A UNIFIED THEORY OF WEAKLY OPEN FUNCTIONS

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Abstract. We introduce a new notion of weakly M-open functions as functions defined between sets satisfying some minimal conditions. We obtain some characterizations and several properties of such functions. The functions enable us to formulate a unified theory of weak openness [36], weak semi-openness [10], weak preopenness [11], and weak β-openness [9].

1. Introduction. In 1984, Rose [36] defined the notion of weakly open functions. Some properties of weakly open functions were studied in [5]. Semi-open sets, preopen sets, and β-open sets play an important role in researching generalizations of open functions in topological spaces. By using these sets, Caldas and Navalagi [7–11] introduced and studied various types of modifications of weakly open functions. Furthermore, the analogy in their definitions and results suggest the need of formulating a unified theory.

In this paper, in order to unify several characterizations and properties of the functions mentioned above, we introduce a new class of functions called weakly M-open functions; these functions are defined between sets satisfying some minimal conditions. We obtain several characterizations and properties of such functions. In Section 3, we obtain several characterizations of weakly M-open functions. In Section 4, we obtain some conditions for a weakly M-open function to be M-open. In the last section, we recall several types of modifications of open sets and point out the possibility for new forms of weakly M-open functions. Moreover, we show that some functions in these new forms are equivalent to each other. As a result, we obtain the following property (stated in Corollary 6.1).

Theorem 1.1. For a function $f\colon (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:

(1) $f\colon (X,\tau) \to (Y,\sigma)$ is weakly open;
(2) $f\colon (X,\tau_s) \to (Y,\sigma)$ is weakly open, where τ_s is the semiregularization of τ;
(3) $f\colon (X,\tau^\alpha) \to (Y,\sigma)$ is weakly open, where τ^α is the family of α-open sets of (X,τ).

2. Preliminaries. Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$). A subset A is said to be *δ-open* [37] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is called a *δ-cluster* point of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $\mathrm{Cl}_\delta(A)$. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$. The *θ-closure* of A, denoted by $\mathrm{Cl}_\theta(A)$, is defined as the set of all points $x \in X$ such that $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. A subset A is said to be θ-closed if $A = \mathrm{Cl}_\theta(A)$ [37]. The complement of a θ-closed set is said to be θ-open. It is shown in [37] that $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set V of X and $\mathrm{Cl}_\theta(S)$ is closed in X for every subset S of X.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *semi-open* [17] (resp. *preopen* [20], *α-open* [23], *β-open* [1] or *semi-preopen* [3]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$),
(2) *δ-preopen* [35] (resp. *δ-semi-open* [28]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open) sets in (X, τ) is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$ or τ^α, $\beta(X)$, $\delta\mathrm{PO}(X)$, $\delta\mathrm{SO}(X)$).

Definition 2.2. The complement of a semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open) set is said to be *semi-closed* [12] (resp. *preclosed* [20], *α-closed* [21], *β-closed* [1] or *semi-preclosed* [3], *δ-preclosed* [35], *δ-semi-closed* [28]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, δ-preclosed, δ-semi-closed) sets of X containing A is called the *semi-closure* [12] (resp. *preclosure* [15], *α-closure* [21], *β-closure* [2], or *semi-preclosure* [3], *δ-preclosure* [35], *δ-semi-closure* [28]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$, $\mathrm{pCl}_\delta(A)$, $\mathrm{sCl}_\delta(A)$).

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior* or *semi-preinterior*, *δ-preinterior*, *δ-semi-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$, $\mathrm{pInt}_\delta(A)$, $\mathrm{sInt}_\delta(A)$).

Throughout the present paper, (X, τ) and (Y, σ) denote topological spaces and $f\colon (X, \tau) \to (Y, \sigma)$ presents a (single valued) function.

Definition 2.5. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be

(1) *semi-open* [6] (resp. *preopen* [20], *α-open* [21], *β-open* [1]) if $f(U)$ is semi-open (resp. preopen, α-open, β-open) for each open set U of X,
(2) *weakly open* [36] (resp. *weakly semi-open* [10], *weakly preopen* [11], *weakly β-open* [9]) if $f(U) \subset \mathrm{Int}(f(\mathrm{Cl}(U)))$ (resp. $f(U) \subset$

sInt(f(Cl(U))), $f(U) \subset$ pInt(f(Cl(U))), $f(U) \subset$ spInt(f(Cl(U)))) for each open set U of X,

(3) *pre-β-open* [18] if $f(U)$ is β-open in Y for each β-open set U of X.

3. Weakly M-open Functions.

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [34] on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Remark 3.1. Let (X, τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, $\beta(X)$, δPO(X), and δSO(X) are all m-structures on X.

Definition 3.2. Let (X, m_X) be an m-space. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [19] as follows:

(1) $m_X\text{-Cl}(A) = \bigcap\{F : A \subset F, X - F \in m_X\}$,
(2) $m_X\text{-Int}(A) = \bigcup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\beta(X)$, δPO(X), δSO(X)), then we have

(1) m_X-Cl(A) = Cl(A) (resp. sCl(A), pCl(A), αCl(A), ${}_\beta$Cl(A), pCl$_\delta$(A), sCl$_\delta$(A)),
(2) m_X-Int(A) = Int(A) (resp. sInt(A), pInt(A), αInt(A), ${}_\beta$Int(A), pInt$_\delta$(A), sInt$_\delta$(A)).

Lemma 3.1. (Maki et al. [19]) Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:

(1) $m_X\text{-Cl}(X - A) = X - (m_X\text{-Int}(A))$ and $m_X\text{-Int}(X - A) = X - (m_X\text{-Cl}(A))$,
(2) If $(X - A) \in m_X$, then $m_X\text{-Cl}(A) = A$ and if $A \in m_X$, then $m_X\text{-Int}(A) = A$,
(3) $m_X\text{-Cl}(\emptyset) = \emptyset$, $m_X\text{-Cl}(X) = X$, $m_X\text{-Int}(\emptyset) = \emptyset$, and $m_X\text{-Int}(X) = X$,
(4) If $A \subset B$, then $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(B)$ and $m_X\text{-Int}(A) \subset m_X\text{-Int}(B)$,
(5) $A \subset m_X\text{-Cl}(A)$ and $m_X\text{-Int}(A) \subset A$,
(6) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ and $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [19] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3.2. (Popa and Noiri [32]) For a minimal structure m_X on a nonempty set X, the following properties are equivalent:

(1) m_X has property $\mathcal{B}$;
(2) If m_X-Int$(V) = V$, then $V \in m_X$;
(3) If m_X-Cl$(F) = F$, then $X - F \in m_X$.

Lemma 3.3. (Noiri and Popa [25]) Let X be a nonempty set and m_X a minimal structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:

(1) $A \in m_X$ if and only if m_X-Int$(A) = A$,
(2) A is m_X-closed if and only if m_X-Cl$(A) = A$,
(3) m_X-Int$(A) \in m_X$ and m_X-Cl(A) is m_X-closed.

Definition 3.4. Let S be a subset of an m-space (X, m_X). A point $x \in X$ is called

(1) an *m_X-θ-adherent point* of S if m_X-Cl$(U) \cap S \neq \emptyset$ for every $U \in m_X$ containing x,
(2) an *m_X-θ-interior point* of S if $x \in U \subset m_X$-Cl$(U) \subset S$ for some $U \in m_X$.

The set of all m_X-θ-adherent points of S is called the *m_X-θ-closure* [25] of S and is denoted by m_X-Cl$_\theta(S)$. If $S = m_X$-Cl$_\theta(S)$, then S is called *m_X-θ-closed*. The complement of an m_X-θ-closed set is said to be *m_X-θ-open*. The set of all m_X-θ-interior points of S is called the *m_X-θ-interior* of S and is denoted by m_X-Int$_\theta(S)$.

Remark 3.3. Let (X, τ) be a topological space and $m_X = \tau$ (resp. SO(X), PO(X), $\beta(X)$), then m_X-Cl$_\theta(S) =$ Cl$_\theta(S)$ [37] (resp. sCl$_\theta(S)$ [13], pCl$_\theta(S)$ [27], spCl$_\theta(S)$ [24]).

Lemma 3.4. (Noiri and Popa [25]) Let A and B be subsets of (X, m_X). Then the following properties hold:

(1) $X - m_X$-Cl$_\theta(A) = m_X$-Int$_\theta(X - A)$ and $X - m_X$-Int$_\theta(A) = m_X$-Cl$_\theta(X - A)$,
(2) A is m_X-θ-open if and only if $A = m_X$-Int $_\theta(A)$,
(3) $A \subset m_X$-Cl$(A) \subset m_X$-Cl$_\theta(A)$ and m_X-Int$_\theta(A) \subset m_X$-Int$(A) \subset A$,
(4) If $A \subset B$, then m_X-Cl$_\theta(A) \subset m_X$-Cl$_\theta(B)$ and m_X-Int$_\theta(A) \subset m_X$-Int$_\theta(B)$,
(5) If A is m_X-open, then m_X-Cl$(A) = m_X$-Cl$_\theta(A)$.

A function $f\colon (X, m_X) \to (Y, m_Y)$ is said to be *weakly M-continuous* [34] at $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset m_Y$-Cl(V). It is shown in Theorem 3.2 of [34] that a function $f\colon (X, m_X) \to (Y, m_Y)$ is weakly M-continuous if and only if $f^{-1}(V) \subset m_X$-Int$(f^{-1}(m_Y$-Cl$(V)))$ for each $V \in m_Y$. For a function $f\colon (X, m_X) \to (Y, m_Y)$, we define the concept of weak M-openness as a natural dual to the concept of weak M-continuity.

Definition 3.5. A function $f\colon (X, m_X) \to (Y, m_Y)$, where X and Y are nonempty sets with m-structures m_X and m_Y, respectively, is said to be *weakly M-open* if for each $U \in m_X$, $f(U) \subset m_Y\text{-Int}(f(m_X\text{-Cl}(U)))$.

Remark 3.4. Let (X, τ) and (Y, σ) be topological spaces and $f\colon (X, m_X) \to (Y, m_Y)$ be a function. If $m_X = \tau$, $m_Y = \sigma$ (resp. SO(Y), PO(Y), $\beta(Y)$), and $f\colon (X, m_X) \to (Y, m_Y)$ is a weakly M-open function, then f is weakly open [36] (resp. weakly semi-open [10], weakly preopen [11], weakly β-open [9]).

Theorem 3.1. For a function $f\colon (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:

(1) f is weakly M-open;
(2) $f(m_X\text{-Int}_\theta(A)) \subset m_Y\text{-Int}(f(A))$ for every subset A of X;
(3) $m_X\text{-Int}_\theta(f^{-1}(B)) \subset f^{-1}(m_Y\text{-Int}(B))$ for every subset B of Y;
(4) $f^{-1}(m_Y\text{-Cl}(B)) \subset m_X\text{-Cl}_\theta(f^{-1}(B))$ for every subset B of Y;
(5) For each $x \in X$ and each m_X-open set U containing x, there exists an m_Y-open set V containing $f(x)$ such that $V \subset f(m_X\text{-Cl}(U))$.

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X and $x \in m_X\text{-Int}_\theta(A)$. Then, there exists a $U \in m_X$ such that $x \in U \subset m_X\text{-Cl}(U) \subset A$. Hence, we have $f(x) \in f(U) \subset f(m_X\text{-Cl}(U)) \subset f(A)$. Since f is weakly M-open, $f(U) \subset m_Y\text{-Int}(f(m_X\text{-Cl}(U))) \subset m_Y\text{-Int}(f(A))$ and $x \in f^{-1}(m_Y\text{-Int}(f(A)))$. Thus, $m_X\text{-Int}_\theta(A) \subset f^{-1}(m_Y\text{-Int}(f(A)))$ and $f(m_X\text{-Int}_\theta(A)) \subset m_Y\text{-Int}(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. By (2), $f(m_X\text{-Int}_\theta(f^{-1}(B))) \subset m_Y\text{-Int}(B)$. Therefore, $m_X\text{-Int}_\theta(f^{-1}(B)) \subset f^{-1}(m_Y\text{-Int}(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. By Lemma 3.4 and (3), we have

$$\begin{aligned} X - m_X\text{-Cl}_\theta(f^{-1}(B)) &= m_X\text{-Int}_\theta(X - f^{-1}(B)) = m_X\text{-Int}_\theta(f^{-1}(Y - B)) \\ &\subset f^{-1}(m_Y\text{-Int}(Y - B)) = f^{-1}(Y - m_Y\text{-Cl}(B)) = X - f^{-1}(m_Y\text{-Cl}(B)). \end{aligned}$$

Therefore, $f^{-1}(m_Y\text{-Cl}(B)) \subset m_X\text{-Cl}_\theta(f^{-1}(B))$.

(4) $\Rightarrow$ (5): Let $x \in X$ and U be any m_X-open set containing x. Let $B = Y - f(m_X\text{-Cl}(U))$. By (4), $f^{-1}(m_Y\text{-Cl}(Y - f(m_X\text{-Cl}(U)))) \subset m_X\text{-Cl}_\theta(f^{-1}(Y - f(m_X\text{-Cl}(U))))$. Now, $f^{-1}(m_Y\text{-Cl}(Y - f(m_X\text{-Cl}(U)))) = X - f^{-1}(m_Y\text{-Int}(f(m_X\text{-Cl}(U))))$. And also we have,

$$\begin{aligned} m_X\text{-Cl}_\theta(f^{-1}(Y - f(m_X\text{-Cl}(U)))) &= m_X\text{-Cl}_\theta(X - f^{-1}(f(m_X\text{-Cl}(U)))) \subset \\ m_X\text{-Cl}_\theta(X - m_X\text{-Cl}(U)) &= X - m_X\text{-Int}_\theta(m_X\text{-Cl}(U)) \subset X - U. \end{aligned}$$

Therefore, we obtain $U \subset f^{-1}(m_Y\text{-Int}(f(m_X\text{-Cl}(U))))$ and $f(U) \subset m_Y\text{-Int}(f(m_X\text{-Cl}(U)))$. Since $f(x) \in f(U)$, there exists $V \in m_Y$ such that $f(x) \in V \subset f(m_X\text{-Cl}(U))$.

(5) $\Rightarrow$ (1): Let $U \in m_X$ and $x \in U$. By (5), there exists an m_Y-open set V containing $f(x)$ such that $V \subset f(m_X\text{-Cl}(U))$. Hence, we have $f(x) \in V \subset m_Y\text{-Int}(f(m_X\text{-Cl}(U)))$ for each $x \in U$. Therefore, we obtain $f(U) \subset m_Y\text{-Int}(f(m_X\text{-Cl}(U)))$. This shows that f is weakly M-open.

Theorem 3.2. Let $f\colon (X, m_X) \to (Y, m_Y)$ be a bijective function, where m_X has property $\mathcal{B}$. Then the following properties are equivalent:

(1) f is weakly M-open;
(2) $m_Y\text{-Cl}(f(m_X\text{-Int}(F))) \subset f(F)$ for each m_X-closed set F of X;
(3) $m_Y\text{-Cl}(f(U)) \subset f(m_X\text{-Cl}(U))$ for each $U \in m_X$.

Proof. (1) $\Rightarrow$ (2): Let F be any m_X-closed set of X. Then $X - F$ is m_X-open and

$$\begin{aligned} Y - f(F) = f(X - F) &\subset m_Y\text{-Int}(f(m_X\text{-Cl}(X - F))) \\ &= m_Y\text{-Int}(f(X - m_X\text{-Int}(F))) \\ &= m_Y\text{-Int}(Y - f(m_X\text{-Int}(F))) = Y - m_Y\text{-Cl}(f(m_X\text{-Int}(F))). \end{aligned}$$

This implies that $m_Y\text{-Cl}(f(m_X\text{-Int}(F))) \subset f(F)$.

(2) $\Rightarrow$ (3): Let $U \in m_X$. By (2), we have

$$\begin{aligned} m_Y\text{-Cl}(f(U)) = m_Y\text{-Cl}(f(m_X\text{-Int}(U))) &\subset m_Y\text{-Cl}(f(m_X\text{-Int}(m_X\text{-Cl}(U)))) \\ &\subset f(m_X\text{-Cl}(U)). \end{aligned}$$

(3) $\Rightarrow$ (1): Let $U \in m_X$. Then, we have

$$\begin{aligned} &Y - m_Y\text{-Int}(f(m_X\text{-Cl}(U))) = m_Y\text{-Cl}(Y - f(m_X\text{-Cl}(U))) \\ &= m_Y\text{-Cl}(f(X - m_X\text{-Cl}(U))) \subset f(m_X\text{-Cl}(X - m_X\text{-Cl}(U))) \\ &= f(X - m_X\text{-Int}(m_X\text{-Cl}(U))) \subset f(X - U) = Y - f(U). \end{aligned}$$

Therefore, we obtain $f(U) \subset m_Y\text{-Int}(f(m_X\text{-Cl}(U)))$. This shows that f is weakly M-open.

Remark 3.5. Let (X, τ) and (Y, σ) be topological spaces and $f\colon (X, \tau) \to (Y, m_Y)$ be a weakly M-open function, where $m_Y = \text{SO}(Y)$ (resp. $\text{PO}(Y)$, $\beta(Y)$). Then by Theorems 3.1 and 3.2, we obtain the characterizations established in Theorem 2.3–2.6 of [10] (resp. Theorem 2.3–2.6 of [11], Theorems 2.4 and 2.5 of [9] and Theorems 2.4–2.6 of [7]).

4. Weak M-openness and M-openness.

Definition 4.1. A function $f\colon (X, m_X) \to (Y, m_Y)$ is said to be

(1) *M-open* [22] if $f(U)$ is m_Y-open in (Y, m_Y) for every $U \in m_X$,
(2) *almost M-open* [22] at $x \in X$ if for each $U \in m_X$ containing x, there exists $V \in m_Y$ containing $f(x)$ such that $V \subset f(U)$. If f is almost M-open at each point $x \in X$, then f is said to be *almost M-open*.

Remark 4.1. Let (X, τ) and (Y, σ) be topological spaces and $f\colon (X, m_X) \to (Y, m_Y)$ be an M-open function.

(1) If $m_X = \tau$ and $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$), then f is semi-open (resp. preopen, α-open, β-open).
(2) If $m_X = \beta(X)$ and $m_Y = \beta(Y)$, then f is pre-β-open.

Lemma 4.1. A function $f\colon (X, m_X) \to (Y, m_Y)$ is almost M-open if and only if $f(U) = \mathrm{Int}_Y(f(U))$ for each $U \in m_X$.

Proof. *Necessity.* Let $U \in m_X$ and $x \in U$. Then, there exists $V_x \in m_Y$ such that $f(x) \in V_x \subset f(U)$; hence, $V_x \subset \mathrm{Int}_Y(f(U))$. Therefore, we have $f(U) \subset \bigcup\{V_x : x \in U\} \subset \mathrm{Int}_Y(f(U))$ and hence, $f(U) = \mathrm{Int}_Y(f(U))$.

Sufficiency. Let $x \in X$ and U be an m_X-open set containing x. Then we have $f(x) \in f(U) = \mathrm{Int}_Y(f(U))$. Therefore, there exists $V \in m_Y$ such that $f(x) \in V \subset f(U)$. This shows that f is almost M-open.

Lemma 4.2. For a function $f\colon (X, m_X) \to (Y, m_Y)$, the following properties hold:

(1) M-openness implies almost M-openness and almost M-openness implies weak M-openness,
(2) M-openness is equivalent to almost M-openness if m_Y has property $\mathcal{B}$.

Proof. (1) It is obvious from Lemma 4.1 that every M-open function is almost M-open. Suppose that f is almost M-open. Let $U \in m_X$. By Lemma 4.1, we have $f(U) = \mathrm{Int}_Y(f(U)) \subset \mathrm{Int}_Y(f(\mathrm{Cl}_X(U)))$. Hence, f is weakly M-open.

(2) This follows from Lemmas 3.2 and 4.1.

Remark 4.2. (a) The converses of Lemma 4.2 (1) are not true in general. There exists an almost M-open function which is not M-open (Example 3.1 of [22]). And also, there exists a weakly M-open function which is not almost M-open (Example 2.19 of [10], Example 2.17 of [11], and Example 2.16 of [9]).

(b) Let $f\colon (X, \tau) \to (Y, \sigma)$ be a function. If $f\colon (X, m_X) \to (Y, m_Y)$, $m_X = \tau$, and $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\beta(Y)$), then we obtain the results established in Theorem 2.18 of [10] (resp. Theorem 2.16 of [11], Theorem 2.15 of [9]).

Definition 4.2. A function $f\colon (X, m_X) \to (Y, m_Y)$ is said to be *strongly M-continuous* if $f(m_X\text{-Cl}(A)) \subset f(A)$ for every subset A of X.

Remark 4.3. If $m_X = \tau$, $m_Y = \sigma$, and $f\colon (X, m_X) \to (Y, m_Y)$ is a strongly M-continuous function, then $f\colon (X, \tau) \to (Y, \sigma)$ is strongly continuous due to Levine [16].

Theorem 4.1. If $f\colon (X, m_X) \to (Y, m_Y)$ is a weakly M-open and strongly M-continuous function, then f is almost M-open.

Proof. Let $U \in m_X$. Since f is weakly M-open and strongly M-continuous, we have $f(U) \subset m_Y\text{-Int}(f(m_X\text{-Cl}(U))) \subset m_Y\text{-Int}(f(U))$. By Lemma 3.1, $f(U) = m_Y\text{-Int}(f(U))$. It follows from Lemma 4.1 that f is almost M-open.

Corollary 4.1. Let $f\colon (X, m_X) \to (Y, m_Y)$ be a strongly M-continuous function and m_Y has property $\mathcal{B}$. Then the following properties are equivalent:

(1) f is M-open;
(2) f is almost M-open;
(3) f is weakly M-open.

Proof. This is an immediate consequence of Lemma 4.2 and Theorem 4.1.

Remark 4.4. (a) There exists a weakly M-open function which is not strongly M-continuous as shown in Example 2.8 of [10], Example 2.8 of [11], and Example 2.7 of [9].

(b) Let (X, τ) and (Y, σ) be topological spaces and $f\colon (X, m_X) \to (Y, m_Y)$ be a function. If $m_X = \tau$ and $m_Y = \text{SO}(Y)$ (resp. $\text{PO}(Y)$, $\beta(Y)$), then by Corollary 4.1 we obtain the results established in Theorem 2.7 of [10] (resp. Theorem 2.6 of [11], Theorem 2.6 of [9]).

Definition 4.3. An m-space (X, m_X) is said to be *m-regular* [25] if for each m_X-closed set F and each $x \notin F$, there exist disjoint m_X-open sets U and V such that $x \in U$ and $F \subset V$.

Remark 4.5. Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\text{SO}(X)$, $\text{PO}(X)$, $\beta(X)$). Then m-regularity coincides with regularity (resp. semi-regularity [14], pre-regularity [27], semi-pre-regularity [24]).

Lemma 4.3. (Noiri and Popa [25]) If an m-space (X, m_X) is m-regular, then for each $x \in X$ and each m_X-open set U containing x, there exists an m_X-open set V such that $x \in V \subset m_X\text{-Cl}(V) \subset U$.

Theorem 4.2. Let (X, m_X) be m-regular. Then a function $f\colon (X, m_X) \to (Y, m_Y)$ is almost M-open if and only if f is weakly M-open.

Proof. If f is almost M-open, then it follows from Lemma 4.2 that f is weakly M-open. Suppose that f is weakly M-open. Let U be any m_X-open set of (X, m_X). By Lemma 4.3, for each $x \in U$ there exists $U_x \in m_X$ such that $x \in U_x \subset m_X\text{-Cl}(U_x) \subset U$. Hence, we obtain $U = \bigcup\{U_x : x \in U\} = \bigcup\{m_X\text{-Cl}(U_x) : x \in U\}$ and hence,

$$\begin{aligned} f(U) &= \bigcup\{f(U_x) : x \in U\} \subset \bigcup\{m_Y\text{-Int}(f(m_X\text{-Cl}(U_x))) : x \in U\} \\ &\subset m_Y\text{-Int}\Big(\bigcup\{f(m_X\text{-Cl}(U_x)) : x \in U\}\Big) \\ &\subset m_Y\text{-Int}\Big(f\Big(\bigcup\{m_X\text{-Cl}(U_x) : x \in U\}\Big)\Big) \\ &= m_Y\text{-Int}(f(U)). \end{aligned}$$

By Lemma 3.1, we have $f(U) = m_Y\text{-Int}(f(U))$. It follows from Lemma 4.1 that f is almost M-open.

Corollary 4.2. Let (X, m_X) be m-regular and m_Y has property $\mathcal{B}$. Then for a function $f\colon (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:

(1) f is M-open;
(2) f is almost M-open;
(3) f is weakly M-open.

Proof. This is an immediate consequence of Lemma 4.2 and Theorem 4.2.

Remark 4.6. Let (X, τ) and (Y, σ) be topological spaces and $f\colon (X, m_X) \to (Y, m_Y)$ be a function. If $m_X = \tau$ and $m_Y = \sigma$ (resp. $\mathrm{SO}(Y)$, $\mathrm{PO}(Y)$, $\beta(Y)$), then by Corollary 4.2 we obtain the results established in Theorem 7 of [36] (resp. Theorem 2.12 of [10], Theorem 2.12 of [11], Theorem 2.3 of [9]).

Definition 4.4. A function $f\colon (X, m_X) \to (Y, m_Y)$ is said to satisfy *the weakly M-open interiority condition* if $m_Y\text{-Int}(f(m_X\text{-Cl}(U))) \subset f(U)$ for every $U \in m_X$.

Theorem 4.3. If a function $f\colon (X, m_X) \to (Y, m_Y)$ is weakly M-open and satisfies the weakly M-open interiority condition, then f is almost M-open.

Proof. Let $U \in m_X$. Since f is weakly M-open, $f(U) \subset m_Y\text{-Int}(f(m_X\text{-Cl}(U))) = m_Y\text{-Int}(m_Y\text{-Int}(f(m_X\text{-Cl}(U)))) \subset m_Y\text{-Int}(f(U)) \subset f(U)$. Hence, $f(U) = m_Y\text{-Int}(f(U))$ and by Lemma 4.1 f is almost M-open.

Corollary 4.3. Let $f\colon (X, m_X) \to (Y, m_Y)$ satisfy the weakly M-open interiority condition and m_Y has property $\mathcal{B}$. Then the following properties are equivalent:

(1) f is M-open;
(2) f is almost M-open;
(3) f is weakly M-open.

Remark 4.7. (a) An M-open function $f\colon (X, m_X) \to (Y, m_Y)$ does not necessarily satisfy the weakly M-open interiority condition as shown by Example 2.10 of [7].

(b) Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$, $m_Y = \beta(Y)$, and $f\colon (X, m_X) \to (Y, m_Y)$ satisfies the weakly M-open interiority condition, then f satisfies the weakly β-open interiority condition [7].

(c) By Corollary 4.3 we obtain the result established in Theorem 2.11 of [7].

Definition 4.5. Let A be a subset of (X, m_X). The *m_X-frontier* [34] of A, m_X-Fr(A), is defined by m_X-Fr$(A) = m_X$-Cl$(A) \cap m_X$-Cl$(X - A)$.

Definition 4.6. A function $f\colon (X, m_X) \to (Y, m_Y)$ is said to be *complementary weakly M-open* if $f(m_X$-Fr$(U))$ is m-closed in (Y, m_Y) for each $U \in m_X$.

Remark 4.8. (a) Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$, $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\beta(Y)$), and $f\colon (X, m_X) \to (Y, m_Y)$ is complementary weakly M-open, then f is complementary weakly semi-open [10] (resp. complementary weakly preopen [11], complementary weakly β-open [9]).

(b) The notions of weakly M-open functions and complementary weakly M-open functions are independent of each other as shown by the following examples: Examples 2.14 and 2.15 of [10], Examples 2.12 and 2.13 of [11], and Examples 2.12 and 2.13 of [9].

Theorem 4.4. If $f\colon (X, m_X) \to (Y, m_Y)$ is a weakly M-open and complementary weakly M-open bijection, where m_X has property $\mathcal{B}$ and m_Y is closed under finite intersection, then f is almost M-open.

Proof. Let $x \in X$ and U be any m-open set in (X, m_X) containing x. Since f is weakly M-open, by Theorem 3.1 there exists $V \in m_Y$ such that $f(x) \in V \subset f(m_X$-Cl$(U))$. Since m_X has property $\mathcal{B}$, we have m_X-Fr$(U) = m_X$-Cl$(U) \cap m_X$-Cl$(X - U) = m_X$-Cl$(U) \cap (X - m_X$-Int$(U)) = m_X$-Cl$(U) \cap (X - U)$. Since $x \in U$, $x \notin m_X$-Fr(U) and hence, $f(x) \notin f(m_X$-Fr$(U))$. Put $W = V \cap (Y - f(m_X$-Fr$(U)))$. Then, since f is complementary weakly M-open and m_Y is closed under finite intersection, we have $f(x) \in W \in m_Y$. Next, we shall show that $W \subset f(U)$. Let $y \in W$. Then $y \in V \subset f(m_X$-Cl$(U))$ and $y \notin f(m_X$-Fr$(U)) = f(m_X$-Cl$(U) \cap (X - U)) = f(m_X$-Cl$(U)) \cap (Y - f(U))$. Therefore, we have $y \in (Y - f(m_X$-Cl$(U))) \cup f(U)$ and hence, $y \in f(U)$. Consequently, we obtain $W \subset f(U)$. This shows that f is almost M-open.

Corollary 4.4. Let $f\colon (X, m_X) \to (Y, m_Y)$ be a weakly M-open and complementary weakly M-open bijection, where m_X has property $\mathcal{B}$ and

m_Y is closed under finite intersection and has property $\mathcal{B}$. Then the following properties are equivalent:

(1) f is M-open;
(2) f is almost M-open;
(3) f is weakly M-open.

Remark 4.9. Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$, $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\beta(Y)$), and $f\colon (X, m_X) \to (Y, m_Y)$ is a function, then by Corollary 4.4, we obtain the results established in Theorem 2.16 of [10] (resp. Theorem 2.16 of [11], Theorem 2.14 of [9]).

5. Some Properties of Weakly M-open Functions.

Definition 5.1. An m-space (X, m_X) is said to be *m-hyperconnected* if m_X-$\mathrm{Cl}(U) = X$ for every m-open set U of (X, m_X).

Remark 5.1. Let (X, τ) be a topological space and $m_X = \tau$. Then an m-hyperconnected space is well-known as a *hyperconnected* space or a D-space.

Theorem 5.1. Let an m-space (X, m_X) be m-hyperconnected and m_Y has property $\mathcal{B}$. Then a function $f\colon (X, m_X) \to (Y, m_Y)$ is weakly M-open if and only if $f(X)$ is m-open in (Y, m_Y).

Proof. *Necessity.* Let f be weakly M-open. Since $X \in m_X$, $f(X) \subset m_Y$-$\mathrm{Int}(f(m_X$-$\mathrm{Cl}(X))) = m_Y$-$\mathrm{Int}(f(X))$ and hence, $f(X) \subset m_Y$-$\mathrm{Int}(f(X))$. Since m_Y has property $\mathcal{B}$, by Lemma 3.3 $f(X) \in m_Y$.

Sufficiency. Suppose that $f(X)$ is m-open in (Y, m_Y). Let $U \in m_X$. Then, $f(U) \subset f(X) = m_Y$-$\mathrm{Int}(f(X)) = m_Y$-$\mathrm{Int}(f(m_X$-$\mathrm{Cl}(U)))$. Therefore, we obtain $f(U) \subset m_Y$-$\mathrm{Int}(f(m_X$-$\mathrm{Cl}(U)))$. This shows that f is weakly M-open.

Remark 5.2. Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$, $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\beta(Y)$), and $f\colon (X, m_X) \to (Y, m_Y)$ is a function, then by Theorem 5.1, we obtain the results established in Theorem 2.25 of [10] (resp. Theorem 2.23 of [11], Theorem 2.21 of [9]).

Definition 5.2. A function $f\colon (X, m_X) \to (Y, m_Y)$ is said to be *contra-M-closed* if $f(F)$ is m-open in (Y, m_Y) for every m-closed set F of (X, m_X).

Remark 5.3. Let (X, τ) and (Y, σ) be topological spaces.

(1) If $m_X = \tau$, $m_Y = \sigma$ (resp. $\mathrm{PO}(Y)$, $\beta(Y)$), and $f\colon (X, m_X) \to (Y, m_Y)$ is a contra-M-closed function, then f is contra-closed [5] (resp. contra-preclosed [11], contra-β-closed [9]),
(2) If $m_X = \mathrm{PO}(X)$, $m_Y = \mathrm{PO}(Y)$, and $f\colon (X, m_X) \to (Y, m_Y)$ is a contra-M-closed function, then f is contra-M-preclosed [11].

Theorem 5.2. If a function $f\colon (X, m_X) \to (Y, m_Y)$ is contra-M-closed and m_X has property $\mathcal{B}$, then f is weakly M-open.

Proof. Let $U \in m_X$. Since m_X has property $\mathcal{B}$, by Lemma 3.3 m_X-Cl(U) is m-closed in (X, m_X). Hence, we have $f(U) \subset f(m_X$-Cl$(U)) = m_Y$-Int$(f(m_X$-Cl$(U)))$. Therefore, we obtain $f(U) \subset m_Y$-Int$(f(m_X$-Cl$(U)))$ and f is weakly M-open.

Remark 5.4. (a) The converse of Theorem 5.2 need not be true as shown in Example 2.11 of [10], Example 2.10 of [11], Example 2.12 of [9].

(b) Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$, $m_Y = \sigma$ (resp. SO(Y), PO(Y), $\beta(Y)$), and $f\colon (X, m_X) \to (Y, m_Y)$ a contra-M-closed function, then by Theorem 5.2, we obtain the results established in Theorem 9 of [5] (resp. Theorem 2.10 of [10], Theorem 2.9 of [11], Theorem 2.9 of [9]).

Definition 5.3. An m-space (X, m_X) is said to be *m-connected* [33] if X cannot be written as the union of two nonempty disjoint sets of m_X.

Remark 5.5. Let (X, τ) be a topological space. If $m_X = \tau$ (resp. SO(X), PO(X), $\beta(Y)$) and (X, m_X) is m-connected, (X, τ) is called connected (resp. semi-connected [29], preconnected [30], β-connected [31]).

Theorem 5.3. If $f\colon (X, m_X) \to (Y, m_Y)$ is a weakly M-open bijection, m_Y has property $\mathcal{B}$, and (Y, m_Y) is m-connected, then (X, m_X) is m-connected.

Proof. Suppose that (X, m_X) is not m-connected. There exist nonempty m-open sets U_1 and U_2 such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = X$. Hence, we have $f(U_1) \cap f(U_2) = \emptyset$ and $f(U_1) \cup f(U_2) = Y$. Since f is weakly M-open, we have $f(U_i) \subset m_Y$-Int$(f(m_X$-Cl$(U_i)))$ for $i = 1, 2$. Since U_i is m-closed, $U_i = m_X$-Cl(U_i) and hence, $f(U_i) \subset m_Y$-Int$(f(U_i))$ for $i = 1, 2$. Hence, we obtain $f(U_i) = m_Y$-Int$(f(U_i))$ for $i = 1, 2$. Since m_Y has property $\mathcal{B}$, by Lemma 3.3 $f(U_i) \in m_Y$ for $i = 1, 2$. Then (Y, m_Y) is decomposed into two nonempty disjoint m-open sets. This is contrary to the hypothesis that (Y, m_Y) is m-connected.

Remark 5.6. Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$ and $m_Y =$ SO(Y) (resp. PO(Y), $\beta(Y)$), then by Theorem 5.3, we obtain the results established in Theorem 2.23 of [10] (resp. Theorem 2.21 of [11], Theorem 2.20 of [9]).

6. New Forms of Weakly M-open Functions. First we recall the relationships among some modifications of open sets. If a subset A of a topological space (X, τ) is semi-open and semi-closed, then it is said to be *semi-regular* [13]. It is shown in [13] that the semi-closure sCl(U) is semi-open and semi-regular for any semi-open set U of (X, τ). This property is very useful. The set of all semi-regular sets of (X, τ) is denoted by SR(X). For a subset A of a topological space (X, τ), we put srCl$(A) = \cap\{F : A \subset F, F \in \text{SR}(X)\}$.

Let A be a subset of a topological space (X, τ). A point x of X is called a *semi-θ-cluster point* of A of $\mathrm{sCl}(U) \cap A \neq \emptyset$ for every $U \in \mathrm{SO}(X)$ containing x. The set of all semi-θ-cluster points of A is called the *semi-θ-closure* [13] of A and is denoted by $\mathrm{sCl}_\theta(A)$. A subset A is said to be *semi-θ-closed* if $A = \mathrm{sCl}_\theta(A)$. The complement of a semi-θ-closed set is said to be *semi-θ-open*. The family of all semi-θ-open sets of (X, τ) is denoted by $\theta\mathrm{SO}(X)$.

A subset A is said to be *b-open* [4] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The *b-interior* of A, $\mathrm{bInt}(A)$, is defined by the union of all b-open sets contained in A. The complement of a b-open set is said to be b-closed [4]. The *b-closure* of A, $\mathrm{bCl}(A)$, is defined by the intersection of all b-closed sets containing A. The family of all b-open sets of (X, τ) is denoted by $\mathrm{BO}(X)$.

For several modifications of open sets, we have the following diagram in which the converses of implications need not be true as shown in [26].

DIAGRAM

$$\begin{tikzcd}
\text{regular open} \arrow[r] \arrow[d] & \delta\text{-open} \arrow[r] \arrow[d] & \text{open} \arrow[r] \arrow[d, phantom, "\dagger"] & \alpha\text{-open} \arrow[r] \arrow[d] & \text{preopen} \arrow[r] \arrow[d] & \delta\text{-preopen} \arrow[d, phantom, "\dagger"] \\
\text{semi-regular} \arrow[r] & \text{semi-}\theta\text{-open} \arrow[r] & \delta\text{-semi-open} \arrow[r] & \text{semi-open} \arrow[r] & b\text{-open} \arrow[r] & \text{semi-preopen}
\end{tikzcd}$$

Remark 6.1. In the diagram above, the following are to be noted.

(1) It is shown in [28] that openness and δ-semi-openness are independent of each other.
(2) It is shown in [26] that δ-preopenness and semi-preopenness are independent of each other.

Let $\mathrm{RO}(X)$ (resp. $\mathrm{RC}(X)$) be the family of all regular open (resp. regular closed) sets of a topological space (X, τ). The family of all δ-open sets of (X, τ) forms a topology for X which is weaker than τ. This topology has $\mathrm{RO}(X)$ as the base. It is called the semiregularization of τ and is denoted by τ_s. Then we have $\mathrm{RO}(X) \subset \tau_s \subset \tau \subset \tau^\alpha$, where $\tau^\alpha = \alpha(X)$. For a subset A of X, we set $\mathrm{rCl}(A) = \cap\{K : A \subset K \text{ and } K \in \mathrm{RC}(X)\}$.

If we take m-structures m_X and m_Y as the families of modified open sets stated in the diagram, we can define a new kind of weakly M-open functions. But, we should notice that the families $\mathrm{RO}(X)$ and $\mathrm{SR}(X)$ do not have property $\mathcal{B}$. By the results established in Sections 3–5, we can obtain those properties. We investigate the relationships among these functions.

Lemma 6.1. Let m_X^1 and m_X^2 be two m-structures on a nonempty set X. If $m_X^1 \subset m_X^2$ and a function $f\colon (X, m_X^2) \to (Y, m_Y)$ is weakly M-open, then $f\colon (X, m_X^1) \to (Y, m_Y)$ is weakly M-open.

Proof. Suppose that $f\colon (X, m_X^2) \to (Y, m_Y)$ is weakly M-open. Let $U \in m_X^1$. Since $m_X^1 \subset m_X^2$, we have $U \in m_X^2$ and $f(U) \subset m_Y$-Int$(f(m_X^2$-Cl$(U)))$. Moreover, we have m_X^2-Cl$(U) \subset m_X^1$-Cl(U) and hence, $f(U) \subset m_Y$-Int$(f(m_X^1$-Cl$(U)))$. This shows that $f\colon (X, m_X^1) \to (Y, m_Y)$ is weakly M-open.

Lemma 6.2. Let (X, τ) be a topological space. Then $\alpha\mathrm{Cl}(U) = \mathrm{rCl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))))$ for every $U \in \alpha(X)$.

Proof. Let U be any α-open set of (X, τ). Since $\mathrm{RO}(X) \subset \tau \subset \tau^\alpha$, we have $\alpha\mathrm{Cl}(U) \subset \mathrm{Cl}(U) \subset \mathrm{rCl}(U)$. Suppose that $x \notin \alpha\mathrm{Cl}(U)$. There exists a $G \in \tau^\alpha$ containing x such that $G \cap U = \emptyset$. Hence, we have $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G))) \cap U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G))) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) = \emptyset$. Since $x \in G \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(G))) \in \mathrm{RO}(X)$, we have $x \notin \mathrm{rCl}(U)$. Therefore, we obtain $\mathrm{rCl}(U) \subset \alpha\mathrm{Cl}(U)$ and $\alpha\mathrm{Cl}(U) = \mathrm{Cl}(U) = \mathrm{rCl}(U)$ for every $U \in \alpha(X)$. Moreover, for every $U \in \alpha(X)$, we have $\mathrm{Cl}(U) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))) = \mathrm{rCl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))))$. Therefore, we obtain $\alpha\mathrm{Cl}(U) = \mathrm{rCl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))))$ for every $U \in \alpha(X)$.

Theorem 6.1. Let (X, τ) be a topological space. For any m-space (Y, m_Y), the following properties are equivalent:

(1) $f\colon (X, \mathrm{RO}(X)) \to (Y, m_Y)$ is weakly M-open;
(2) $f\colon (X, \tau_s) \to (Y, m_Y)$ is weakly M-open;
(3) $f\colon (X, \tau) \to (Y, m_Y)$ is weakly M-open;
(4) $f\colon (X, \tau^\alpha) \to (Y, m_Y)$ is weakly M-open.

Proof. Since $\mathrm{RO}(X) \subset \tau_s \subset \tau \subset \tau^\alpha$, by Lemma 6.1 we have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4): Let U be any α-open set of (X, τ). Since $U \in \tau^\alpha$, we have $U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) \in \mathrm{RO}(X)$. By (1),

$$f(U) \subset f(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))) \subset m_Y\text{-}\mathrm{Int}(f(\mathrm{rCl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))))).$$

By Lemma 6.2, we have $f(U) \subset m_Y$-Int$(f(\alpha\mathrm{Cl}(U)))$. This shows that $f\colon (X, \tau^\alpha) \to (Y, m_Y)$ is weakly M-open.

Remark 6.2. In Theorem 6.1, let (Y, σ) be a topological space and $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\beta(Y)$). Then we obtain the following characterizations of weakly semi-open (resp. weakly preopen, weakly β-open) functions.

Corollary 6.1. The following properties are equivalent:

(1) $f\colon (X, \tau) \to (Y, \sigma)$ is weakly open;
(2) $f\colon (X, \tau_s) \to (Y, \sigma)$ is weakly open;
(3) $f\colon (X, \tau^\alpha) \to (Y, \sigma)$ is weakly open.

Proof. This is an immediate consequence of Theorem 6.1.

Theorem 6.2. For any m-space (Y, m_Y) and any function $f\colon (X,\tau) \to (Y, m_Y)$, the following properties are equivalent:

(1) $f\colon (X, \mathrm{SR}(X)) \to (Y, m_Y)$ is weakly M-open;
(2) $f\colon (X, \theta\mathrm{SO}(X)) \to (Y, m_Y)$ is weakly M-open;
(3) $f\colon (X, \delta\mathrm{SO}(X)) \to (Y, m_Y)$ is weakly M-open;
(4) $f\colon (X, \mathrm{SO}(X)) \to (Y, m_Y)$ is weakly M-open.

Proof. Since $\mathrm{SR}(X) \subset \theta\mathrm{SO}(X) \subset \delta\mathrm{SO}(X) \subset \mathrm{SO}(X)$, by Lemma 6.1 we have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4): Suppose that $f\colon (X, \mathrm{SR}(X)) \to (Y, m_Y)$ is weakly M-open. Let $U \in \mathrm{SO}(X)$. Then $\mathrm{sCl}(U) \in \mathrm{SR}(X)$ and we have $f(\mathrm{sCl}(U)) \subset m_Y - \mathrm{Int}(f(\mathrm{srCl}(\mathrm{sCl}(U))))$. We have $\mathrm{srCl}(\mathrm{sCl}(U)) = \mathrm{sCl}(U)$. Therefore, we obtain $f(U) \subset f(\mathrm{sCl}(U)) \subset m_Y - \mathrm{Int}(f(\mathrm{sCl}(U)))$. This shows that $f\colon (X, \mathrm{SO}(X)) \to (Y, m_Y)$ is weakly M-open.

First we recall the relationships among some modifications of semi-preopen (β-open) sets. If a subset A of a topological space (X,τ) is semi-preopen and semi-preclosed, then it is said to be *semi-pre-regular* [24]. It is shown in [24] that the semi-preclosure $\mathrm{spCl}(U)$ is semi-preopen and semi-pre-regular for any semi-preopen set U of (X,τ). This property is very useful. The family of all semi-pre-regular sets of (X,τ) is denoted by $\mathrm{SPR}(X)$. For a subset A of a topological space (X,τ), we put $\mathrm{sprCl}(A) = \cap\{F : A \subset F, F \in \mathrm{SPR}(X)\}$.

Let A be a subset of a topological space (X,τ). A point x of X is called a *semi-pre-θ-cluster point* of A if $\mathrm{spCl}(U) \cap A \neq \emptyset$ for every $U \in \mathrm{SPO}(X)$ containing x. The set of all semi-pre-θ-cluster points of A is called the *semi-pre-θ-closure* [24] of A and is denoted by $\mathrm{spCl}_\theta(A)$. A subset A is said to be *semi-pre-θ-closed* (briefly *sp-θ-closed*) if $A = \mathrm{spCl}_\theta(A)$. The complement of a semi-pre-θ-closed set is said to be *semi-pre-θ-open* (briefly *sp-θ-open*). The family of all semi-pre-θ-open sets of (X,τ) is denoted by $\theta\mathrm{SPO}(X)$.

Lemma 6.3. (Noiri [24]) For a subset A of a topological space (X,τ), the following properties hold:

(1) $A \in \beta(X)$ if and only if $\mathrm{spCl}(A) \in \mathrm{SPR}(X)$,
(2) $\mathrm{SPR}(X) \subset \theta\mathrm{SPO}(X) \subset \beta(X)$.

Theorem 6.3. For any m-space (Y, m_Y) and any function $f\colon (X,\tau) \to (Y, m_Y)$, the following properties are equivalent:

(1) $f\colon (X, \mathrm{SPR}(X)) \to (Y, m_Y)$ is weakly M-open;
(2) $f\colon (X, \theta\mathrm{SPO}(X)) \to (Y, m_Y)$ is weakly M-open;
(3) $f\colon (X, \beta(X)) \to (Y, m_Y)$ is weakly M-open.

Proof. By Lemmas 6.1 and 6.3 (2), we have (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) ⇒ (3): Let U be any β-open set of (X,τ). By Lemma 6.3, $U \subset \mathrm{spCl}(U) \in \mathrm{SPR}(X)$ and by (1) we have

$$\begin{aligned} f(U) \subset f(\mathrm{spCl}(U)) \subset m_Y\text{-Int}(f(\mathrm{sprCl}(\mathrm{spCl}(U)))) &= m_Y\text{-Int}(f(\mathrm{spCl}(U))) \\ &= m_Y\text{-Int}(f({}_\beta\mathrm{Cl}(U))). \end{aligned}$$

This shows that $f\colon (X,\beta(X)) \to (Y, m_Y)$ is weakly M-open.

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