mu_i : X \longrightarrow T_i(X)$ is, in general, not a quasihomeomorphism.

Proof.

(1) see [6].
(2) It is sufficient to consider a T_0-space which is not T_i. Indeed, suppose that $\mu_i : X \longrightarrow T_i(X)$ is a quasihomeomorphism. Then, by Remarks 2.1 (c), μ_i is a homeomorphism which is impossible.

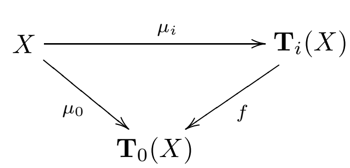
□

For a given $i \in \{1, 2, 3\frac{1}{2}\}$, the following result characterizes topological spaces such that the canonical surjection $\mu_i : X \longrightarrow T_i(X)$ is a quasihomeomorphism.

Proposition 2.3. *Let X be a topological space and $i \in \{1, 3\frac{1}{2}, 2\}$. Then the following statements are equivalent:*

(a) *X is a $T_{(0,i)}$-space;*
(b) *The canonical surjection $\mu_i : X \longrightarrow T_i(X)$ is a quasihomeomorphism.*

Proof. $(a) \Longrightarrow (b)$. Since X is a $T_{(0,i)}$-space, then $T_0(X)$ is a T_i-space and consequently there exists a unique continuous map $f : T_i(X) \longrightarrow T_0(X)$ making the following diagram commute



That is $f \circ \mu_i = \mu_0$. On the other hand since $T_i(X)$ is a T_0-space, there is a unique continuous map $g : T_0(X) \longrightarrow T_i(X)$ such that $g \circ \mu_0 = \mu_i$. Now combining the previous equalities we get easily $f \circ g = 1_{T_0(X)}$ and $g \circ f = 1_{T_i(X)}$ which means that f and g are homeomorphisms and finally μ_i is a quasihomeomorphism.

$(b) \Longrightarrow (a)$. Consider the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_i} & \mathbf{T}_i(X) \\ \downarrow{\scriptstyle \mu_0} & & \| {\scriptstyle 1} \\ \mathbf{T}_0(X) & \xrightarrow[\mathbf{T}_0(\mu_i)]{} & \mathbf{T}_0(\mathbf{T}_i(X)) = \mathbf{T}_i(X) \end{array}$$

Clearly, $\mathbf{T}_0(\mu_i)$ is a quasihomeomorphism between a T_0-space and T_i-space. Now, since for any $i \in \{1, 2, 3\frac{1}{2}\}$ $T_i(X)$ is a T_D-space, then according to Remarks 2.1 (c) $\mathbf{T}_0(\mu_i)$ is a homeomorphism which implies that $\mathbf{T}_0(X)$ is a T_i-space. $\square$

In [1] and [4] the authors give some characterizations of $T_{(S,1)}$, $T_{(S,2)}$, $T_{(S,D)}$, and $T_{(S,3\frac{1}{2})}$ without mentioning their relationships with $T_{(0,1)}$, $T_{(0,2)}$, and $T_{(0,3\frac{1}{2})}$. The following result does the job.

Theorem 2.4. *Let $i \in \{1, 2, 3\frac{1}{2}, D\}$. The following properties hold.*

(1) *If $i = 2$ or $3\frac{1}{2}$, then we have $T_{(S,i)} = T_{(0,i)}$.*

(2) *If $i = 1$ or D, then we have $T_{(S,i)} = T_{(0,i)} + T_{(0,S)}$.*

Proof. (1) Let X be a $T_{(S,i)}$-space, where $i = 2$ or $3\frac{1}{2}$. Then the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\theta_X} & \mathbf{S}(X) \\ \downarrow{\scriptstyle \mu_0} & & \| {\scriptstyle 1} \\ \mathbf{T}_0(X) & \xrightarrow[\mathbf{T}_0(\theta_X)]{} & \mathbf{T}_0(\mathbf{S}(X)) = \mathbf{S}(X) \end{array}$$

Hence, $\mathbf{T}_0(\theta_X) : \mathbf{T}_0(X) \longrightarrow \mathbf{S}(X)$ is a quasi-homeomorphism. Since $\mathbf{T}_0(X)$ is a T_0-space and $\mathbf{S}(X)$ is a T_i-space and consequently a T_D-space, then according to Remarks 2.1 (c) $\mathbf{T}_0(\theta_X)$ is a homeomorphism which implies that $\mathbf{T}_0(X)$ is a T_i-space.

Conversely, let X be a $T_{(0,i)}$-space. Since $T_0(X)$ is T_i, then it is sober and $\mathbf{S}(\mathbf{T}_0(X))$ is homeomorphic to $\mathbf{T}_0(X)$. Hence the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\mu_0} & \mathbf{T}_0(X) \\ {\scriptstyle \theta_X}\downarrow & & \Vert\,{\scriptstyle 1} \\ \mathbf{S}(X) & \xrightarrow[\mathbf{S}(\mu_0)]{} & \mathbf{S}(\mathbf{T}_0(X)) = \mathbf{T}_0(X) \end{array}$$

On the other hand, according to Remarks 2.1 (d), $\mathbf{S}(\mu_0) : \mathbf{S}(X) \longrightarrow \mathbf{T}_0(X)$ is a homeomorphism. Therefore $\mathbf{S}(X)$ is a T_i-space and finally X is a $T_{(S,i)}$-space.

(2) Let X be a $T_{(S,i)}$-space, where $i = 1$ or D. Then the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\theta_X} & \mathbf{S}(X) \\ {\scriptstyle \mu_0}\downarrow & & \Vert\,{\scriptstyle 1} \\ \mathbf{T}_0(X) & \xrightarrow[\mathbf{T}_0(\theta_X)]{} & \mathbf{T}_0(\mathbf{S}(X)) = \mathbf{S}(X) \end{array}$$

Hence, $\mathbf{T}_0(\theta_X) : \mathbf{T}_0(X) \longrightarrow \mathbf{S}(X)$ is a quasi-homeomorphism. Since $\mathbf{T}_0(X)$ is a T_0-space and $\mathbf{S}(X)$ is a T_i-space which is a T_D-space, then according to Remarks 2.1 (c) $\mathbf{T}_0(\theta_X)$ is a homeomorphism which implies that $\mathbf{T}_0(X)$ is a sober T_i-space.

Conversely, let X be a $T_{(0,i)}$ and $T_{(0,S)}$ space. Then the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\mu_0} & \mathbf{T}_0(X) \\ {\scriptstyle \theta_X}\downarrow & & \Vert\,{\scriptstyle 1} \\ \mathbf{S}(X) & \xrightarrow[\mathbf{S}(\mu_0)]{} & \mathbf{S}(\mathbf{T}_0(X)) = \mathbf{T}_0(X) \end{array}$$

Since X is a $T_{(0,S)}$-space, then $\mathbf{S}(\mathbf{T}_0(X))$ is homeomorphic to $\mathbf{T}_0(X)$. Now according to Remarks 2.1 (d), $\mathbf{S}(\mu_0)$ is an homeomorphism and so that $\mathbf{S}(X)$ is homeomorphic to $\mathbf{T}_0(X)$ which is a T_i-space (because X is a $T_{(0,i)}$-space). Finally X is a $T_{(S,i)}$-space. □

3. Lattice Equivalent Spaces

For any topological space X, let us denote by $\Gamma(X)$ the lattice of all closed sets of X.

Two topological spaces X and Y are said to be *lattice equivalent* if there exists a bijective map $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ such that φ and φ^{-1} are order-preserving maps. The map φ is called *lattice equivalence*.

A lattice equivalence $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ is said to be *induced by a homeomorphism* if there is a homeomorphism $f : X \longrightarrow Y$ such that $\varphi(C) = f(C)$, for each $C \in \Gamma(X)$.

In [10], Thron was concerned with lattice equivalence induced by a homeomorphism, so let us recall some interesting results induced by Thron.

Proposition 3.1. [10, Theorem 2.1] *Every lattice equivalence between two T_D-spaces is induced by a homeomorphism.*

Proposition 3.2. [10, Corollary 2.1] *Every lattice equivalence between a T_0-space and a T_2-space is induced by a homeomorphism.*

In this section we are interested in characterizing topological spaces X such that X and $F(X)$ are lattice equivalent for any covariant functor $F \in \{T_0, T_1, T_2, T_{3\frac{1}{2}}, S\}$.

Remark 3.3. Let $q : X \longrightarrow Y$ be a quasihomeomorphism. Then the map $\varphi : \Gamma(Y) \longrightarrow \Gamma(X)$, which is defined by $\varphi(C) = q^{-1}(C)$, for any closed set C of Y, is a lattice equivalence.

As an immediate consequence of Remark 3.3, we have the following result.

Proposition 3.4. *Let X be a topological space.*

(1) *X and $\mathbf{T}_0(X)$ are lattice equivalent.*
(2) *X and $\mathbf{S}(X)$ are lattice equivalent.*

Now, we give the main result of this section.

Theorem 3.5. *Let X be a topological space and $i \in \{1, 2, 3\frac{1}{2}\}$. Then the following statements are equivalent:*

(1) *X and $T_i(X)$ are lattice equivalent;*
(2) *X is a $T_{(0,i)}$-space.*

Proof. (2) $\Longrightarrow$ (1). Suppose that X is a $T_{(0,i)}$-space. Then, by Proposition 2.3, $\mu_i : X \longrightarrow \mathbf{T}_i(X)$ is a quasihomeomorphism and consequently Remark 3.3 does the job.

(1) $\Longrightarrow$ (2).

• For $i = 1$, let X be a topological space such that X and $\mathbf{T}_1(X)$ are lattice equivalent. Then, by [3, Theorem 2 (f)], X is a $T_{(0,1)}$-space.

• For $i = 2, 3\frac{1}{2}$, let X be a topological space such that X and $\mathbf{T}_i(X)$ are lattice equivalent. Since X and $\mathbf{T}_0(X)$ are lattice equivalent, then $\mathbf{T}_i(X)$ and $\mathbf{T}_0(X)$ are lattice equivalent. Now, since every Tychonoff space is T_2, then Proposition 3.2 shows that $\mathbf{T}_0(X)$ and $\mathbf{T}_i(X)$ are homeomorphic, which means that X is a $T_{(0,i)}$-space. □

4. Lattice Invariant Properties

Definition 4.1. *A topological property P is said to be lattice invariant if a topological space lattice-equivalent to a topological space having P has also P.*

In [10] Thron has proved that regularity and normality are lattice-invariant properties but T_0 and T_1 are not.

Subsequently, Wong has proved that complete regularity, compactness, local compactness, Lindelöf, second countability and connectedness are lattice-invariant properties but T_2, complete normality, separability and first countability are not [13].

In this section, we are interested in separation axioms studied in the previous sections.

Theorem 4.2. *$T_{(0,2)}$, $T_{(S,D)}$, $T_{(S,1)}$, and $T_{(0,3\frac{1}{2})}$ are lattice invariant properties but S, $T_{(0,1)}$, $T_{(0,S)}$, $T_{(1,2)}$, $T_{(1,S)}$, $T_{(1,3\frac{1}{2})}$, and $T_{(0,D)}$ are not.*

Proof. • $T_{(0,2)}$ and $T_{(0,3\frac{1}{2})}$ are lattice invariant properties.

Let $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ be a lattice equivalence and suppose that X is a $T_{(0,2)}$-space (resp., $T_{(0,3\frac{1}{2})}$-space).

Since every topological space and its T_0-reflection are lattice equivalent, then we conclude that $\mathbf{T}_0(X)$ and $\mathbf{T}_0(Y)$ are lattice equivalent.

Now, according to the fact that $\mathbf{T}_0(X)$ is a T_2-space (resp., $\mathbf{T}_0(X)$ is a Tychonoff space and consequently a T_2-space) and $\mathbf{T}_0(Y)$ is a T_0-space, then Proposition 3.2 shows that $\mathbf{T}_0(X)$ and $\mathbf{T}_0(Y)$ are homeomorphic, which implies that $\mathbf{T}_0(Y)$ is a T_2-space (resp., a Tychonoff space) and finally Y is a $T_{(0,2)}$-space (resp., $T_{(0,3\frac{1}{2})}$-space).

• $T_{(S,D)}$ and $T_{(S,1)}$ are lattice invariant properties.

Let $\varphi : \Gamma(X) \longrightarrow \Gamma(Y)$ be a lattice equivalence such that X is a $T_{(S,D)}$-space (resp., $T_{(S,1)}$-space). Since every topological space and its Sober-reflection are lattice equivalent, then we conclude that $\mathbf{S}(X)$ and $\mathbf{S}(Y)$ are lattice equivalent. Now according to [5, Corollary 3.11], $\mathbf{S}(X)$ and $\mathbf{S}(Y)$ are homeomorphic, so $\mathbf{S}(Y)$ is a T_D-space (resp., T_1-space), therefore Y is a $T_{(S,D)}$-space (resp., $T_{(S,1)}$-space).

• S, $T_{(0,1)}$, $T_{(0,S)}$, $T_{(1,2)}$, $T_{(1,S)}$, $T_{(1,3\frac{1}{2})}$, and $T_{(0,D)}$ are not lattice invariant properties.

To see this, consider the following example.

Let X be an infinite set equipped with the cofinite topology. Let $\alpha \notin X$, and $Y = X \cup \{\alpha\}$. We equip Y with the topology whose closed sets are Y and the finite sets of X. Clearly the canonical embedding $X \hookrightarrow Y$ is a quasihomeomorphism; thus it induces a lattice equivalence φ.

Clearly, we get the following properties.

(a) X is a T_1-space which is not sober.

(b) Y is a sober space which is not T_D (note that $\{\alpha\}$ is not a locally closed set of Y).

(c) $T_1(Y)$ is a one point space.

Now, we are in a position to conclude that:

(1) The property S is not a lattice invariant property.

(2) $T_0(X) = X$ is a T_1-space, so X is a $T_{(0,1)}$-space but $T_0(Y) = Y$ is not T_D. Now, since every T_1-space is a T_D-space, then $T_{(0,D)}$ and $T_{(0,1)}$ are not lattice invariant properties.

(3) $T_1(Y)$ is a one point space, so it is Tychonoff and consequently Y is a $T_{(1,3\frac{1}{2})}$-space but $T_1(X) = X$ is not sober. Now, since $T_{3\frac{1}{2}} \Longrightarrow T_2 \Longrightarrow S$, then $T_{(1,S)}$, $T_{(1,2)}$, and $T_{(1,3\frac{1}{2})}$ are not lattice invariant properties.

(4) $T_0(X) = X$ is not sober, so X is not $T_{(0,S)}$, however $T_0(Y) = Y$ is sober then Y is $T_{(0,S)}$ and consequently $T_{(0,S)}$ is not a lattice invariant property.

□

Questions 4.3. (1) *Is $T_{(2,3\frac{1}{2})}$ a lattice invariant property.*

(2) *In the second section, we have proved that X and $F(X)$ are lattice equivalent if and only if X is a $T_{(0,F)}$-space for $F \in \{T_0, T_1, T_2, T_{3\frac{1}{2}}\}$ but not for $F = S$. Two immediate questions arise:*

(a) *Given a topological space X, characterize covariant functors F such that X and $F(X)$ are lattice equivalent.*

(b) *Given a covariant functor F, characterize topological spaces X such that X and $F(X)$ are lattice equivalent.*

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