ON WEAK FORMS OF SEMI-OPEN AND SEMI-CLOSED FUNCTIONS

Miguel Caldas and Govindappa Navalagi

Abstract. Weakly semi-open functions between topological spaces are defined as dual to the weak semi-continuous. Connections between this function and other existing topological notions are established and separate examples are given. It is shown that weakly semi-open functions on regular spaces are semi-open. We also introduce and study the concept of weakly semi-closed functions.

1. Introduction and Preliminaries. N. Levine [20] and N. Biswas [6], introduced and investigated the notions of semi-open sets and semi-closed sets in topological spaces, respectively. Since then, a lot of work has been done using these notions and many interesting results have been obtained [2, 6, 10, 13, 15, 26, 27]. D. A. Rose [28] and D. A. Rose with D. S. Janckovic [29] have defined the notions of weakly open and weakly closed functions and investigated some of the fundamental properties of these types of functions. In this paper, we introduce the notions of weakly semi-open and weakly semi-closed functions as a new generalization of weakly open and weakly closed functions, respectively. We investigate some properties of these functions comparing with the other related functions. In this connection, we obtain a new decomposition of semi-openness and closedness. It is also shown that a weakly semi-open inverse image of a semi-connected space is connected.

Throughout this paper, (X, τ) and (Y, σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If S is any subset of a space X, then $Cl(S)$ and $Int(S)$ denote the closure and the interior of S, respectively. Recall that a set S is called regular open (respectively regular closed) if $S = Int(Cl(S))$ (respectively $S = Cl(Int(S))$. A point $x \in X$ is called a θ-cluster [32] point of S if $S \cap Cl(U) \neq \phi$ for each open set U containing x. The set of all θ-cluster points of S is called the θ-closure of S and is denoted by $Cl_\theta(S)$. Hence, a subset S is called θ-closed [32] if $Cl_\theta(S) = S$. The complement of a θ-closed set is called a θ-open set. The θ-interior of a subset S of X is the union of all open subsets of X whose closures are contained in S, and is denoted by $Int_\theta(S)$. A subset $S \subset X$ is called semi-open [20] (respectively preopen [22], α-open [24], and β-open [1] (or semi-preopen [2])), if $S \subset Cl(Int(S))$ (respectively $S \subset Int(Cl(S))$, $S \subset Int(Cl(Int(S))$ and $S \subset Cl(Int(Cl(S)))$. The complement of a semi-open set is called a semi-closed [6] set. The family of all semi-open (respectively semi-closed) sets of a space X is denoted by $SO(X,\tau)$ (respectively $SC(X,\tau)$). The intersection of all semi-closed sets containing S is called the semi-closure of S [6,13] and is denoted by $sCl(S)$. The semi-interior [13] of S is defined by the union of all semi-open sets contained in S and is denoted by $sInt(S)$.

A space X is called extremally disconnected (E.D.) [33] if the closure of each open set in X is open. The space X is called semi-connected [27] if X cannot be expressed as the union of two nonempty disjoint semi-open sets.

A function $f:(X,\tau)\to(Y,\sigma)$ is called:

(i) semi-continuous [20] if for each open subset V of Y, $f^{-1}(V)\in SO(X,\tau)$.

(ii) weakly semi-continuous [12, 17] if for each $x\in X$ and each open subset V of Y containing $f(x)$, there exists a semi-open subset U of X such that $x\in U$ and $f(U)\subset Cl(V)$.

(iii) semi-open [7] (respectively semi-closed [25]) if $f(U)\in SO(Y,\sigma)$ (respectively $f(U)\in SC(Y,\sigma)$) for each open (respectively closed) subset U of X.

(iv) weakly open [28] if $f(U)\subset Int(f(Cl(U)))$ for each open subset U of X.

(v) weakly closed [29] if $Cl(f(Int(F)))\subset f(F)$ for each closed subset F of X.

(vi) almost open in the sense of Singal and Singal, written as (a.o.S) [31] if the image of each regular open subset U of X is an open set in Y.

(vii) preopen [22] (respectively preclosed [14] , β-open [1], α-open [23]) if for each open subset U (respectively closed subset F, open subset U, open subset U) of X, $f(U)$ is a preopen (respectively $f(F)$ is a preclosed, $f(U)$ is a β-open, $f(U)$ is an α-open) set in Y.

(viii) pre-semi-closed [16] if for each semi-closed subset B of X, $f(B)$ is a semi-closed in Y.

(ix) contra-open [4] (respectively contra-closed [4]) if $f(U)$ is closed (respectively open) in Y for each open (respectively closed) subset U of X.

(x) completely continuous [3] if for each open subset V of Y, $f^{-1}(V)$ is a regular open set in X.

(xi) complementary semi-weakly continuous [17] if for each open subset V of Y, $f^{-1}(Fr(V))$ is semi-closed in X, where $Fr(V)$ denotes the frontier on V.

2. Weakly Semi-open Functions. Since semi-continuity [20] is dual to semi-openness [7], we define in this section the concept of weak semi-openness as a natural dual to the weak semi-continuity due to G. Costovici [12] and A. Kar et. al. [19].

Definition 2.1. A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be weakly semi-open if $f(U) \subset sInt(f(Cl(U)))$ for each open subset U of X.

Clearly, every semi-open function is weakly semi-open, but the converse is not generally true.

Example 2.2. A weakly semi-open function need not be semi-open. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a,b\}, \{bc\}, X\}$. Then the identity function $f\colon (X,\tau) \to (X,\sigma)$ is weakly semi-open which is not semi-open, since for $U = \{a\}$, $f(U)$ is not a semi-open set in (X,σ).

Theorem 2.3. For a function $f\colon (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent:

(i) f is weakly semi-open.
(ii) $f(Int_\theta(A)) \subset sInt(f(A))$ for every subset A of X.
(iii) $Int_\theta(f^{-1}(B)) \subset f^{-1}(sInt(B))$ for every subset B of Y.
(iv) $f^{-1}(sCl(B)) \subset Cl_\theta(f^{-1}(B))$ for every subset B of Y.

Proof.

$(i) \to (ii)$: Let A be any subset of X and $x \in Int_\theta(A)$. Then, there exists an open set U such that $x \in U \subset Cl(U) \subset A$. Then, $f(x) \in f(U) \subset f(Cl(U)) \subset f(A)$. Since f is weakly semi-open, $f(U) \subset sInt(f(Cl(U))) \subset sInt(f(A))$. This implies that $f(x) \in sInt(f(A))$. This shows that $x \in f^{-1}(sInt(f(A)))$. Thus, $Int_\theta(A) \subset f^{-1}(sInt(f(A)))$, and so, $f(Int_\theta(A)) \subset sInt(f(A))$.

$(ii) \to (i)$: Let U be an open set in X. As $U \subset Int_\theta(Cl(U))$ implies, $f(U) \subset f(Int_\theta(Cl(U))) \subset sInt(f(Cl(U)))$. Hence, f is weakly semi-open.

$(ii) \to (iii)$: Let B be any subset of Y. Then by (ii), $f(Int_\theta(f^{-1}(B)) \subset sInt(B)$. Therefore, $Int_\theta(f^{-1}(B)) \subset f^{-1}(sInt(B))$.

$(iii) \to (ii)$: This is obvious.

$(iii) \to (iv)$: Let B be any subset of Y. Using (iii), we have $X - Cl_\theta(f^{-1}(B)) = Int_\theta(X - f^{-1}(B)) = Int_\theta(f^{-1}(Y - B)) \subset f^{-1}(sInt(Y - B)) = f^{-1}(Y - sCl(B)) = X - (f^{-1}(sCl(B))$. Therefore, we obtain $f^{-1}(sCl(B)) \subset Cl_\theta(f^{-1}(B))$.

$(iv) \to (iii)$: Similarly we obtain, $X - f^{-1}(sInt(B)) \subset X - Int_\theta(f^{-1}(B))$, for every subset B of Y, i.e., $Int_\theta(f^{-1}(B)) \subset f^{-1}(sInt(B))$.

Furthermore, we can prove the following theorem.

Theorem 2.4. For a function $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent.

(i) f is weakly semi-open
(ii) For each $x\in X$ and each open subset U of X containing x, there exists a semi-open set V containing $f(x)$ such that $V\subset f(Cl(U))$.

Proof.

$(i)\to(ii)$: Let $x\in X$ and U be an open set in X with $x\in U$. Since f is weakly semi-open, $f(x)\in f(U)\subset sInt(f(Cl(U)))$. Let $V=sInt(f(Cl(U)))$. Hence, $V\subset f(Cl(U))$, with V containing $f(x)$.

$(ii)\to(i)$: Let U be an open set in X and let $y\in f(U)$. It follows from (ii) that $V\subset f(Cl(U))$ for some V semi-open in Y containing y. Hence, we have $y\in V\subset sInt(f(Cl(U)))$. This shows that $f(U)\subset sInt(f(Cl(U)))$, i.e., f is a weakly semi-open function.

Theorem 2.5. Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective function. Then the following statements are equivalent.

(i) f is weakly semi-open,
(ii) $sCl(f(U))\subset f(Cl(U))$ for each U open in X,
(iii) $sCl(f(Int(F))\subset f(F)$ for each F closed in X.

Proof.

$(i)\to(iii)$: Let F be a closed set in X. Then we have $f(X-F)=Y-f(F)\subset sInt(f(Cl(X-F)))$ and so $Y-f(F)\subset Y-sCl(f(Int(F)))$. Hence, $sCl(f(Int(F)))\subset f(F)$.

$(iii)\to(ii)$: Let U be an open set in X. Since $Cl(U)$ is a closed set and $U\subset Int(Cl(U))$ by (iii) we have $sCl(f(U))\subset sCl(f(Int(Cl(U)))\subset f(Cl(U))$.

$(ii)\to(iii)$: Similar to $(iii)\to(ii)$.

$(iii)\to(i)$: Clear.

The proof of the following theorem is straightforward and thus is omitted.

Theorem 2.6. For a function $f:(X,\tau)\to(Y,\sigma)$ the following conditions are equivalent.

(i) f is weakly semi-open.
(ii) $f(Int(F))\subset sInt(f(F))$ for each closed subset F of X,
(iii) $f(Int(Cl(U)))\subset sInt(f(Cl(U)))$ for each open subset U of X,
(iv) $f(U)\subset sInt(f(Cl(U)))$ for each preopen subset U of X,
(v) $f(U)\subset sInt(f(Cl(U)))$ for each α-open subset U of X.

Now, we recall a definition of strong continuity. This definition when combined with weak semi-openness implies semi-openness.

A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be strongly continuous [22], if for every subset A of X, $f(Cl(A)) \subset f(A)$.

Theorem 2.7. If $f\colon (X,\tau) \to (Y,\sigma)$ is weakly semi-open and strongly continuous, then f is semi-open.

Proof. Let U be an open subset of X. Since f is weakly semi-open $f(U) \subset sInt(f(Cl(U)))$. However, because f is strongly continuous, $f(U) \subset sInt(f(U))$ and therefore, $f(U)$ is semi-open.

Example 2.8. A semi-open function need not be strongly continuous. Let $X = Y = \{a,b,c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a,b\}, Y\}$. Then the identity function $f\colon (X,\tau) \to (Y,\sigma)$ is semi-open but is not strongly continuous since, if $U = \{a\}$, $f(Cl(U)) = f(X) = X \not\subset f(U)$.

Recall that $f\colon (X,\tau) \to (Y,\sigma)$ is said to be relatively weakly open [5] provided that $f(U)$ is open in $f(Cl(U))$ for every open subset U of X.

Theorem 2.9. A function $f\colon (X,\tau) \to (Y,\sigma)$ is semi-open if f is weakly semi-open and relatively weakly open.

Proof. Assume f is weakly semi-open and relatively weakly open. Let U be an open subset of X and let $y \in f(U)$. Since f is relatively weakly open there is an open subset V of Y for which $f(U) = f(Cl(U)) \cap V$. Because f is weakly semi-open, it follows that $f(U) \subset sInt(f(Cl(U)))$. Then $y \in sInt(f(Cl(U))) \cap V \subset f(Cl(U)) \cap V = f(U)$ and therefore, $f(U)$ is semi-open.

Theorem 2.10.

(i) If $f\colon (X,\tau) \to (Y,\sigma)$ is contra-closed, then f is a weakly semi-open function.
(ii) If $f\colon (X,\tau) \to (Y,\sigma)$ is preopen and contra-open, then f is weakly semi-open.

Proof.

(i) Let U be an open subset of X. Then, we have $f(U) \subset f(Cl(U))) = sInt(f(Cl(U)))$.
(ii) Let U be an open subset of X. Since f is preopen $f(U) \subset Int(Cl(f(U)))$ and since f is contra-open $f(U)$ is closed. Therefore, $f(U) \subset Int(Cl(f(U))) = Int(f(U)) \subset Int(f(Cl(U))) \subset sInt(f(Cl(U)))$.

The converse of Theorem 2.10 does not hold.

Example 2.11. A weakly semi-open function need not be contra-closed is given from Example 2.8.

Theorem 2.12. Let X be a regular space. Then $f\colon (X,\tau) \to (Y,\sigma)$ is weakly semi-open if and only if f is semi-open.

Proof. The sufficiency is clear. For the necessity, let W be a nonempty open subset of X. For each x in W, let U_x be an open set such that $x \in U_x \subset Cl(U_x) \subset W$. Hence, we obtain that $W = \cup\{U_x \ : \ x \in W\} = \cup\{Cl(U_x) \ : \ x \in W\}$ and, $f(W) = \cup\{f(U_x) \ : \ x \in W\} \subset \cup\{sInt(f(Cl(U_x))) : x \in W\} \subset sInt(f(\cup\{Cl(U_x) : x \in W\}) = sInt(f(W))$. Thus, f is semi-open.

Next, we define the dual form of a complementary semi-weakly continuous function called a complementary weakly semi-open function.

Definition 2.13. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called complementary weakly semi-open (written as c.w.s.o.) if for each open set U of X, $f(Fr(U))$ is semi-closed in Y, where $Fr(U)$ denotes the frontier of U.

The following two examples show that weakly semi-open and c.w.s.o. are independent of each other.

Example 2.14. A weakly semi-open function need not be c.w.s.o. Let $X = \{a,b\}$, $\tau = \{\emptyset, \{b\}, X\}$, $Y = \{x,y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. Let $f\colon (X,\tau) \to (Y,\sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is clearly weakly semi-open, but is not c.w.s.o., since $F_r(\{b\}) = Cl(\{b\}) - \{b\} = \{a\}$ and $f(F_r(\{b\})) = \{x\}$ is not a semi-closed set in Y.

Example 2.15. c.w.s.o. does not imply weakly semi-open. Let $X = \{a,b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X\}$, $Y = \{x,y\}$ and $\sigma = \{\emptyset, \{y\}, Y\}$. Let $f\colon (X,\tau) \to (Y,\sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is not weakly semi-open, but f is c.w.s.o., since the frontier of every open set is the empty set and $f(\emptyset) = \emptyset$ is semi-closed.

Theorem 2.16. Let Y be an extremally disconnected space. If $f\colon (X,\tau) \to (Y,\sigma)$ is bijective weakly semi-open and c.w.s.o., then f is semi-open.

Proof. Let U be an open subset in X with $x \in U$, since f is weakly semi-open, by Theorem 2.4 there exists a semi-open set V containing $f(x) = y$ such that $V \subset f(Cl(U))$. Now $Fr(U) = Cl(U) - U$ and thus $x \notin Fr(U)$. Hence, $y \notin f(Fr(U))$ and therefore, $y \in V - f(Fr(U))$. Put $V_y = V - f(Fr(U))$, a semi-open set since f is c.w.s.o and Y is extremally disconnected. Since $y \in V_y$, $y \in f(Cl(U))$. But $y \notin f(Fr(U))$ and thus, $y \notin f(Fr(U)) = f(Cl(U)) - f(U)$ which implies that $y \in f(U)$. Therefore, $f(U) = \cup\{V_y : V_y \in SO(Y,\sigma), y \in f(U)\}$. Hence, f is semi-open.

The following theorem is a variation of a result of C. Baker [4] in which contra-openness is replaced with weakly semi-open and closed by contra-pre-semi-closed, where, $f\colon(X,\tau)\to(Y,\sigma)$ is said to be contra-pre-semi-closed [9] provided that $f(F)$ is semi-open for each semi-closed subset F of X.

Theorem 2.17. If $f\colon(X,\tau)\to(Y,\sigma)$ is weakly semi-open, $SC(Y,\sigma)$ is closed under arbitrary unions and if for each semi-closed subset F of X and each fiber $f^{-1}(y)\subset X-F$ there exists an open subset U of X for which $F\subset U$ and $f^{-1}(y)\cap Cl(U)=\phi$, then f is contra-pre-semi-closed.

Proof. Assume F is a semi-closed subset of X and let $y\in Y-f(F)$. Thus, $f^{-1}(y)\subset X-F$ and hence, there exists an open subset U of X for which $F\subset U$ and $f^{-1}(y)\cap Cl(U)=\phi$. Therefore, $y\in Y-f(Cl(U))\subset Y-f(F)$. Since f is weakly semi-open $f(U)\subset sInt(f(Cl(U)))$. By complement, we obtain $y\in sCl(Y-f(Cl(U)))\subset Y-f(F)$. Let $B_y=sCl(Y-f(Cl(U)))$. Then B_y is a semi-closed subset of Y containing y. Hence, $Y-f(F)=\cup\{B_y : y\in Y-f(F)\}$ is semi-closed and therefore, $f(F)$ is semi-open.

Theorem 2.18. If $f\colon(X,\tau)\to(Y,\sigma)$ is an a.o.S function, then it is a weakly semi-open function.

Proof. Let U be an open set in X. Since f is a.o.S and $Int(Cl(U))$ is regular open, $f(Int(Cl(U)))$ is open in Y and hence, $f(U)\subset f(Int(Cl(U))\subset Int(f(Cl(U)))\subset sInt(f(Cl(U)))$. This shows that f is weakly semi-open.

The converse of Theorem 2.18 is not true in general.

Example 2.19. A weakly semi-open function need not be a.o.S. Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},\{c\},\{a,c\},X\}$, and $\sigma=\{\emptyset,\{b\},\{a,b\},\{b,c\},X\}$. Let $f\colon(X,\tau)\to(Y,\sigma)$ be the identity. Then f is not a.o.S since $Int(f(Int(Cl(\{a\}))))=\emptyset$. But f is weakly semi-open.

Lemma 2.20. If $f\colon(X,\tau)\to(Y,\sigma)$ is a continuous function, then for any subset U of X, $f(Cl(U))\subset Cl(f(U))$ [28].

Theorem 2.21. If $f:(X,\tau)\to(Y,\sigma)$ is a weakly semi-open and continuous function, then f is a β-open function.

Proof. Let U be an open set in X. Then by weak semi-openness of f, $f(U)\subset sInt(f(Cl(U)))$. Since f is continuous $f(Cl(U))\subset Cl(f(U))$. Hence, we obtain that $f(U)\subset sInt(f(Cl(U)))\subset sInt(Cl(f(U)))\subset Cl(Int(Cl(f(U))))$. Therefore, $f(U)\subset Cl(Int(Cl(f(U))))$ which shows that $f(U)$ is a β-open set in Y. Thus, f is a β-open function.

Since every strongly continuous function is continuous we have the following corollary.

Corollary 2.22. If $f: (X, \tau) \to (Y, \sigma)$ is a weakly semi-open and strongly continuous function, then f is a β-open function.

Theorem 2.23. If $f: (X, \tau) \to (Y, \sigma)$ is an injective weakly semi-open function of a space X onto a semi-connected space Y, then X is connected.

Proof. Let us assume that X is not connected. Then there exist nonempty open sets U_1 and U_2 such that $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = X$. Hence, we have $f(U_1) \cap f(U_2) = \phi$ and $f(U_1) \cup f(U_2) = Y$. Since f is weakly semi-open, we have $f(U_i) \subset sInt(f(Cl(U_i)))$ for $i = 1, 2$, and since U_i is open and also closed, we have $f(Cl(U_i)) = f(U_i)$ for $i = 1, 2$. Hence, $f(U_i)$ is semi-open in Y for $i = 1, 2$. Thus, Y has been decomposed into two nonempty disjoint semi-open sets. This is contrary to the hypothesis that Y is a semi-connected space. Thus, X is connected.

Definition 2.24. A space X is said to be hyperconnected [26] if every nonempty open subset of X is dense in X.

Theorem 2.25. If X is a hyperconnected space, then a function $f: (X, \tau) \to (Y, \sigma)$ is weakly semi-open if and only if $f(X)$ is semi-open in Y.

Proof. The sufficiency is clear. For the necessity observe that for any open subset U of $X, f(U) \subset f(X) = sInt(f(X)) = sInt(f(Cl(U)))$.

3. Weakly semi-closed Functions. Now, we define a generalized form of semi-closed functions.

Definition 3.1. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be weakly semi-closed if $sCl(f(Int(F))) \subset f(F)$ for each closed set F in X.

Clearly,

(i) Every closed function is α-closed and every α-closed function is semi-closed, but the reverse implications are not true in general [23].
(ii) every semi-closed function is a weakly semi-closed function, but the converse is not generally true, as the next example shows.

Example 3.2.

(i) Let $X = \{x, y, z\}$ and $\tau = \{\emptyset, \{x\}, \{x, y\}, X\}$. Then a function $f: (X, \tau) \to (X, \tau)$ which is defined by $f(x) = x$, $f(y) = z$ and $f(z) = y$ is α-open and α-closed but neither open nor closed [23].
(ii) Let $f: (X, \tau) \to (Y, \sigma)$ be the function from Example 2.2. Then f is weakly semi-closed but not semi-closed.

From the observation above and Example 3.2, we have the following diagram.

$$\begin{array}{ccc}
\text{closed (open) func.} & \longrightarrow & \alpha\text{-closed } (\alpha\text{-open}) \text{ func.} \\
\downarrow & & \downarrow \\
\text{weakly semi-closed (semi-open) func.} & \longleftarrow & \text{semi-closed (semi-open) func.}
\end{array}$$

<u>Theorem 3.3</u>. For a function $f\colon (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent.

(i) f is weakly semi-closed.
(ii) $sCl(f(U)) \subset f(Cl(U))$ for every open subset U of X.

<u>Proof</u>.

$(i) \to (ii)$. Let U be any open subset of X. Then $sCl(f(U)) = sCl(f(Int(U))) \subset sCl(f(Int(Cl(U)))) \subset f(Cl(U))$.

$(ii) \to (i)$. Let F be any closed subset of X. Then, $sCl(f(Int(F))) \subset f(Cl(Int(F))) \subset f(Cl(F)) = f(F)$.

The proof of the following theorem is straightforward and is therefore omitted.

<u>Theorem 3.4</u>. For a function $f\colon (X,\tau) \to (Y,\sigma)$ the following conditions are equivalent:

(i) f is weakly semi-closed,
(ii) $sCl(f(U)) \subset f(Cl(U))$ for each open subset U of X,
(iii) $sCl(f(Int(F))) \subset f(Cl(Int(F)))$ for each closed subset F of X,
(iv) $sCl(f(Int(F))) \subset f(Cl(Int(F)))$ for each regular closed subset F of X,
(v) $sCl(f(Int(F))) \subset f(F)$ for each pre-closed subset F of X,
(vi) $sCl(f(Int(F))) \subset f(F)$ for every α-closed subset F of X.

<u>Remark 3.5</u>. By Theorem 2.5, if $f\colon (X,\tau) \to (Y,\sigma)$ is a bijective function, then f is weakly semi-open if and only if f is weakly semi-closed.

<u>Theorem 3.6</u>.

(i) If $f\colon (X,\tau) \to (Y,\sigma)$ is contra-open, then f is weakly semi-closed.
(ii) If $f\colon (X,\tau) \to (Y,\sigma)$ is preclosed and contra-closed, then f is weakly semi-closed.

Proof.

(i) Let F be a closed subset of X. Then $sCl(f(Int(F))) \subset f(Int(F)) \subset f(F)$.

(ii) Let F be a closed subset of X. Since f is preclosed $Cl(Int(f(F))) \subset f(F)$ and since f is contra-closed $f(F)$ is open. Therefore, $sCl(f(Int(F))) \subset sCl(f(F)) \subset Cl(Int(f(F))) \subset f(F)$.

G. L. Garg and D. Sivaraj in [16] showed that if f is a pre-semi-closed function then $sCl(f(A)) \subset f(sCl(A))$ for every subset A of X. Therefore, every pre- semi-closed function is weakly semi-closed.

Theorem 3.7. If $f\colon (X,\tau) \to (Y,\sigma)$ is one-to-one and weakly semi-closed, then for every subset F of Y and every open set U in X with $f^{-1}(F) \subset U$, there exists a semi-closed set B in Y such that $F \subset B$ and $f^{-1}(B) \subset Cl(U)$.

Proof. Let F be a subset of Y and let U be an open subset of X with $f^{-1}(F) \subset U$. Put $B = sCl(f(Int(Cl(U))))$, then B is a semi-closed subset of Y such that $F \subset B$ since $F \subset f(U) \subset f(Int(Cl(U))) \subset sCl(f(Int(Cl(U)))) = B$. And since f is weakly semi-closed, $f^{-1}(B) \subset Cl(U)$.

Taking the set F in Theorem 3.7 to be y for $y \in Y$ we obtain the following result.

Corollary 3.8. If $f\colon (X,\tau) \to (Y,\sigma)$ is one-to-one and weakly semi-closed, then for every point y in Y and every open set U in X with $f^{-1}(y) \subset U$, there exists a semi-closed set B in Y containing y such that $f^{-1}(B) \subset Cl(U)$.

Recall, that two nonempty sets A and B of X are strongly separated [29], if there exist open sets U and V in X with $A \subset U$ and $B \subset V$ and $Cl(U) \cap Cl(V) = \phi$. If A and B are singleton sets we may speak of points being strongly separated. We will use the fact that in a normal space, disjoint closed sets are strongly separated.

A space X is said to be Ultra semi-Hausdorff or in short Ultra semi-T_2 if for every pair of distinct points x and y, there exist two semi-closed sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.

Theorem 3.9. If $f\colon (X,\tau) \to (Y,\sigma)$ is a weakly semi-closed surjection and all pairs of disjoint fibers are strongly separated, then Y is Ultra semi-T_2.

Proof. Let y and z be two points in Y. Let G and H be open sets in X such that $f^{-1}(y) \in G$ and $f^{-1}(z) \in H$ with $Cl(G) \cap Cl(H) = \phi$. By weak semi-closedness (Theorem 3.7) there are semi-closed sets U and V in

Y such that $y \in U$ and $z \in V$, $f^{-1}(U) \subset Cl(G)$ and $f^{-1}(V) \subset Cl(H)$. Therefore, $U \cap V = \phi$ because $Cl(G) \cap Cl(H) = \phi$ and f surjective. Then Y is Ultra semi-T_2.

Corollary 3.10. If $f\colon (X,\tau) \to (Y,\sigma)$ is a weakly semi-closed surjection with all fibers closed and X is normal, then Y is Ultra semi-T_2.

Corollary 3.11. If $f\colon (X,\tau) \to (Y,\sigma)$ is a continuous weakly semi-closed surjection with X a compact T_2 space and Y a T_1 space, then Y is a compact Ultra semi-T_2 space.

Proof. Since f is a continuous surjection and Y is a T_1 space, Y is compact and all fibers are closed. Since X is normal Y is also Ultra semi-T_2 space.

Definition 3.12. A topological space X is said to be quasi H-closed [11] (respectively SC-closed [8]), if every open (respectively semi-closed) cover of X has a finite subfamily whose closures cover X. A subset A of a topological space X is quasi H-closed relative to X (respectively SC-closed relative to X) if every cover of A by open (respectively semi-closed) subsets of X has a finite subfamily whose closures cover A.

Remark 3.13. D. S. Jankovic in [18] showed that a space X is extremally disconnected if and only if the intersection of two semi-open sets is semi-open, consequently the union of two semi-closed sets is semi-closed.

Lemma 3.14. A function $f\colon (X,\tau) \to (Y,\sigma)$ is open if and only if for each $B \subset Y$, $f^{-1}(Cl(B)) \subset Cl(f^{-1}(B))$ [30].

Theorem 3.15. Let X be an extremally disconnected space. Let $f\colon (X,\tau) \to (Y,\sigma)$ be an open, weakly semi-closed function which is one-to-one and such that $f^{-1}(y)$ is quasi H-closed relative to X for each y in Y. If G is SC-closed relative to Y then $f^{-1}(G)$ is quasi H-closed.

Proof. Let $\{V_\beta : \beta \in I\}$, I being the index set, be an open cover of $f^{-1}(G)$. Then for each $y \in G \cap f(X)$, $f^{-1}(y) \subset \cup\{Cl(V_\beta) : \beta \in I(y)\} = H_y$ for some finite subfamily $I(y)$ of I. Since X is extremally disconnected each $Cl(V_\beta)$ is open, hence, H_y is open in X. So by Corollary 3.8, there exists a semi-closed set U_y containing y such that $f^{-1}(U_y) \subset Cl(H_y)$. By Remark 3.13 $\{U_y : y \in G \cap f(X)\} \cup \{Y - f(X)\}$ is a semi-closed cover of G, $G \subset \cup\{Cl(U_y) : y \in K\} \cup \{Cl(Y - f(X))\}$ for some finite subset K of $g \cap f(X)$. Hence and by Lemma 3.14, $f^{-1}(G) \subset \cup\{f^{-1}(Cl(U_y)) : y \in K\} \cup \{f^{-1}(Cl(Y - f(X)))\} \subset \cup\{Cl(f^{-1}(U_y)) : y \in K\} \cup \{Cl(f^{-1}(Y - f(X)))\} \subset \{Cl(f^{-1}(U_y)) : y \in K\}$, so $f^{-1}(G) \subset \cup\{Cl(V_\beta) : \beta \in I(y), y \in K\}$. Therefore, $f^{-1}(G)$ is quasi H-closed.

Corollary 3.16. Let $f\colon (X,\tau) \to (Y,\sigma)$ be as in Theorem 3.15. If Y is SC-closed, then X is quasi H-closed.

Acknowledgment. The authors are grateful to the referee for his remarkable work which improved the quality of this paper.

References

1. M. E. Abd-El-Monsef, S. N. El-Deeb, and R. A. Mahmoud, "β-open and β-continuous Mappings," *Bull. Fac. Sci. Assiut Univ.,* 12 (1983), 70–90.
2. D. Andrijevic, "Semipreopen Sets," *Mat. Vesnik,* 38 (1986), 24–32 .
3. S. P. Arya and R. Gupta, "Strongly Continuous Mappings," *Kyungpook Math. J.,* 14 (1974), 131–143.
4. C. W. Baker, "Contra-open Functions and Contra-closed Functions," *Math. Today,* 15 (1997), 19–24.
5. C. W. Baker, "Decomposition of Openness," *Internat. J. Math. & Math. Sci.,* 17 (1994), 413–415.
6. N. Biswas, "Characterizations of Semicontinuous Functions," *Atti. Accad. Naz. Lince Rend. Cl. Sci. Fis. Mat. Natur.,* 48 (1970), 399–402.
7. N. Biswas, "On Some Mappings in Topological Spaces," *Bull. Calcutta Math. Soc.,* 61 (1969), 127–135.
8. M. Caldas and C. W. Baker, "Contra Pre-semiopen Maps," *Kyungpook Math. Journal,* 40 (2000), 379–389.
9. M. Caldas, "Weak and Strong Forms of Irresolute Maps," *Internat. J. Math. & Math. Sci.,* 23 (2000), 253–259.
10. M. Caldas, "Semigeneralized Continuous Maps in Topological Spaces," *Portugaliae Math.,* 52 (1995), 399–407.
11. D. E. Cameron, "Some Maximal Topologies Which are Q. H. C.," *Proc. Amer. Math. Soc.,* 75 (1979), 149–156.
12. G. Costovici, "Other Elementary Properties of the Mappings of Topological Spaces," *Bul. Inst. Politehn. lasi. Sect.,* 26 (1980), 19–21.
13. S. G. Crossley and S. K. Hilderbrand, "Semi-closure," *Texas J. Sci.,* 22 (1971), 99–112.
14. S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour, and T. Noiri, "On p-regular Spaces," *Bull. Math. de la Soc. Sci. Math. de la. R. S. Roumanie,* 27 (1983), 311–315.
15. J. Dontched and M. Ganster, "More on sg-compact Saces," *Portugaliae Math.,* 55 (1998), 457–464.

16. G. L. Garg and D. Shivaraj, "Presemiclosed Mappings," *Periodica Math. Hung.,* 19 (1988), 97–106.

17. S. Jafari, "Semicontinuity in Topological Spaces.," (submitted).

18. D. S. Jankovick, "On Locally Irreducible Spaces," *Ann. Soc. Sci. Bruxelle,* 97 (1983), 59–72.

19. A. Kar and P. Bhattacharya, "Weakly Semicontinuous Functions," *J. Indian Acad. Math.,* 8 (1986), 83–93.

20. N. Levine, "Semiopen Sets and Semicontinuity in Topological Spaces," *Amer. Math. Monthly,* 70 (1963), 36–41.

21. N. Levine, "Strong Continuity in Topological Spaces," *Amer. Math. Monthly,* 67 (1960), 269.

22. A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deeb, "Precontinuous and Weak Precontinuous Mappings," *Proc. Math. Phys. Soc. Egypt.,* 53 (1982), 47–53.

23. A. S. Mashhour, I. A. Hasanein, and S. N. EL-Deeb, "α-continuous and α-open Mappings," *Acta Math. Hung.,* 41 (1983), 213–218.

24. O. Njastad, "On Some Classes of Nearly Open Sets," *Pacific J. Math.,* 15 (1965), 961–970.

25. T. Noiri, "A Generalization of Closed Mappings," *Atti. Accad. Naz. Lince Rend. Cl. Sci. Fis. Mat. Natur.,* 8 (1973), 210–214.

26. T. Noiri, "On θ-semicontinuous Functions," *India J.Math.,* 21 (1990), 410–415.

27. V. Pipitone and G. Russo, "Spazi Semiconnessi e Spazi Semiaperti," *Rend. Circ. Mat. Palermo,* 24 (1975), 273–285.

28. D. A. Rose, "On Weak Openness and Almost Openness," *Internat. J. Math. & Math. Sci.,* 7 (1984), 35–40.

29. D. A. Rose and D. S. Jankovic, "Weakly Closed Functions and Hausdorff Spaces," *Math. Nachr.,* 130 (1987), 105–110.

30. R. Sikorski, "Closure Homomorphisms and Interior Mappings," *Fund. Math.,* 41 (1954), 12–20.

31. M. K. Singal and A. R. Singal, "Almost Continuous Mappings," *Yokohama Math. J.,* 16 (1968), 63–73.

32. N. V. Velicko, "H-closed Topological Spaces," *Amer. Math. Soc. Transl.,* 78 (1968), 103–118.

33. S. Willard, *General Topology,* Addison-Wesley Publishing Company (1970).

Mathematics Subject Classification (2000): 54A40, 54C10, 54D10, 54C08

Miguel Caldas
Departamento de Matematica Aplicada
Universidade Federal Fluminense
Rua Mario Santos Braga, s/n
CEP: 24020-140
Niteroi, RJ. Brasil
email: gmamccs@vm.uff.br

G. B. Navalagi
Department of Mathematics
KLE Societys
G. H. College, Haveri-581110,
Karnataka, India
e-mail: gnavalagi@hotmail.com